A Truncated Bivariate Cauchy Distribution

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Abstract. A truncated version of the bivariate Cauchy distribution is introduced. Explicit expressions for its moments and estimation procedures are derived. Unlike the Cauchy distribution, this possesses finite moments of all orders and could therefore be a better model for certain practical situations. An application with real data is discussed to show one such situation.

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1. Introduction

The main weakness of Cauchy distributions is that they have no moments. Nadarajah and Kotz [3] suggested a truncated univariate Cauchy distribution that overcomes this weakness. In this note, we study the corresponding bivariate version. The bivariate Cauchy distribution is given by the joint probability density function (pdf):

\[ g(x, y) = \frac{1}{2\pi} \left( 1 + x^2 + y^2 \right)^{-3/2} \]  

for \(-\infty < x < \infty\) and \(-\infty < y < \infty\). Let \(G\) denote the corresponding joint cumulative distribution function (cdf) given by:

\[ G(x, y) = \frac{1}{4} + \frac{1}{2\pi} \left( \arctan x + \arctan y + \arctan \frac{xy}{\sqrt{1 + x^2 + y^2}} \right). \]  

Then the truncated version of (1.1) is given by the joint pdf:

\[ f(x, y) = \frac{1}{2\pi \Omega} \left( 1 + x^2 + y^2 \right)^{-3/2} \]  

for \(-\infty < B \leq x \leq A < \infty\) and \(-\infty < D \leq y \leq C < \infty\), where \(\Omega = G(A, C) - G(A, D) - G(B, C) + G(B, D)\). The cdf associated with (1.3) is:

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\begin{align}
F(x, y) &= \frac{1}{\Omega} \{ G(x, y) - G(x, D) - G(B, y) + G(B, D) \}. \\
\text{The corresponding marginal cdfs are:} \\
F_X(x) &= \frac{1}{\Omega} \{ G(x, C) - G(x, D) - G(B, C) + G(B, D) \} \\
\text{and} \\
F_Y(y) &= \frac{1}{\Omega} \{ G(A, y) - G(A, D) - G(B, y) + G(B, D) \}. \\
\text{The corresponding marginal pdfs are:} \\
f_X(x) &= \begin{cases} \\
\frac{|C|}{2\Omega\pi} (1 + x^2)^{3/2} \left\{ \frac{(2C^* + 5) \sqrt{C^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{C^*} \right)}{8\sqrt{C^*}} \right\}, & (D > 0), \\
\frac{|C|}{2\Omega\pi} (1 + x^2)^{3/2} \left\{ \frac{(2C^* + 5) \sqrt{C^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{C^*} \right)}{8\sqrt{C^*}} \right\}, & (D < 0, C > 0), \\
\frac{|C|}{2\Omega\pi} (1 + x^2)^{3/2} \left\{ \frac{(2C^* + 5) \sqrt{C^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{C^*} \right)}{8\sqrt{C^*}} \right\}, & (C < 0) \\
\end{cases} \\
\text{and} \\
f_Y(y) &= \begin{cases} \\
\frac{|A|}{2\Omega\pi} (1 + y^2)^{3/2} \left\{ \frac{(2A^* + 5) \sqrt{A^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{A^*} \right)}{8\sqrt{A^*}} \right\}, & (B > 0), \\
\frac{|A|}{2\Omega\pi} (1 + y^2)^{3/2} \left\{ \frac{(2A^* + 5) \sqrt{A^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{A^*} \right)}{8\sqrt{A^*}} \right\}, & (B < 0, A > 0), \\
\frac{|A|}{2\Omega\pi} (1 + y^2)^{3/2} \left\{ \frac{(2A^* + 5) \sqrt{A^*} + 1}{8} + \frac{3\text{arcsinh} \left( \sqrt{A^*} \right)}{8\sqrt{A^*}} \right\}, & (A < 0), \\
\end{cases}
\end{align}
where $A^* = A^2/(1 + y^2)$, $B^* = B^2/(1 + y^2)$, $C^* = C^2/(1 + x^2)$, $D^* = D^2/(1 + x^2)$ and $_2F_1$ denotes the Gauss hypergeometric function defined by

$$
_2F_1 (a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},
$$

where $(e)_k = e(e + 1) \cdots (e + k - 1)$ denotes the ascending factorial, see Prudnikov et al. [5] and Gradshteyn and Ryzhik [1]. We refer to (1.3) as the truncated bivariate Cauchy distribution. Because (1.3) is defined over a finite interval, the truncated bivariate Cauchy distribution has all its moments. Thus, (1.3) may prove to be a better model for certain practical situations than one based on just the bivariate Cauchy distribution, see Section 4.

The rest of this paper is organized as follows: explicit expressions for the moments of (1.3) are derived in Section 2, estimation issues are discussed in Section 3, and an application to consumer price indices data is illustrated in Section 4.

2. Moments

We derive three representations for the product moment $E(X^\alpha Y^\beta)$. Theorems 2.1 and 2.2 provide general representations while Theorem 2.3 considers the particular case that $\beta$ is an odd integer.

**Theorem 2.1.** If $(X, Y)$ has the joint pdf (1.3) then

$$
E(X^\alpha Y^\beta) = \begin{cases} 
H(A, C) - H(B, C) - H(A, D) + H(B, D), & \text{if } B > 0 \text{ and } D > 0, \\
(-1)^\alpha \{H(-B, C) - H(-A, C) - H(-B, D) + H(-A, D)\}, & \text{if } A < 0 \text{ and } D > 0, \\
(-1)^{\alpha+\beta} \{H(-B, -D) - H(-B, -C) - H(-A, -D) + H(-A, -C)\}, & \text{if } A < 0 \text{ and } C < 0, \\
(-1)^{\beta} \{H(A, -D) - H(A, -C) - H(B, -D) + H(B, -C)\}, & \text{if } B > 0 \text{ and } C < 0, \\
H(A, C) + (-1)^\alpha H(-B, C) + (-1)^{\beta} H(A, -D) + (-1)^{\alpha+\beta} H(-B, -D), & \text{if } B < 0, A > 0, D < 0 \text{ and } C > 0, \\
H(A, C) - H(A, D) + (-1)^\alpha \{H(-B, C) - H(-B, D)\}, & \text{if } B < 0, A > 0 \text{ and } D > 0, \\
H(A, C) - H(B, C) + (-1)^{\beta} \{H(A, -D) - H(B, -D)\}, & \text{if } B > 0, D < 0 \text{ and } C > 0, \\
(-1)^\alpha \{H(-B, C) - H(-A, C)\} + (-1)^{\alpha+\beta} \{H(-B, -D) - H(-A, -D)\}, & \text{if } A < 0, D < 0 \text{ and } C > 0, \\
(-1)^{\beta} \{H(A, -D) - H(A, -C)\} + (-1)^{\alpha+\beta} \{H(-B, -D) - H(-B, -C)\}, & \text{if } B < 0, A > 0 \text{ and } C < 0,
\end{cases}
$$

for $\alpha \geq 1$ and $\beta \geq 1$, where

$$
H(P, Q) = \frac{1}{2\pi} \int_0^P \int_0^Q \frac{x^\alpha y^\beta}{(1 + x^2 + y^2)^{3/2}} dy dx.
$$
Proof. The result of the theorem is obvious. For example, if \( B > 0 \) then \( D > 0 \) one can write

\[
E \left( X^\alpha Y^\beta \right) = \frac{1}{2\pi} \int_B^A \int_D^C x^\alpha y^\beta \left( 1 + x^2 + y^2 \right)^{-3/2} dy dx
\]

\[
= \frac{1}{2\pi} \left\{ \int_0^A \int_0^C x^\alpha y^\beta \left( 1 + x^2 + y^2 \right)^{-3/2} dy dx \\
- \int_0^A \int_0^D x^\alpha y^\beta \left( 1 + x^2 + y^2 \right)^{-3/2} dy dx \\
- \int_0^B \int_0^C x^\alpha y^\beta \left( 1 + x^2 + y^2 \right)^{-3/2} dy dx \\
+ \int_0^B \int_0^D x^\alpha y^\beta \left( 1 + x^2 + y^2 \right)^{-3/2} dy dx \right\}
\]

\[
= H(A, C) - H(B, C) - H(A, D) + H(B, D).
\]

The remaining cases can be established similarly.

Theorem 2.2. If \((X, Y)\) has the joint pdf (1.3) then (2.2) can be expressed in the form

\[
H(P, Q) = \frac{P^{\alpha+1}Q^{1+2\beta}}{2\pi(\alpha+1)(1+2\beta)} \sum_{k=0}^{\infty} \frac{(-1)^k(\beta + 1/2)k(3/2)_k Q^2}{(\beta + 3/2)_k k!} \\
\times 2F_1 \left( \frac{\alpha + 1}{2}, \frac{3}{2} + k; \frac{\alpha + 3}{2}; -P^2 \right).
\]

Proof. Setting \( z = 1 + x^2 + y^2 \), (2.2) can be reexpressed as

\[
(2.3) \quad H(P, Q) = \frac{1}{4\pi} \int_0^P \int_{1+x^2}^{1+x^2+Q^2} x^\alpha (z - 1 - x^2)^{(\beta-1)/2} z^{-3/2} dz dx.
\]

Now, an application of equation (2.2.6.1) in Prudnikov et al. [5] to calculate the inner integral in (2.3) shows that

\[
H(P, Q) = \frac{Q^{1+2\beta}}{2\pi(1+2\beta)} \int_0^P x^\alpha (1 + x^2)^{-3/2} 2F_1 \left( \beta + 1, \frac{3}{2}; \beta + \frac{3}{2}; -\frac{Q^2}{1+x^2} \right) dx
\]

\[
= \frac{Q^{1+2\beta}}{2\pi(1+2\beta)} \int_0^P x^\alpha (1 + x^2)^{-3/2} \sum_{k=0}^{\infty} \frac{(\beta + 1/2)k(3/2)_k}{(\beta + 3/2)_k k!} \left( -\frac{Q^2}{1+x^2} \right)^k dx
\]

\[
(2.4) = \frac{Q^{1+2\beta}}{2\pi(1+2\beta)} \sum_{k=0}^{\infty} \frac{(\beta + 1/2)k(3/2)_k(-1)^k Q^{2k}}{(\beta + 3/2)_k k!} \int_0^P x^\alpha (1 + x^2)^{-3/2-k} dx,
\]

where we have used the definition of the Gauss hypergeometric function. The result of the theorem follows by applying equation (2.2.6.1) in Prudnikov et al. [5] again to calculate the integral in (2.4).
Theorem 2.3. If \((X, Y)\) has the joint pdf (1.3) and if \((\beta - 1)/2 \geq 1\) is an integer then (2.2) can be expressed in the form

\[
H(P, Q) = \frac{1}{4\pi} \sum_{k=0}^{(\beta-1)/2} \binom{(\beta - 1)/2}{k} (-1)^{(\beta-1)/2-k} \{S(k, Q) - S(k, 0)\},
\]

where

\[
S(k, Q) = \sum_{l=0}^{(\beta-1)/2-k} \binom{(\beta - 1)/2 - k}{l} \frac{P^{\alpha+1+2l} (1 + Q^2)^{k-1/2}}{(k - 1/2)(\alpha + 1 + 2l)} \times \frac{\alpha + 1}{2} + l, \frac{1}{2} - k; \frac{\alpha + 3}{2} + l; -\frac{P^2}{1 + Q^2}.
\]

Proof. If \(\beta\) is an odd integer then one can write (2.3) as

\[
H(P, Q) = \frac{1}{4\pi} \sum_{k=0}^{(\beta-1)/2} \binom{(\beta - 1)/2}{k} \int_0^P \int_{1+x^2+Q^2} \binom{(\beta - 1)/2}{k} (-1)^{(\beta-1)/2-k} \frac{x^\alpha (-1 - x^2)^{(\beta-1)/2-k} z^{k-3/2}}{z^{k - 1/2}} dz dx
\]

\[
= \frac{1}{4\pi} \sum_{k=0}^{(\beta-1)/2} \binom{(\beta - 1)/2}{k} (-1)^{(\beta-1)/2-k} \int_0^P x^\alpha (1 + x^2)^{(\beta-1)/2-k} \frac{(1 + Q^2 + x^2)^{k-1/2}}{k - 1/2} dx - \frac{(1 + x^2)^{k-1/2}}{k - 1/2}
\]

\[
= \frac{1}{4\pi} \sum_{k=0}^{(\beta-1)/2} \binom{(\beta - 1)/2}{k} \{S(k, Q) - S(k, 0)\},
\]

where

\[
(2.5) \quad S(k, Q) = \frac{1}{k - 1/2} \int_0^P x^\alpha (1 + x^2)^{(\beta-1)/2-k} (1 + Q^2 + x^2)^{k-1/2} dx.
\]

Note that (2.5) can be expanded as

\[
(2.6) \quad S(k, Q) = \frac{1}{k - 1/2} \sum_{l=0}^{(\beta-1)/2-k} \binom{(\beta - 1)/2 - k}{l} \int_0^P x^{\alpha+2l} (1 + Q^2 + x^2)^{k-1/2} dx
\]

\[
= \frac{1}{2k - 1} \sum_{l=0}^{(\beta-1)/2-k} \binom{(\beta - 1)/2 - k}{l} \int_0^{P^2} y^{(\alpha-1)/2+l} (1 + Q^2 + y)^{k-1/2} dy.
\]

The result of the theorem follows by using equation (2.2.6.1) in Prudnikov et al. [5] to calculate the integral in (2.6).
3. Estimation

Here, we consider estimation by the method of maximum likelihood and the method of moments. Suppose \( \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \) is a random sample from (1.3). It is easy to show that the maximum likelihood estimators of \( A, B, C \) and \( D \) are \( \hat{A} = \max x_i, \hat{B} = \min x_i, \hat{C} = \max y_i \) and \( \hat{D} = \min y_i \), respectively. The method of moments estimators can be determined by equating, say

\[
E(XY) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i,
\]

\[
E(X^2Y) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 y_i,
\]

\[
E(XY^2) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i^2,
\]

and

\[
E(X^2Y^2) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 y_i^2,
\]

where the expectations can be calculated by using the results of Theorems 2.1 to 2.3.

4. Application

The bivariate Cauchy distribution has received applications in many areas, including biological analyses, clinical trials, stochastic modelling of decreasing failure rate life components, study of labour turnover, queueing theory, and reliability (see, for example, Nayak [4] and Lee and Gross [2]). For data from these areas, there is no reason to believe that empirical moments of any order should be infinite. Thus, the choice of the bivariate Cauchy distribution as a model is unrealistic since its product moments \( E(X^\alpha Y^\beta) \) are not finite for all \( \alpha \) and \( \beta \). The alternative given by (1.3) will be a more appropriate model for the kind of data mentioned. The choice of the limits \( A, B, C \) and \( D \) could be based on historical records or the methods discussed in Section 3. As an example, consider modelling the dependence between the consumer price indices of Australia and the United States. The data for the years from 1901 to 2005 extracted from the website www.globalfindata.com/ are shown in Table 1. We fitted the models given by (1.1) and (1.3) to this data set. The method of maximum likelihood was used. Prior to fitting, a probability integral transformation was used to standardize the data to have Cauchy marginals. The two models yielded the following maximized log likelihoods: \(-494.7993\) and \(-446.8655\). It follows by the likelihood ratio test that the truncated model is a significant improvement. The fitted joint densities for the two models are shown in Figures 1 and 2.
Table 1. Consumer price index data for Australia and the United States for the years 1901–2005.

<table>
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<th>OZ CPI</th>
<th>US CPI</th>
<th>Year</th>
<th>OZ CPI</th>
<th>US CPI</th>
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Figure 1. Fitted joint density of (1) for the data on consumer price indices of Australia and the United States

Figure 2. Fitted joint density of (3) for the data on consumer price indices of Australia and the United States

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A Truncated Bivariate Cauchy Distribution

References


