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A Stochastic Paris-Erdogan Model for Fatigue Crack Growth Using Two-State Model

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Abstract. The one-state stochastic Paris-Erdogan crack model, $\frac{da}{dn} = C(\Delta K)^m$ is extended and replaced by two-state model for fatigue crack growth. This is done by means of memoryless transformation and introduction of an auxilliary process. The mean and variance of the crack size are then derived analytically and the validity of the expressions obtained is verified by comparing them with simulation results. It is found that the results obtained from simulation and analytic expression are sufficiently in good agreement to allow us to conclude the validity of the analytic expression and to validate the numerical simulation procedure.

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1. Introduction

During the last four decades, numerous mathematical models for predicting fatigue crack growth have been proposed and analysed and most are validated to some extent by reference to experimental results [6]. Under conditions of linear-elastic fracture, the most widely used and recognized empirical law is the classical Paris-Erdogan model [3]. This model is chosen as a starting point to develop a simulation model for structural damage or crack growth because of its simplicity. Furthermore, it has been accepted for decades as a basic and widely applicable framework in fracture mechanics [7]. The model relates the rate of change of crack size (length) a with number of load cycles n in the form

$$\frac{da}{dn} = C(\Delta K)^m \tag{1.1}$$

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This model can be considered as a one-state crack propagation model. Here $C > 0$, $2 \leq m \leq 4$ [6] and $\Delta K = K_{max} - K_{min}$, where K is the crack tip stress intensity factor. A simple model for K is:

$$K = B\sqrt{\pi a}\, S \tag{1.2}$$

where S is the stress at the crack tip and B is the geometry correction factor which depends on the crack shape, the crack size (crack length a) and the shape of the component. In this paper, B is assumed to be constant and independent of the crack length even though this is only practical for small crack growth. Since the problem of estimating crack length is in practice important for relatively large crack growth, the immediate engineering application of this paper is limited. This geometrical factor B is often taken to be 1.1 as suggested by Paris and Sih [4]. With this, equation (1.1) becomes

$$\frac{da}{dn} = \lambda a^{m/2} (\Delta S)^m \tag{1.3}$$

where $\lambda = C(1.1)^m \pi^{m/2}$ and $\Delta S = S_{max} - S_{min}$ [6].

If we want to rewrite equation (1.3) in a different form suitable for a simulation model in which the existing crack size (a_{N+1}) depends on the previous size (a_N), and in which crack-growth (damage) accumulates relatively slowly and continuously with the load cycles N, we can have a backward difference approximation of the form [1]

$$a_{N+1} - a_N = \lambda a_N^{m/2} (\Delta s_N)^M. \tag{1.4}$$

For system in which crack growth per cycle is 'small', we can introduce a frequency $\omega = \dfrac{dn}{dt}$ and replace (1.3) by

$$\frac{da}{dt} = \omega \lambda a^{m/2} (\Delta S)^m. \tag{1.5}$$

If the loading is not deterministic, we can model ΔS via some non-negative stochastic process $X(t)$ [1]. Since ΔS is random in nature, it can be of any distribution which has positive values. With the interpretation of ω as some mean frequency the stochastic process $A(t)$, modelling the crack length, is then determined by the stochastic differential equation

$$\dot{A} = \omega \lambda A^{m/2} X^m, \quad A(t_0) = a_0 \tag{1.6}$$

The initial crack size a_0 may also be random; $A(t_0) = A_0$.

In Section 2, we shall see how (1.6) can be simplified by means of a memoryless transformation. By using this technique, the crack propagation model is easily handled by a two-state crack growth model by introducing an auxilliary process, Z. This approach is discussed in detail in Section 3.

2. Memoryless transformation

Define the stochastic process $W(t)$ via the memoryless transform [2].

$$W(t) = \int_{A_0}^{A(t)} \frac{1}{\omega\lambda a^{m/2}} da + w_0 \tag{2.1}$$

This defines A in terms of W, so that,

$$W(t) - w_0 = \int_{A_0}^{A(t)} \frac{1}{\omega\lambda a^{m/2}} da = \begin{cases} \frac{1}{\omega\lambda} ln[A/A_0] & m = 2 \\ \frac{2}{2-m}\frac{1}{\omega\lambda}\left(A^{\frac{2-m}{2}} - A_0^{\frac{2-m}{2}}\right) & m \neq 2 \end{cases} \tag{2.2}$$

where w_0 is interpreted as an arbitrary initial state for W which is independent of A_0. w_0 is assumed to be fixed(although it may also be taken as random variable with some appropriate distribution). Then we have

$$\dot{W} = \frac{1}{\omega\lambda}\frac{1}{A^{m/2}}\dot{A} = X^m \tag{2.3}$$

Note that the right hand side of (2.3) is independent of process W, which is an advantage if (2.3) needs to be solved. Note also that $W(t_0) = w_0$.

3. Two-state crack growth model

If we present the stochastic Paris-Erdogan model, equation (1.6) as

$$\dot{W} = \frac{1}{\omega\lambda}\frac{1}{A^{m/2}}\dot{A} = X^m = Y$$

then, when the driving process $Y = X^m = (\Delta S)^m$ is non-Gaussian, an exact Fokker-Planck type equation for W is not possible [5]. To overcome this, we introduce a subsidiary process Z defined by another stochastic differential equation, driven by Gaussian white-noise, and then define the non-negative process Y, which is modelled as a lognormal distribution, via a memoryless transformation $G(Z)$. Specifically we consider a coupled system of stochastic differential equations

$$\dot{W} = G(Z), \quad W(t_0) = w_0 \tag{3.1}$$

$$\dot{Z} = -kZ + q\Gamma, \quad Z(t_0) = Z_0 \tag{3.2}$$

where $k > 0$, q is the intensity fluctuation of the Gaussian white-noise $\Gamma(t)$: $\Gamma(t) \sim N(0,1)$ and $E[\Gamma(t)\Gamma(s)] = \delta(t-s)$ and w_0 , Z_0 are general random variables characterizing the initial state of W and the subsidiary random process, Z, respectively.

Equation (3.2) defines an Ornstein-Uhlenbeck process: $Z(t)$ is Gaussian with

$$\overline{Z} = E[Z] = Z_0 \exp(-kt), \quad Z_0 \text{ fixed.} \tag{3.3}$$

$$E[(Z - \overline{Z})^2] = \frac{q^2}{2k}[1 - \exp(-2kt)]. \tag{3.4}$$

So, for $kt \gg 1$, $Z(t)$ is a stationary Gaussian process: $Z(t) \sim N(0, \frac{1}{2}q^2k^{-1})$. Further for $kt \gg 1$,

$$E[(Z(t) - \overline{Z})(Z(s) - \overline{Z})] \approx \frac{q^2}{2k} \exp[-k \mid t - s \mid]. \tag{3.5}$$

Therefore, with $q = \sqrt{2k}$ we generate an asymptotically stationary process $Z(t) \sim N(0, 1)$ with $E[Z(t)Z(s)] = \exp[-k|t - s|]$. It follows that (3.2) becomes

$$\dot{Z} = -kZ + \sqrt{2k}\Gamma, \quad Z(t_0) = z_0. \tag{3.6}$$

In what follows we are going to interpret Z as the stationary process generated by (4.1). In the next section, we derive analytic expressions for the mean and variance of W based on two-state model, equations (3.1) and (4.1). This is important because by using the formula for the mean and variance of W, computations involving functions of distributions (such as for the reliability functions) can easily be accomplished without having to simulate the two-state crack model.

4. Analytic expression for mean of W and variance of W

4.1. Mean of W. Since Y is lognormal, let $L = \ln[Y] \sim N(\mu, \sigma^2)$, so that $Y = \exp(L)$. If we let $Z \sim N(0, 1)$, then obviously $L = \mu + \sigma Z$. Therefore,

$$Y = G(Z) = \exp(\mu + \sigma Z). \tag{4.1}$$

From (3.1) and (4.1), the mean values of W and Z satisfy

$$\dot{\overline{W}} = E[G(Z)] = E[\exp(\mu + \sigma Z)] = \exp(\mu)E[\exp(\sigma Z)] \tag{4.2}$$

and

$$\dot{\overline{Z}} = -k\overline{Z}. \tag{4.3}$$

From the fact that

$$E[\exp(X)] = \exp\left[\frac{1}{2}E(X^2)\right] \tag{4.4}$$

for zero-mean Gaussian X we have

$$E[\exp(\sigma Z)] = \exp\left[\frac{1}{2}E(\sigma^2 Z^2)\right] \tag{4.5}$$

$$= \exp\left[\frac{1}{2}\sigma^2 E(Z^2)\right]. \tag{4.6}$$

Since $E[Z^2] = 1$, (4.2) becomes

$$\dot{\overline{W}} = \exp\left[\mu + \frac{1}{2}\sigma^2\right] \tag{4.7}$$

giving(on integration)

$$\overline{W}(t) = w_0 + (t - t_0)\exp\left[\mu + \frac{1}{2}\sigma^2\right], \quad w_0 \text{ fixed.} \tag{4.8}$$

Note that if we are to stimulate W via (3.1) using an explicitly stationary form for Z, then (4.8) will be exact.

4.2. Variance of W. Defining $W' = W - \overline{W}$ and $\vartheta = Var(W) = E[W'W']$, we then have

$$\dot{\vartheta} = \frac{d}{dt}E[W'W'] = E\left[\frac{d}{dt}(W'W')\right]. \tag{4.9}$$

Using product rule for differentiation, we have

$$\dot{\vartheta} = 2E[W'\dot{W}]. \tag{4.10}$$

Using the definition of W' and the fact that $\dot{W} = G(Z)$, this can be written as

$$\dot{\vartheta} = 2E[W'G(Z)]. \tag{4.11}$$

To determine a closed form for right hand side of (4.11), we make use of the representation from equation(3.1),

$$W(t) = w_0 + \int_{t_0}^{t} G(Z(s))ds. \tag{4.12}$$

So that we have

$$E[G(Z)W'] = E[G(Z)W] - E[G(Z)]\overline{W} \tag{4.13}$$

$$= E\left[G(Z(t))\left\{w_0 + \int_{t_0}^{t} G(Z(s))ds\right\}\right] - E[G(Z(t))]\overline{W} \tag{4.14}$$

$$= \int_{t_0}^{t} E[G(Z(t))G(Z(s))]ds - E[G(Z(t))](\overline{W} - w_0). \tag{4.15}$$

With $G(Z) = \exp(\mu + \sigma Z)$, this becomes

$$E[G(Z)W'] = \exp[2\mu]\int_{t_0}^{t} E[\exp\{\sigma(Z(t) + Z(s))\}]ds - (\overline{W} - w_0)\exp\left[\mu + \frac{1}{2}\sigma^2\right]. \tag{4.16}$$

Now, since $Z(t)$, $Z(s) \sim N(0,1)$ with $E[Z(t)Z(s)] = \exp[-k(t-s)] = \rho(t-s)$, we have $Z(t) + Z(s) \sim N(0, 2(1+\rho))$.

So, therefore

(4.17)
$$E[G(Z)W'] = \exp[2\mu] \int_{t_0}^{t} \exp[\sigma^2(1+\rho(t-s)ds - (\overline{W} - w_0) \exp\left[\mu + \frac{1}{2}\sigma^2\right]$$

$$= \exp[2\mu + \sigma^2] \int_{t_0}^{t} \exp[\sigma^2\rho(t-s)]ds - (\overline{W} - w_0) \exp\left[\mu + \frac{1}{2}\sigma^2\right] \tag{4.18}$$

$$= \exp\left[\mu + \frac{1}{2}\sigma^2\right] \left\{ \int_{t_0}^{t} \exp[\sigma^2\rho(t-s)]ds - (\overline{W} - w_0) \right\}. \tag{4.19}$$

From (4.11), we have

$$\dot{\vartheta} = 2E[G(Z)W'] = 2\exp\left[\mu + \frac{1}{2}\sigma^2\right] \times \left\{ \exp\left[\mu + \frac{1}{2}\sigma^2\right] \int_{t_0}^{t} \exp[\sigma^2\rho(t-s)]ds - (\overline{W} - w_0) \right\}. \tag{4.20}$$

Writing $\overline{G} = \exp\left[\mu + \frac{1}{2}\sigma^2\right]$ and recalling that $\overline{W} = w_0 + (t - t_0)\exp\left[\mu + \frac{1}{2}\sigma^2\right]$ from equation(4.8), we can write

$$\dot{\vartheta} = 2 \times \overline{G} \times \left\{ \overline{G} \times \int_{t_0}^{t} \exp[\sigma^2\rho(t-s)]ds - \overline{G}(t - t_0) \right\}. \tag{4.21}$$

Equivalently, (4.21) can be written as

$$\dot{\vartheta} = 2 \times \overline{G} \times \left\{ \overline{G} \times \int_{0}^{t-t_0} \exp[\sigma^2\rho(s)]ds - \overline{G}(t - t_0) \right\} \tag{4.22}$$

$$= 2 \times \overline{G}^2 \times \left\{ \int_{0}^{t-t_0} \exp[\sigma^2\rho(s)]ds - (t - t_0) \right\}. \tag{4.23}$$

Differentiating (4.23), we have

$$\ddot{\vartheta} = 2 \times \overline{G}^2 \times \{\exp[\sigma^2\rho(t - t_0)] - 1\}. \tag{4.24}$$

To solve (4.24) for ϑ, a two-step Euler method is used [see Section (5.2)].

In order to verify that the expression for $\overline{W}$ and ϑ are valid, we compare them with simulation results carried out using equations (3.1) and (4.1).

5. Comparisons of mean and variance of W between simulation and analytic expression

We simulate W from (3.1) and (4.1) and compute the mean and variance of W for each time step t. The results obtained are then compared with those obtained using equation (4.8) for the mean, and equations (4.23) and (4.24) for the variance of the analytic expression.

For simplicity, we use the initial conditions $w_0 = 0$, $z_0 = 0$ and $t_0 = 0$, and choose $\mu = 0.1$ and the time step $\Delta t = 1.0 \times 10^{-4}$. For purpose of verification different values of $\sigma(0.1,\ 0.5,\ 1.0,\ 1.5)$ are used in the computations. To ensure that the stationary asymptotic form of Z; $Z(t) \sim N(0,1)$ with $E[Z(t)Z(s)] = \exp[-k|t-s|]$, is obtained k or t should be sufficiently large. In this work, we have chosen a large value of k so that to ensure the stationary form is attained very rapidly. Several values of k are tested and finally a reasonable value chosen for k is 10.

5.1. Simulation. From equation (3.1),

$$\dot{W} = G(Z), \quad W(t_0) = w_0$$

we simulate a sequence of random variables $W(n), n = 0, 1, ...$ according to

$$W(n+1) = W(n) + G(Z(n))dt \tag{5.1}$$

where $W(0) = 0, G(Z(n)) = \exp[\mu + \sigma Z(n)]$.

The sequence of $Z(n)$ is obtained via equation (4.1) by simulating

$$Z(n+1) = Z(n) + (-kZ(n)dt + \sqrt{2k}\ d\Gamma(n)) \tag{5.2}$$

where $Z(0) = 0$ and $d\Gamma(n) \sim N(0, dt)$ [8]. Random numbers for $d\Gamma(n)$ are generated independently using NAG routine G05DDF. The mean and variance of W are computed for different values of σ for each $t_n = ndt$.

5.2. Evaluating the analytical expressions. Mean of W, $\overline{W}$ for each t_n is computed using equation (4.8),

$$\overline{W}(t) = w_0 + (t - t_0) \exp\left[\mu + \frac{1}{2}\sigma^2\right].$$

We solve for ϑ using Euler method applied to equation (4.24)

$$\ddot{\vartheta} = 2 \times \overline{G}^2 \times \{\exp[\sigma^2 \rho(t - t_0)]\},$$

that is by computing ϑ and $\dot{\vartheta}$ according to

$$\vartheta(n+1) = \vartheta(n) + \dot{\vartheta}(n)dt \tag{5.3}$$

and

$$\dot{\vartheta}(n+1) = \dot{\vartheta}(n) + \ddot{\vartheta}(n)dt \tag{5.4}$$

with $\vartheta(0) = 0$ and $\dot{\vartheta}(0) = 0$.

6. Results and conclusion

Here we present results of $\overline{W}$ and ϑ obtained for $n = 100\,000$ number of cycles from both simulation and analytic expressions. Graphs of $\overline{W}$ and ϑ are plotted against t for different values of σ. Figures 1–4 show results for $\overline{W}$ and Figures 5–8 for ϑ. Figures 1–4 shows good agreement between the results from simulation and analytic expression for $\overline{W}$.

However, for variance of W(Figures 5–8) there is very slight disagreement when σ is above 1.0. This disagreement is probably because the calculation of variance by simulation, as well as by analytic expression involved estimations: even though the formula is exact, it could only be solved numerically. Since the variance ϑ for large σ is correspondingly large, any deviation from the estimation can give greater effect to the differences between the analysis and the simulation results. However, these results are still in sufficiently good agreement to allow us to conclude the validity of the analytic expression and to validate the numerical simulation procedure.

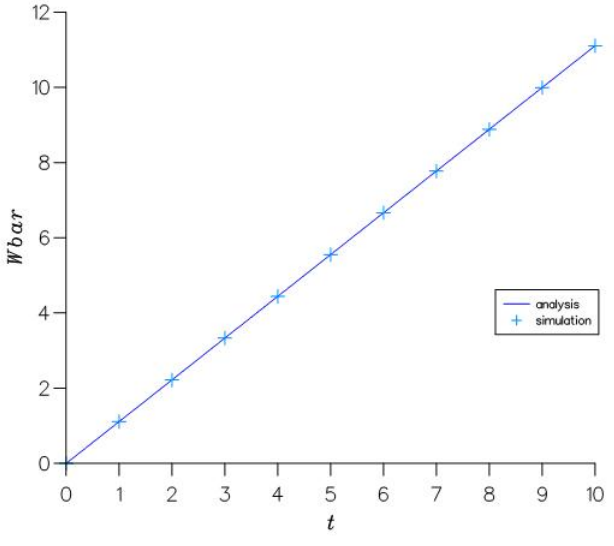

Figure 1. Graph of $\overline{W}$ against time step t for $\sigma = 0.1$

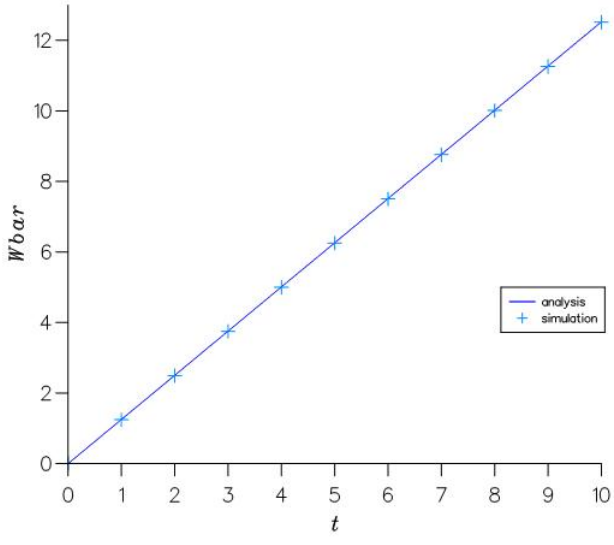

Figure 2. Graph of $\overline{W}$ against time step t for $\sigma = 0.5$

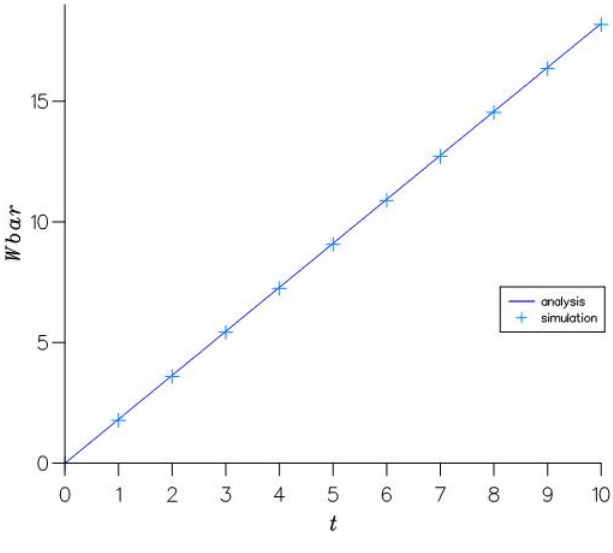

Figure 3. Graph of $\overline{W}$ against time step t for $\sigma = 1.0$

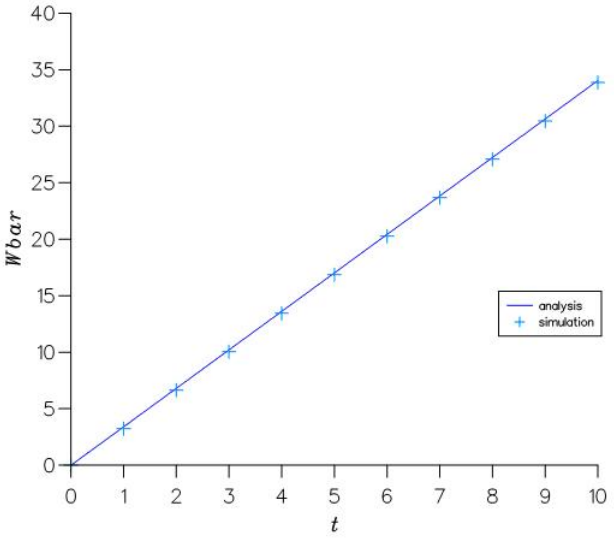

Figure 4. Graph of $\overline{W}$ against time step t for $\sigma = 1.5$

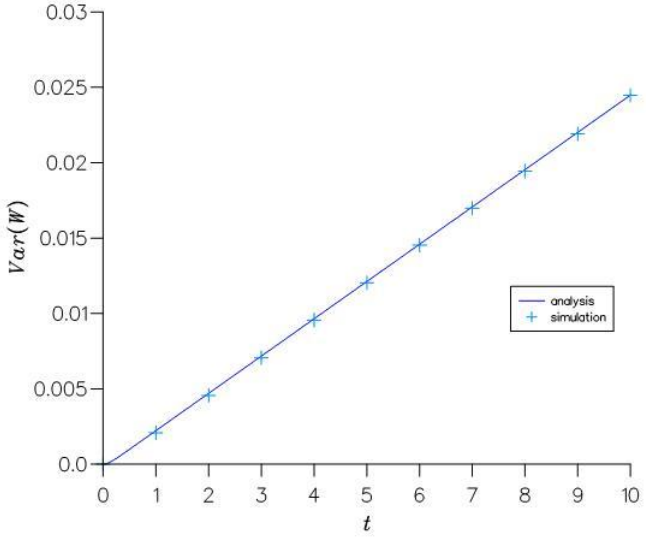

Figure 5. Graph of ϑ against time step t for $\sigma = 0.1$

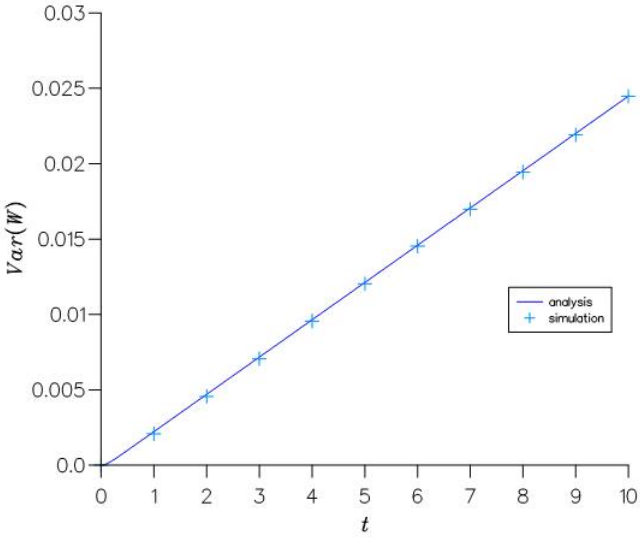

Figure 6. Graph of ϑ against time step t for $\sigma = 0.5$

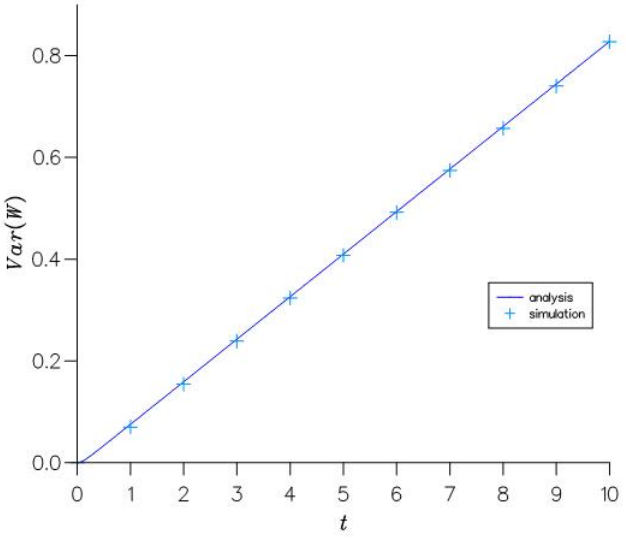

Figure 7. Graph of ϑ against time step t for $\sigma = 1.0$

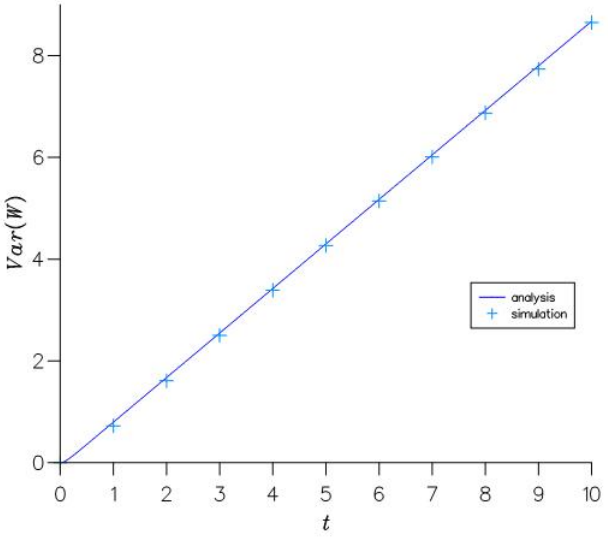

Figure 8. Graph of ϑ against time step t for $\sigma = 1.5$

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