Uniqueness of Meromorphic Functions Sharing Three Weighted Values

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Abstract. In this paper, we study the uniqueness of meromorphic functions sharing three values and improve some previous results.

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1. Introduction, definitions and results

Let $f$ and $g$ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \{\infty\} \cup \mathbb{C}$ we say that $f$ and $g$ share the value $a$ CM (counting multiplicities) if $f$ and $g$ have the same $a$-points with the same multiplicities. If the multiplicities are not taken into account, we say that $f$ and $g$ share the value $a$ IM (ignoring multiplicities). For standard definitions and notations of the value distribution theory, we refer to [2]. We denote by $E$ a set of non-negative real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $f$, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty (r \notin E)$. We use $N_0(r)(N_0(r))$ to denote the counting function (reduced counting function) of those zeros of $f-g$ which are not the zeros of $g(g-1)$, $\frac{1}{g}$ and $N_0^*(r)(N_0^*(r))$ to denote the counting function (reduced counting function) of those zeros of $f-g$ which are not the zeros of $g(g-1)$.

In 1999, Q. C. Zhang proved the following results:

Theorem 1.1. [10] Let $f$ and $g$ be two non-constant meromorphic functions sharing $0, 1, \infty$ CM. If

$$\limsup_{r \to \infty, r \notin E} \frac{N_0(r)}{T(r,f)} > \frac{1}{2},$$

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then \( f \) is a bilinear transformation of \( g \) and one of the following relations holds:

(i) \( f \equiv g \),
(ii) \( f + g \equiv 1 \),
(iii) \( (f - 1)(g - 1) \equiv 1 \), and
(iv) \( fg \equiv 1 \).

**Theorem 1.2.** [10] Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing \( 0, 1, \infty \) CM. If

\[
0 < \limsup_{r \to \infty, r \notin E} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},
\]

then \( N_0(r) = \frac{1}{k} T(r, f) + s(r, f) \) and \( f \) is not any fractional linear transformation of \( g \) and assume one of the following forms:

(i) \( f = \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1} \) and \( g = \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}, 1 \leq s \leq k; \)
(ii) \( f = \frac{e^{(k+1-s)\gamma} - 1}{e^{(k+1)\gamma} - 1} \) and \( g = \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}, 1 \leq s \leq k; \)
(iii) \( f = \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1} \) and \( g = \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}, 1 \leq s \leq k; \)

where \( s \) and \( k \) are positive integers such that \( s \) and \( k + 1 \) are relatively prime and \( \gamma \) is a non-constant for entire function.

In 2003, H. X. Yi and Y. H. Li proved the following theorem.

**Theorem 1.3.** [8] Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \( 0, 1, \infty \) CM. Then

\[
\frac{1}{2} + o(1) \leq \frac{T(r, f)}{T(r, g)} \leq 2 + o(1)
\]

and

\[
\frac{1}{2} + o(1) \leq \frac{T(r, g)}{T(r, f)} \leq 2 + o(1)
\]

as \( r \to \infty (r \notin E) \). If, in particular, \( f \) and \( g \) are entire, then \( T(r, f) \sim T(r, g) \) as \( r \to \infty (r \notin E) \).

In 2005, Qi Han [3] used the notion of weighted value sharing, introduced in [4], to improve the above results. We now explain the notion of weighted sharing of values which measures how close a shared value is being shared IM or being shared CM.

**Definition 1.1.** [4] Let \( k \) be a non-negative integer or infinity. For \( a \in \{\infty\} \cup \mathbb{C} \), we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f \) and \( g \) share the value \( a \) with weight \( k \).
The definition implies that if \( f \) and \( g \) share a value \( a \) with weight \( k \) then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(\leq k) \) if and only if it is a zero of \( g - a \) with multiplicity \( m(\leq k) \) and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(> k) \) if and only if it is a zero of \( g - a \) with multiplicity \( n(> k) \) where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \( (a, k) \) to mean that \( f, g \) share the value \( a \) with weight \( k \).

Clearly if \( f, g \) share \( (a, k) \) then \( f, g \) share \( (a, p) \) for all integers \( p, 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \( (a, 0) \) or \( (a, \infty) \) respectively.

We now state the results of Qi Han.

**Theorem 1.4.** [3] Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((0, 1)\), \((1, \infty)\) and \((\infty, \infty)\). If

\[
\limsup_{r \to \infty, r \notin E_N} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},
\]

then the conclusion of Theorem 1.1 holds.

**Theorem 1.5.** [3] Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing \((0, 1)\), \((1, \infty)\) and \((\infty, \infty)\). If

\[
0 < \limsup_{r \to \infty, r \notin E_N} \frac{N_0(r)}{T(r, f)} < \frac{1}{2},
\]

then the conclusion of Theorem 1.2 holds.

**Theorem 1.6.** [3] If \( f \) and \( g \) are two non-constant meromorphic functions sharing \((0, 1)\), \((1, \infty)\) and \((\infty, \infty)\), then the conclusion of Theorem 1.3 holds.

In this paper, we prove the following results which improve the above theorems.

**Theorem 1.7.** Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((0, k_1)\), \((1, k_2)\) and \((\infty, k_3)\), where \( k_j (j = 1, 2, 3) \) are positive integers satisfying

\[
(1.1) \quad k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2
\]

If

\[
\limsup_{r \to \infty, r \notin E_N} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},
\]

then the conclusion of Theorem 1.1 holds.

**Theorem 1.8.** Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing \((0, k_1)\), \((1, k_2)\) and \((\infty, k_3)\), where \( k_j (j = 1, 2, 3) \) are positive integers satisfying \((1.1)\). If

\[
0 < \limsup_{r \to \infty, r \notin E_N} \frac{N_0(r)}{T(r, f)} < \frac{1}{2},
\]

then the conclusion of Theorem 1.2 holds.

**Theorem 1.9.** Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((0, k_1)\), \((1, k_2)\) and \((\infty, k_3)\), where \( k_j (j = 1, 2, 3) \) are positive integers satisfying \((1.1)\). Then the conclusion of Theorem 1.3 holds.

We now give some more necessary definitions.
Definition 1.2. Let \( f \) and \( g \) share \((a,0)\) and \( z \) be an \( a \)-point of \( f \) and \( g \) with multiplicities \( p_f(z) \) and \( p_g(z) \) respectively. We put

\[
\nu_f(z) = \begin{cases} 
1 & \text{if } p_f(z) > p_g(z) \\
0 & \text{if } p_f(z) \leq p_g(z)
\end{cases}
\]

and

\[
\mu_f(z) = \begin{cases} 
1 & \text{if } p_f(z) < p_g(z) \\
0 & \text{if } p_f(z) \geq p_g(z)
\end{cases}
\]

Let \( \overline{\pi}(r,a; f > g) = \sum_{|z| \leq r} \nu_f(z) \) and \( \overline{\pi}(r,a; f < g) = \sum_{|z| \leq r} \mu_f(z) \). We now denote by \( \overline{N}(r,a; f > g) \) and \( \overline{N}(r,a; f < g) \) the integrated counting functions obtained from \( \overline{\pi}(r,a; f > g) \) and \( \overline{\pi}(r,a; f < g) \) respectively. Finally, we put

\[
\overline{N}_*(r,a; f, g) = \overline{N}(r,a; f > g) + \overline{N}(r,a; f < g)0
\]

Definition 1.3. Let \( p \) be a positive integer and \( a \in \{\infty\} \cup \mathbb{C} \). By \( N(r,a; f \mid \leq p) \), we denote the counting function of those \( a \)-points of \( f \) (counted with multiplicities) whose multiplicities are not greater than \( p \). By \( N(r,a; f \mid > p) \), we denote the corresponding reduced counting function. In an analogous manner, we define \( N(r,a; f \mid \geq p) \) and \( N(r,a; f \mid = p) \).

2. Lemmas

In this section, we present some necessary lemmas.

Lemma 2.1. \([1]\) Let \( f \) and \( g \) be two non-constant meromorphic functions sharing three values \( IM \). Then

\[
T(r,f) \leq 3T(r,g) + S(r,f)
\]

and

\[
T(r,g) \leq 3T(r,f) + S(r,g).
\]

From Lemma 2.1, we see that \( S(r,f) = S(r,g) \), which we denote by \( S(r) \) in the sequel.

Lemma 2.2. \([9]\) Let \( f \) and \( g \) be two distinct non-constant meromorphic functions sharing \((0,k_1),(1,k_2)\) and \((\infty,k_3)\), where \( k_j(j = 1,2,3) \) are positive integers satisfying \((1.1)\). Then

\[
\overline{N}(r,a; f \mid \geq 2) = S(r)
\]

and

\[
\overline{N}(r,a; g \mid \geq 2) = S(r)
\]

for \( a = 0,1,\infty \).

Lemma 2.3. \([5]\) Let \( f \) and \( g \) share \((0,0),(1,0)\) and \((\infty,0)\) and \( f \neq g \). If \( \alpha = \frac{f - 1}{g - 1} \) and \( h = \frac{g}{f} \), then

(i) \( \overline{N}(r,0; \alpha) = \overline{N}(r,\infty; f < g) + \overline{N}(r,1; f > g) \),

(ii) \( \overline{N}(r,\infty; \alpha) = \overline{N}(r,\infty; f > g) + \overline{N}(r,1; f < g) \),

(iii) \( \overline{N}(r,0; h) = \overline{N}(r,0; f < g) + \overline{N}(r,\infty; f > g) \),
(iv) $\overline{N}(r, \infty; h) = \overline{N}(r, 0; f > g) + \overline{N}(r, \infty; f < g)$.

**Lemma 2.4.** Let $f$ and $g$ share $(0, k_1), (1, k_2)$ and $(\infty, k_3)$ and $f \not\equiv g$, where $k_j (j = 1, 2, 3)$ are positive integers satisfying (1.1). If $\alpha$ and $h$ are defined as in Lemma 2.3, then $\overline{N}(r, a; \alpha) = S(r)$ and $\overline{N}(r, a; h) = S(r)$ for $a = 0, \infty$.

**Proof.** The lemma follows from Lemma 2.2 and Lemma 2.3 because

$$\overline{N}_*(r, a; f, g) \leq \overline{N}(r, a; |f| \geq 2)$$

for $a = 0, 1, \infty$.

Using Lemma 2.2 and Lemma 2.4, we can prove the following lemma in the line of Lemma 2.5 [5].

**Lemma 2.5.** Let $f$ and $g$ share $(0, k_1), (1, k_2)$ and $(\infty, k_3)$ and $f \not\equiv g$, where $k_j (j = 1, 2, 3)$ are positive integers satisfying (1.1). If $f$ is not a bilinear transformation of $g$, then each of the following equalities holds:

- (i) $T(r, f) + T(r, g) = N(r, 0; |f| \leq 1) + N(r, 1; |f| \leq 1) + N(r, \infty; |f| \leq 1) + N_0(r) + S(r)$,
- (ii) $T(r, f) + T(r, g) = N(r, 0; |g| \leq 1) + N(r, 1; |g| \leq 1) + N(r, \infty; |g| \leq 1) + N_0(r) + S(r)$,
- (iii) $N(r, 0; f - g|f = \infty) = S(r)$ and $N(r, 0; f - g|g = \infty) = S(r)$,
- (iv) $N(r, 0; |f - g| \geq 2) = S(r)$,

where $N(r, 0; f - g|f = \infty)$ denotes the counting function of those zeros of $f - g$ which are poles of $f$.

**Lemma 2.6.** [7] Let $f$ be a non-constant meromorphic function and

$$R(f) = \frac{\sum_{i=0}^{m} a_i f^i}{\sum_{j=0}^{n} b_j f^j}$$

be a non-constant irreducible rational in $f$ with constant coefficient $\{a_i\}$ and $\{b_j\}$ satisfying $a_m \neq 0$ and $b_n \neq 0$. Then

$$T(r, R(f)) = \max\{m, n\} T(r, f) + O(1).$$

In particular, if $f$ is a bilinear transformation of $g$, then we have $T(r, f) \sim T(r, g)$ as $r \to \infty$.

**Lemma 2.7.** [6] Let $f_1$ and $f_2$ be two distinct non-constant meromorphic functions satisfying

$$\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$$

for $i = 1, 2$. If $f_1^s f_2^t - 1$ is not identically zero for all integers $s$ and $t(|s| + |t| > 0)$, then for any positive $\varepsilon$ we have

$$\overline{N}_0(r, 1; f_1, f_2) \leq \varepsilon T(r; f_1, f_2) + S(r; f_1, f_2),$$

where $\overline{N}_0(r, 1; f_1, f_2)$ denotes the reduced counting function of $f_1$ and $f_2$ related to the common $1$ -points and $T(r; f_1, f_2) = T(r, f_1) + T(r, f_2), S(r; f_1, f_2) = o\{T(r; f_1, f_2)\}$ as $r \to \infty (r \notin E)$. 

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3. Proof of the theorems

Proof of Theorems 1.7 and 1.8. Suppose that $f$ is not a bilinear transformation of $g$, otherwise, we obtain that $f$ and $g$ share $(0, \infty), (1, \infty), (\infty, \infty)$ and hence Theorem 1.7 and Theorem 1.8 follow from Theorem 1.1 and Theorem 1.2 respectively.

From the definitions of $N^*_0(r)$ and $N_0(r)$, we see that $N^*_0(r) - N_0(r)$ is the counting function of those zeros of $f - g$ which are the poles of $f$. So by (iii) of Lemma 2.5 we get

$$\tag{3.1} N^*_0(r) - N_0(r) = S(r).$$

Now we prove that

$$\tag{3.2} \mathcal{N}_0(r, 1; \alpha, h) = N_0(r) + S(r).$$

We consider the following cases.

Case 1. Let $z_0$ be a common simple zero of $f$ and $g$ such that $\alpha(z_0) = h(z_0) = 1$. Since $h - 1 = \frac{g - f}{f}$, it follows that $z_0$ is a multiple zero of $f - g$. Hence by Lemma 2.2 and (iv) of Lemma 2.5 we see that the reduced counting function of the zeros of $f$ and $g$ for which $\alpha(z) = h(z) = 1$ is $S(r)$.

Case 2. Let $z_1$ be a common simple 1-point of $f$ and $g$ such that $\alpha(z_1) = h(z_1) = 1$. Since $\alpha - 1 = \frac{f - g}{g - 1}$, it follows that $z_1$ is a multiple zero of $f - g$. Hence by Lemma 2.2 and (iv) of Lemma 2.5 we see that the reduced counting function of the 1-points of $f$ and $g$ for which $\alpha(z) = h(z) = 1$ is $S(r)$.

Case 3. Let $z_2$ be a common simple pole of $f$ and $g$ such that $\alpha(z_2) = h(z_2) = 1$. Since $\alpha - 1 = \frac{f - g}{g - 1}$ and $h - 1 = \frac{g - f}{f}$, it follows that $z_2$ is not a pole of $f - g$.

Hence, $z_2$ is a zero of $\alpha + h - 2 = \frac{(f - g)(f - g + 1)}{f(g - 1)}$ with multiplicity $\geq 2$ and so $z_2$ is a zero of $\frac{\alpha'}{\alpha} + \frac{h'}{h}$. Also

$$\alpha' + \frac{h'}{h} = (\alpha' + \frac{h'}{h}) \frac{\alpha + h - 1}{\alpha h} - \frac{(\alpha - 1)\alpha' + (h - 1)h'}{\alpha h}.$$

Since $f$ is not a bilinear transformation of $g$, $\frac{\alpha'}{\alpha} + \frac{h'}{h} \neq 0$. From the preceding identity, we see that $z_2$ is a zero of $\frac{\alpha'}{\alpha} + \frac{h'}{h}$. Now by Lemma 2.4, we get

$$N \left( r, 0; \frac{\alpha'}{\alpha} + \frac{h'}{h} \right) \leq T \left( r, \frac{\alpha'}{\alpha} + \frac{h'}{h} \right) = \frac{N \left( r, \frac{\alpha'}{\alpha} \right) + N(r, \frac{h'}{h}) + S(r)}{\mathcal{N}(r, 0; \infty) + \mathcal{N}(r, \infty, \alpha) + \mathcal{N}(r, 0; h) + \mathcal{N}(r, \infty; h) + S(r)} = S(r).$$
Therefore, by Lemma 2.2 we see that the reduced counting function of the poles of \( f \) and \( g \) for which \( \alpha(z) = h(z) = 1 \) is \( S(r) \).

Also by (iv) of Lemma 2.5 we have

\[
N_0^*(r) = \overline{N}_0^*(r) + S(r) \quad \text{and} \quad N_0^*(r) = N_0^*(r) + S(r).
\]

Hence from above we get by (3.1)

\[
\overline{N}_0^*(r, 1; \alpha, h) = N_0^*(r) + S(r) = N_0^*(r) + S(r),
\]

which is (3.2). From the definitions of \( \alpha \) and \( h \) and from Lemma 2.1 we get

\[
T(r, \alpha) + T(r, h) \leq 2T(r, f) + 2T(r, g) + O(1)
\]

(3.3)

Hence by (3.2) we obtain

\[
\limsup_{r \to \infty, r \notin E} \frac{\overline{N}_0^*(r, 1; \alpha, h)}{T(r; \alpha, h)} \geq \frac{1}{8} \limsup_{r \to \infty, r \notin E} \frac{N_0^*(r)}{T(r, f)} > 0.
\]

If we put

\[
a = \limsup_{r \to \infty, r \notin E} \frac{\overline{N}_0^*(r, 1; \alpha, h)}{T(r; \alpha, h)},
\]

we see that the following inequality does not hold for any \( \varepsilon (0 < \varepsilon < a) \)

\[
\overline{N}_0^*(r, 1; \alpha, h) \leq \varepsilon T(r; \alpha, h) + S(\alpha, h)
\]

as \( r \to \infty (r \notin E) \), where \( T(r; \alpha, h) = T(r, \alpha) + T(r, h) \).

Since \( f = \frac{1 - \alpha}{1 - \alpha h} \) and \( g = \frac{h(1 - \alpha)}{1 - \alpha h} \), we get

\[
T(r, f) \leq 2T(r, \alpha) + 2T(r, h) + O(1)
\]

and

\[
T(r, g) \leq 2T(r, \alpha) + 2T(r, h) + O(1).
\]

This together with (3.3) implies that \( S(r) = S(r; \alpha, h) \). Hence by Lemma 2.4 and Lemma 2.7, there exist two integers \( s \) and \( t (|s| + |t| > 0) \) such that

\[
\alpha^s h^t \equiv 1.
\]

Hence

\[
(f - 1)^t \equiv (f - 1)^s
\]

(3.4)

and so

\[
\left( \frac{1 - f}{1 - g} \right)^t \equiv \left( \frac{1 - f}{1 - g} \right)^{s-t}
\]

(3.5)

If \( st = 0 \) or \( s - t = 0 \), then from (3.4) and (3.5) we see that \( f \) is bilinear transformation of \( g \), which is a contradiction. Therefore \( st \) and \( s - t \) are not equal to zero. Consequently, (3.4) and (3.5) imply that \( f \) and \( g \) share \((0, \infty), (1, \infty), (\infty, \infty)\). Now,
Theorem 1.7 and Theorem 1.8 follow respectively from Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.9.** We consider the following cases.

**Case 1.** Let
\[
\limsup_{r \to \infty, r \notin E} \frac{N_0(r)}{T(r, f)} > 0.
\]
If $f$ is a bilinear transformation of $g$, then one of the relations of Theorem 1.7 holds. So by Lemma 2.6 we get $T(r, f) \sim T(r, g)$ as $r \to \infty$. If $f$ is not a bilinear transformation of $g$, then one of the relations of Theorem 1.8 holds. Since $s$ and $k + 1$ are positive integers which are relatively prime, by Lemma 2.6 we get $T(r, f) \sim T(r, g)$ as $r \to \infty$.

**Case 2.** Let
\[
\limsup_{r \to \infty, r \notin E} \frac{N_0(r)}{T(r, f)} = 0.
\]
If $f$ is a bilinear transformation of $g$, by Lemma 2.6 we have $T(r, f) \sim T(r, g)$ as $r \to \infty$.

If $f$ is not a bilinear transformation of $g$ by Lemma 2.5(i) and (ii) we get
\[
T(r, g) \leq T(r, f) + N(r, \infty; f| \leq 1) + S(r)
\]
and
\[
T(r, f) \leq T(r, g) + N(r, \infty; g| \leq 1) + S(r).
\]
From (3.6) and (3.7), we obtain
\[
T(r, g) \leq 2T(r, f) + S(r)
\]
and
\[
T(r, f) \leq 2T(r, g) + S(r).
\]
If, in particular, $f$ and $g$ are entire from (3.6) and (3.7) we get $T(r, f) \sim T(r, g)$ as $r \to \infty (r \notin E)$. This proves the theorem.

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