BULLETIN *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **31***(1) (2008), 25–33*

Subgroup Separability and Conjugacy Separability of Certain HNN Extensions

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Abstract. In this note, we give characterizations for certain HNN extensions of polycyclic-by-finite groups with central associated subgroups to be subgroup separable and conjugacy separable. We shall do this by showing the equivalence of subgroup separability and conjugacy separability in this type of HNN extensions.

2000 Mathematics Subject Classification: Primary 20E06, 20E26; Secondary 20F10, 20F19

Key words and phrases: HNN extensions, Polycyclic-by-finite groups, Subgroup separable, Conjugacy separable, Residually finite

1. Introduction

In this note, we give characterizations for certain HNN extensions of polycyclic-by-finite groups with central associated subgroups to be subgroup separable and conjugacy separable. We shall do this by showing the equivalence of subgroup separability and conjugacy separability in this type of HNN extensions.

A group G is called subgroup separable if, for each finitely generated subgroup M and for each $x \in G \backslash M$, there exists a normal subgroup N of finite index in G such that $x \notin MN$. It is well known that free groups, polycyclic groups and surface groups are subgroup separable (M. Hall [10], Mal'cev [11], Scott [16, 17]). Finite extensions of subgroup separable groups are again subgroup separable. Hence, polycyclic-by-finite groups, free-by-finite groups and Fuchsian groups are subgroup separable.

A group G is said to be conjugacy separable if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a normal subgroup N of finite index in G such that Nx and Ny are not conjugate in G/N. Blackburn [4] proved that the finitely generated nilpotent groups are conjugacy separable. Later, Formanek [9] and Remeslennikov [14] independently extended this result by proving

Received: May 18, 2006; *Revised:* August 17 2007.

that polycyclic-by-finite groups are conjugacy separable. Dyer in [8] proved that free-by-finite groups are conjugacy separable.

We shall investigate the subgroup separability and conjugacy separability of HNN extensions since not many HNN extensions are known to be subgroup separable and conjugacy separable. Indeed one of the simplest type of HNN extensions, the Baumslag-Solitar group, $\langle h, t; t^{-1}h^2t = h^3\rangle$ is not even residually finite [3]. However, more recently, Metaftsis and Raptis [12] gave a characterization for the subgroup separability of HNN extensions of finitely generated abelian groups. Before that, Raptis, Talelli and Varsos [13] proved that the equivalence of residual finiteness and conjugacy separability in HNN extensions of finitely generated abelian groups and certain HNN extensions of finitely generated torsion free nilpotent groups.

Our main results will be Corollary 3.2, Corollary 3.3 and Theorem 3.3. Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group with H and K being proper finitely generated central subgroups of A. Corollary 3.6 and Corollary 3.3 give characterizations for G to subgroup separable and conjugacy separable respectively. Theorem 3.3 establishes the equivalence of subgroup separability and conjugacy separability in the HNN extension G.

Theorem 3.3 is proved via Theorem 3.1 and Theorem 3.2. Theorem 3.1 states that $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ where H and K are proper central subgroups of A, is subgroup separable if and only if both the group A and the subgroup (HNN extension) $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ are subgroup separable. Theorem 3.1 partially extends the results of Metaftsis and Raptis [12].

Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension as in Theorem 3.3. Theorem 3.2 states that if G is subgroup separable and A is a group such that A/D is conjugacy separable for each finitely generated central subgroup, D, then G is conjugacy separable.

The notation used here is standard. In addition, the following will be used for any group G:

$N \lhd_f G$ means N is a normal subgroup of finite index in G.

$x \sim_G y$ means x and y are conjugate in G.

$\{y\}^G = \{g^{-1}yg \ ; \ g \in G\}$ denotes the conjugacy class of y in G.

$G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ shall denote an HNN extension where A is the base group, H, K are the associated subgroups, and φ is the associated isomorphism $\varphi : H \longrightarrow K$.

$\|g\|$ denotes the length of the element g in $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$.

2. Preliminaries

We begin with two definitions.

Definition 2.1. *A group G is called H-separable for the subgroup H if for each $x \in G\backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.*

Equivalently, G is H-separable if $\bigcap_{N \lhd_f A} NH = H$.

G is termed subgroup separable if G is H-separable for every finitely generated subgroup H.

Definition 2.2. *A group G is said to be conjugacy separable if for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists $N \lhd_f G$, such that Nx and Ny are not conjugate in G/N.*
Equivalently, G is conjugacy separable if for each pair $x, y \in G$ and $x \notin \{y\}^G$, there exists $N \lhd_f G$ such that $x \notin \{y\}^G N$.

Next, we state Theorem 2.1 which will be used in the proof of Theorem 3.1.

Theorem 2.1. [7] *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. Suppose $x, y \in G$ and x, y are cyclically reduced. If $x \sim_G y$, then $\|x\| = \|y\|$ and one of the following holds;*

(a) $\|x\| = \|y\| = 0$ *and there exists a finite sequence $z_1, z_2, \ldots, z_n$ of elements in $H \cup K$ such that $x \sim_A z_1 \sim_{A,t^*} z_2 \sim_{A,t^*} \cdots \sim_{A,t*} z_n \sim_A y$ where $u \sim_{A,t^*} v$ means one of: ($u \sim_A v$) or ($u \in H$ and $v = t^{-1}ut \in K$) or ($u \in K$ and $v = tut^{-1} \in H$).*

(b) $\|x\| = \|y\| = n \geq 1$ *and $x = c^{-1}y^*c$ where y^* is a cyclic permutation of y and $c \in X$ with $X = H$ if $\epsilon_n = -1$ and $X = K$ if $\epsilon_n = 1$, where ϵ_n is the power of t in the last term of x expressed in cyclically reduced form, $x = x_0 t^{\epsilon_1} \ldots x_{n-1} t^{\epsilon_n}$.*

3. The main results

We begin by stating three lemmas (Lemmas 3.1–3.3) on subgroup separability which are easily proved and well known to researchers in this area. The three lemmas will be used in the proof of Theorem 3.1.

Lemma 3.1. *Let A be a subgroup separable group and H is a finitely generated normal subgroup of A. Then A/H is subgroup separable.*

Next we state a result given by Romanovskii [15].

Lemma 3.2. [15] *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. If $K = A$ and $H \neq A$, then G is not H-separable.*

Remark 3.1. In this note, we consider the HNN extension $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ where H and K are subgroups in the center of A. Note that if $H = A = K$, then A is abelian. Furthermore, A is normal in G and $G/A \cong \langle t \rangle$. Hence, G is polycyclic and by [11], G is subgroup separable. From now on we assume that $H \neq A \neq K$.

Lemma 3.3 is well known and very useful. We also provide a proof.

Lemma 3.3. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. If A is subgroup separable and H and K are finite, then G is subgroup separable.*

Proof. Since A is residually finite, then by [2], G is residually finite. Then there exists $M \lhd_f G$ such that $M \cap H = 1$ as H is finite. Now let $(\mathcal{G}, X)$ be a graph of groups with a single vertex v and a single edge e where $G_v = A$ and $G_e = H$. It is well known that its fundamental group $\pi_1((\mathcal{G}, X)) = G$. From the structure theorem of Bass-Serre Theory (see [6]), we know that M is the fundamental group of a graph of groups where the vertex groups are $M \cap x^{-1}G_v x = M \cap x^{-1}Ax$, where x runs over a suitable finite set of (M, A)-double coset representatives and the edge groups are

$M \cap y^{-1} G_e y = M \cap y^{-1} H y$, where y runs over a suitable finite set of (M, H)-double coset representatives. But the edge groups $M \cap y^{-1} H y = 1$ since $M \lhd_f G$. So, M is a free product of $M \cap x^{-1} A x$ and possibly a free group. By the choice of M, $M \cap x^{-1} A x$ is some conjugate of $M \cap A$ and $M \cap y^{-1} H y = 1$. Therefore, M is a free product of a finite number of subgroup separable groups and possibly a free group. By Burns [5] and Romanovskii [15], a free product of a finite number of subgroup separable groups is again subgroup separable. Therefore, M is subgroup separable. Since M is of finite index in G, then by [16], G is subgroup separable. ∎

We prove our first main result which is a criterion on subgroup separability. We shall also use Theorem 2 of Metaftsis and Raptis [12].

Theorem 3.1. *Let $G = \langle t, A; t^{-1} H t = K, \varphi \rangle$ be an HNN extension. Suppose H and K are finitely generated subgroups in the center of A and $H \neq A \neq K$. Then G is subgroup separable if and only if A and $G_1 = \langle t, HK; t^{-1} H t = K, \varphi \rangle$ are subgroup separable.*

Proof. Suppose G is subgroup separable. Since A and G_1 are subgroups of G, A and G_1 are subgroup separable.

Suppose A and G_1 are subgroup separable. In order to show that G is subgroup separable, we shall show that $\bigcap\limits_{N \lhd_f G} NL = L$ for every finitely generated subgroup L of G. We will do this in several steps.

First, we show that there exists a subgroup $U \lhd G$ such that $U^n \lhd G$ and G/U^n is subgroup separable for each $n \geq 1$. Since G_1 is subgroup separable, then by Theorem 2 of [12], there exists a finitely generated subgroup U such that $U \lhd G_1$ and U has finite index in H. Furthermore, $\varphi(U) = U$ since $U \lhd G_1$. So as U is in the centre of G and $\varphi(U) = U$, we have $U \lhd G$ and U has finite index in K. Since U^n is a characteristic subgroup of U, we have $U^n \lhd G$ and hence $G/U^n = \langle t, A/U^n; t^{-1} H/U^n t = K/U^n, \varphi \rangle$. By Lemma 3.1, A/U^n is subgroup separable since A is subgroup separable and U^n is finitely generated. Furthermore H/U^n and K/U^n are finite. Therefore by Lemma 3.3, G/U^n is subgroup separable. Thus we have shown that there exists a subgroup $U \lhd G$ such that $U^n \lhd G$ and G/U^n is subgroup separable for each $n \geq 1$.

Next, we show that if $P = \bigcap\limits_{n \geq 1} U^n L$, then $P \cap H = L \cap H$. Now $P \cap H = \bigcap\limits_{n \geq 1} (U^n L \cap H) = \bigcap\limits_{n \geq 1} U^n (L \cap H)$ since $U^n \subseteq H$. Since H is a finitely generated abelian group and $L \cap H$ is finitely generated, $\bigcap\limits_{N_H \lhd_f H} N_H (L \cap H) = L \cap H$. This implies that $\bigcap\limits_{n \geq 1} U^n (L \cap H) = L \cap H$ since given any $N_H \lhd_f H$ there exists an integer $n \geq 1$ such that $U^n \subseteq N_H$. Thus $P \cap H = L \cap H$.

Next, we show that $\bigcap\limits_{n \geq 1} U^n L = L$, for every finitely generated subgroup L of G. Let $p \in P = \bigcap\limits_{n \geq 1} U^n L$. Then $p = u_n l_n$ for some $u_n \in U^n$ and $l_n \in L$. So $p l_n^{-1} = u_n \in H$. But $p l_n^{-1} \in P$ since $l_n \in L \subseteq P$. This implies that $p l_n^{-1} \in P \cap H = L \cap H$ and therefore $p \in L$. Hence $\bigcap\limits_{n \geq 1} U^n L = L$.

Finally, we show that $\bigcap_{N \triangleleft_f G} NL = L$ for every finitely generated subgroup L of G. Since G/U^n is subgroup separable, $\bigcap_{N \triangleleft_f G} (U^n N/U^n)(U^n L/U^n) = (U^n L/U^n)$. But $\bigcap_{N \triangleleft_f G} (U^n N/U^n)(U^n L/U^n) = \left(\bigcap_{N \triangleleft_f G} U^n NL \right) /U^n$. This implies that $\bigcap_{N \triangleleft_f G} NL \subseteq U^n L$ for every $n \geq 1$ and therefore $\bigcap_{N \triangleleft_f G} NL \subseteq \bigcap_{n \geq 1} U^n L = L$. Hence, $\bigcap_{N \triangleleft_f G} NL = L$. The proof of the theorem is now completed. ∎

Again by applying the results of Metaftsis and Raptis [12], we obtain the following characterisations on subgroup separability.

Corollary 3.1. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension. Suppose H and K are finitely generated subgroups in the center of A and $H \neq A \neq K$. Then G is subgroup separable if and only if A is subgroup separable and $H \cap K$ is a subgroup of finite index in H and K and there exists a finitely generated subgroup U such that U has finite index in $H \cap K$ and $\varphi(U) = U$.*

Proof. By Theorem 3.1, G is subgroup separable if and only if A and $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ are subgroup separable. By Theorem 2 of [12], G_1 is subgroup separable if and only if $H \cap K$ is a subgroup of finite index in H and K and there exists a finitely generated normal subgroup U of G_1 such that U has finite index in $H \cap K$. Since U is normal in G_1, we have $\varphi(U) = U$. ∎

Since polycyclic-by-finite groups, free-by-finite groups and Fuchsian groups are subgroup separable (M. Hall [10], Mal'cev [11], Scott [16, 17], we have the following corollary.

Corollary 3.2. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is a polycyclic-by-finite group or a free-by-finite group or a Fuchsian group. Suppose H and K are finitely generated subgroups in the center of A and $H \neq A \neq K$. Then G is subgroup separable if and only if $H \cap K$ is a subgroup of finite index in H and K and there exists a finitely generated subgroup U of finite index in H such that $\varphi(U) = U$.*

Next we prove our second main result which shows a connection between subgroup separability and conjugacy separability in certain HNN extensions.

Theorem 3.2. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where A is a group such that A/D is conjugacy separable for all finitely generated subgroups D in the center of A. Suppose H and K are finitely generated subgroups in the center of A and $H \neq A \neq K$. If G is subgroup separable then G is conjugacy separable.*

Proof. Since G is subgroup separable, then by Corollary 3.1, A is subgroup separable and there exists a finitely generated subgroup U of finite index in H such that $\varphi(U) = U$. Note that $U^n \triangleleft_f H$, $U^n \triangleleft G$ and $G/U^n = \langle t, A/U^n; t^{-1}H/U^n t = K/U^n, \varphi \rangle$ for all $n \geq 1$. Furthermore G/U^n is conjugacy separable by [8]. We shall write $\overline{G}_n = G/U^n, \overline{A}_n = A/U^n, \overline{H}_n = H/U^n$ and $\overline{K}_n = K/U^n$. Hence, $\overline{G}_n = \langle t, \overline{A}_n; t^{-1}\overline{H}_n t = \overline{K}_n, \varphi \rangle$.

Let $x, y \in G$ and $x \notin \{y\}^G$. We may assume x and y have minimal length in $\{x\}^G$ and $\{y\}^G$, respectively. If $x \notin \{y\}^G U^n$ for some $n \geq 1$, we are done, for G/U^n is conjugacy separable by [8].

Suppose that $x \in \{y\}^G U^n$ for all $n \geq 1$. We shall show that this case cannot occur.

Case 1. $x, y \in H \cup K$. Since $x \in \{y\}^G U$, we have $x = g^{-1}ygu$ for some $u \in U$, $g \in G$. But $\{y\}^G = \{g^{-1}yg\}^G$ and so we may assume that $x = yu$.

Since $\overline{x} \sim_{\overline{G}_m} \overline{y}$ in $\overline{G}_m$, for each $m \geq 1$, then by Theorem 2.1, there exists a finite sequence $\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n$ of elements in $\overline{H}_m \cup \overline{K}_m$ such that $\overline{x} \sim_{\overline{A}_m} \overline{z}_1 \sim_{\overline{A}_m, t^*} \overline{z}_2 \sim_{\overline{A}_m, t^*} \cdots \sim_{\overline{A}_m, t*} \overline{z}_n \sim_{\overline{A}_m} \overline{y}$. Since $\overline{H}_m$ and $\overline{K}_m$ are in the center of $\overline{A}_m$, $\overline{x} = t^{-r_m}\overline{y}t^{r_m}$ where $r_m \in \mathbb{Z}$. So for each $m \geq 1$,

$$yu = x = t^{-r_m} y t^{r_m} u_m, \text{ where } r_m \in \mathbb{Z} \text{ and } u_m \in U^m. \tag{3.1}$$

Suppose the set of integers $W = \{r_m\}$ is finite. Since G is subgroup separable, there exists $N \lhd_f G$ such that $x \notin t^{-w}yt^w N$ for all $w \in W$. But then $U^p \subseteq N$ for some integer p. From (3.1), $x = t^{-r_p}yt^{r_p}u_p$ where $r_p \in W$, $u_p \in U^p$. Hence $x \in t^{-r_p}yt^{r_p}N$, a contradiction.

We assume the set W is infinite. From (3.1), $t^{-r_m}yt^{r_m} = yuu_m^{-1} \in H \cup K$ and $t^{r_m}yt^{-r_m} = yt^{r_m}u_m u^{-1}t^{-r_m} \in H \cup K$, since $U \lhd G$. Suppose $t^{-s}yt^s \notin H \cup K$ for some $s > 0$. Since W is infinite, there exists a m_1 such that $c = |r_{m_1}| > s$. But $t^{-c}yt^c \in H \cup K$, a contradiction. Similarly if $t^{-s}yt^s \notin H \cup K$ for some $s < 0$, we also get a contradiction.

Therefore $t^{-s}yt^s \in H \cup K$ for all integer s. Let P_1 be the subgroup generated by $t^{-s}yt^s$, $s \in \mathbb{Z}$. Suppose $x \notin P_1$. Since G is subgroup separable, there exists $N_1 \lhd_f G$ such that $x \notin P_1N_1$. But then $U^{p_1} \subseteq N_1$ for some integer p_1. From (3.1), $x = t^{-r_{p_1}}yt^{r_{p_1}}u_{p_1}$ where $r_{p_1} \in \mathbb{Z}$, $u_{p_1} \in U^{p_1}$. Hence, $x \in P_1U^{p_1} \subseteq P_1N_1$, a contradiction.

Suppose $x \in P_1$. Let P_2 be the subgroup generated by P_1, U and t. Then $x \in P_1 \subsetneq P_2$. Since P_1U is a subgroup of HK, P_1U is finitely generated abelian. Furthermore $t^{-1}P_1Ut = P_1U$ and hence $P_2/P_1U \cong < t >$. Therefore, P_2 is polycyclic and thus is conjugacy separable ([9, 14]). Hence there exists $N_2 \lhd_f P_2$ such that $x \notin \{y\}^{P_2}N_2$. But then $U^{p_2} \subseteq N_2$ for some integer p_2. From (3.1), $x = t^{-r_{p_2}}yt^{r_{p_2}}u_{p_2}$ where $r_{p_2} \in \mathbb{Z}$, $u_{p_2} \in U^{p_2}$. Hence $x \in P_2U^{p_2} \subseteq P_2N_2$, a contradiction.

Case 2. $x \in A - (H \cup K), y \in A$ or $x \in A, y \in A - (H \cup K)$. We shall only show the case $x \in A - (H \cup K), y \in A$. Since A is subgroup separable and conjugacy separable, there exists $M_A \lhd_f A$ such that $x \notin HM_A \cup KM_A$ and $x \notin \{y\}^A M_A$. But then $U^p \subseteq M_A$ for some integer p. Since $\overline{x} \sim_{\overline{G}_p} \overline{y}$ in $\overline{G}_p$, then by Theorem 2.3, $\overline{x} = \overline{a}^{-1}\overline{ya}$ for some $a \in A$, since $\overline{H}_p$ and $\overline{K}_p$ are in the center of $\overline{A}_p$. But then $x \in \{y\}^A U^p \subseteq \{y\}^A M_A$, a contradiction.

Case 3. Suppose that $\|x\| = n \geq 1$, $\|y\| = m \geq 1$ where $n \neq m$. By the choice of x and y as elements of minimal length in their respective conjugacy classes, x and y are cyclically reduced. Let $x = x_0t^{\epsilon_1}x_1 \ldots t^{\epsilon_{n-1}}x_{n-1}t^{\epsilon_n}$ and $y = y_0t^{\epsilon'_1}y_1 \ldots t^{\epsilon'_{m-1}}y_{m-1}t^{\epsilon'_m}$ in cyclically reduced forms. Since A is subgroup separable, there exists $M_A \lhd_f A$ such that $x_i, y_j \notin HM_A$ for those $x_i, y_j \notin H$ and $x_i, y_j \notin KM_A$ for those $x_i, y_j \notin K$. But then $U^p \subseteq M_A$ for some integer p. By assumption $x \in \{y\}^G U^p$. Hence in $\overline{G}_p$, $\overline{x}$ and $\overline{y}$ are cyclically reduced with $\|\overline{x}\| = n$ and $\|\overline{y}\| = m$ and $\overline{x} \in \{\overline{y}\}^{\overline{G}_p}$. But $\overline{x} \notin \{\overline{y}\}^{\overline{G}_p}$ since $\overline{x}$ has length n while the cyclically reduced elements in $\{\overline{y}\}^{\overline{G}_p}$ have length $m \neq n$. Hence, we have a contradiction.

Case 4. Suppose $\|x\| = \|y\| = n \geq 1$. Let $x = x_0 t^{\epsilon_1} x_1 \ldots t^{\epsilon_{n-1}} x_{n-1} t^{\epsilon_n}$ and $y = y_0 t^{\epsilon'_1} y_1 \ldots t^{\epsilon'_{n-1}} y_{n-1} t^{\epsilon'_n}$ in cyclically reduced forms. Note that $x \in \{y\}^G$ if and only if $x \in \{y^{(j)}\}^X$ for some $j = 0 \ldots, n-1$ and $y^{(j)} = y_j t^{\epsilon'_{j+1}} y_{j+1} \ldots t^{\epsilon'_{n+j-1}} y_{n+j-1} t^{\epsilon'_{n+j}}$, the subscripts are taken modulo n, where $\epsilon_i = \epsilon'_{j+i}$, for all i and $X = H$ if $\epsilon_n = -1$ and $X = K$ if $\epsilon_n = 1$ (see Theorem 2.1). For each $j = 0 \ldots, n-1$, we shall show that there is an integer $p_j \geq 1$ such that $x \notin \{y^{(j)}\}^X U^{p_j}$. For simplicity, we shall only show the case $j = 0$, that is $y^{(j)} = y^{(0)} = y$. The other cases are similar.

Clearly $x \notin \{y\}^{HK}$. Suppose that $x \in \{y\}^{HK} U^p$ for all $p \geq 1$. Since $x \in \{y\}^{HK} U$ we have $x = v^{-1} y v u = v^{-1} y u v$ for some $v \in HK$, $u \in U$. Therefore, $yu \notin \{y\}^{HK}$ and $yu \in \{y\}^{HK} U^p$ for all $p \geq 1$. Hence, $yu = v_p^{-1} y v_p u_p$ where $v_p \in HK$, $u_p \in U^p$ for all $p \geq 1$.

Let Q be the subgroup generated by all the v_p. Then Q is a finitely generated abelian subgroup in the center of A. Note that $[y, v_p] = y^{-1} v_p^{-1} y v_p = u u_p^{-1} \in U$ for all $p \geq 1$. By using the identity $[y, v_{p_1} v_{p_2}] = [y, v_{p_2}] v_{p_2}^{-1} [y, v_{p_1}] v_{p_2}$ and the fact that U is in the center of A, we deduce that $[y, v_{p_1} v_{p_2}] = [y, v_{p_1}][y, v_{p_2}]$. Furthermore $[y, v_p]^{-1} = (u u_p^{-1})^{-1} = u_p u^{-1} = [y, v_p^{-1}]$. Therefore $[y, q_1 q_2] = [y, q_1][y, q_2]$ and $[y, q_1]^{-1} = [y, q_1^{-1}]$ for all $q_1, q_2 \in Q$. Thus $[y, Q] = \{[y, q] \ ; \ q \in Q\}$ is a subgroup in U. Clearly $[y, Q]$ is finitely generated. Now if $u \in [y, Q]$, then $u = y^{-1} q^{-1} y q$ for some $q \in Q$ and hence $yu = q^{-1} y q \in \{y\}^{HK}$, a contradiction. Therefore $u \notin [y, Q]$. Since G is subgroup separable, there exists $N \lhd_f G$ such that $u \notin [y, Q] N$. But then $U^{p'} \subseteq N$ for some integer $p' \geq 1$. From above, $u = y^{-1} v_{p'}^{-1} y v_{p'} u_{p'}$ where $v_{p'} \in Q$, $u_{p'} \in U^{p'}$. Hence $u \in [y, Q] U^{p'} \subseteq [y, Q] N$, a contradiction. Hence $x \notin \{y\}^{HK} U^p$ for some $p \geq 1$. In particular $x \notin \{y\}^X U^p$ for some $p \geq 1$.

Thus, we have found integers $p_j \geq 1$ such that $x \notin \{y^{(j)}\}^X U^{p_j}$ for each $j = 0 \ldots, n-1$. Let $p_n \geq 1$ be the integer such that $x_i, y_j \notin H U^{p_n}$ for those $x_i, y_j \notin H$ and $x_i, y_j \notin K U^{p_n}$ for those $x_i, y_j \notin K$. Let $p = p_0 p_1 \ldots p_{n-1} p_n$. By assumption $x \in \{y\}^G U^p$. Then in $\overline{G}_p$, $\overline{x}$ and $\overline{y}$ are cyclically reduced with $\|\overline{x}\| = \|\overline{y}\| = n$ and $\overline{x} \in \{\overline{y}\}^{\overline{G}_p}$. But this implies that $x \in \{y^{(j)}\}^X U^p \subseteq \{y^{(j)}\}^X U^{p_j}$, a contradiction. The proof is now completed. ∎

Lemma 3.4. *Let F be a polycyclic-by-finite group. Then F/D is conjugacy separable for each subgroup D in the center of F.*

Proof. Let A be the polycyclic group such that F/A is finite. Note that F/D is a finite extension of AD/D and $AD/D \cong A/(A \cap D)$. Since $A/(A \cap D)$ is also polycyclic, F/D is polycyclic-by-finite. Hence F/D is conjugacy separable ([9, 14]). ∎

Now we prove Theorem 3.3 which shows the equivalence of subgroup separability and conjugacy separability in this type of HNN extensions.

Theorem 3.3. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group. Suppose H and K are finitely generated subgroups in the center of A, $H \neq A \neq K$ and $H \cap K$ is a subgroup of finite index in H and K. Then G is subgroup separable if and only if G is conjugacy separable.*

Proof. Suppose G is subgroup separable. Then by Lemma 3.4 and Theorem 3.2, G is conjugacy separable.

Suppose G is conjugacy separable. Then G is residually finite. By [18], either $H = K$ or $H \nsubseteq K$ and $K \nsubseteq H$. Suppose $H = K$. By taking $U = H$ in Corollary 3.2, G is subgroup separable.

Suppose $H \nsubseteq K$ and $K \nsubseteq H$. This implies that $H \neq HK \neq K$. Then $G' = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ is a subgroup of G and is residually finite. Since H and K have finite index in HK, by [1], there exists a finitely generated subgroup U of finite index in H such that $\varphi(U) = U$. Hence, G is subgroup separable by Corollary 3.2. ∎

From Corollary 3.2 and Theorem 3.3, we obtain the following characterization on conjugacy separability.

Corollary 3.3. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group. Suppose H and K are finitely generated subgroups in the center of A, $H \neq A \neq K$ and $H \cap K$ is a subgroup of finite index in H and K. Then G is conjugacy separable if and only if there exists a finitely generated subgroup U of finite index in H such that $\varphi(U) = U$.*

By analyzing its proof, Theorem 3.3 can be stated as follows:

Corollary 3.4. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be as in Theorem 3.3. Then the followings are equivalent:*

(a) *G is residually finite,*
(b) *G is subgroup separable,*
(c) *G is conjugacy separable.*

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