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Some Common Fixed Point Theorems in Complete $\mathcal{L}$-Fuzzy Metric Spaces

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Abstract. In this paper, we prove a common fixed point theorem in $\mathcal{L}$-fuzzy metric space which is a generalization of some results in Adibi et. al. [1].

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1. Introduction and preliminaries

The notion of fuzzy sets was introduced by Zadeh [23]. Various concepts of fuzzy metric spaces were considered in [7, 8, 13, 14]. Many authors have studied fixed point theory in fuzzy metric spaces; see for example [3, 4, 11, 12, 16, 17]. In the sequel, we shall adopt the usual terminology, notation and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [19] which are a generalization of fuzzy metric spaces [10] and intuitionistic fuzzy metric spaces [18, 20].

Definition 1.1. [11] *Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called a universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.*

Lemma 1.1. [5, 6] *Consider the set L^* and the operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice .

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = x$, for all $x \in [0,1]$.

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These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 1.2. *A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:*

(1) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$; *(boundary condition)*
(2) $(\forall (x, y) \in L^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$; *(commutativity)*
(3) $(\forall (x, y, z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$; *(associativity)*
(4) $(\forall (x, x', y, y') \in L^4)(x \leq_L x'$ *and* $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$. *(monotonicity)*

A t–norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y)$$

For example, $\mathcal{T}(x, y) = \min(x, y)$ and $\mathcal{T}(x, y) = xy$ are two continuous t–norms on $[0, 1]$. A t-norm can also be defined recursively as an $(n + 1)$-ary operation $(n \in \mathbb{N})$ by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, \cdots, x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

Definition 1.3. *A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation.*

Definition 1.4. *The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times]0, +\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0, +\infty[$:*

(a) $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ *for all* $t > 0$ *if and only if* $x = y$;
(c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(d) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$;
(e) $\mathcal{M}(x, y, \cdot) :]0, \infty[\to L$ *is continuous and* $\lim_{t\to\infty} \mathcal{M}(x, y, t) = 1_{\mathcal{L}}$.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in X$ and a fixed radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L\setminus\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$*.

Example 1.1. [21] Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2\times]0, +\infty[$ be defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right).$$

Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.2. [1] Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)}\right),$$

for all $t, h, m, n \in \mathbf{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Lemma 1.2. [10] *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Definition 1.5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_\mathcal{L}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,*

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_\mathcal{L}$ whenever $n \to +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Henceforth, we assume that $\mathcal{T}$ is a continuous t-norm on the lattice $\mathcal{L}$ such that for every $\mu \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, there is a $\lambda \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu).$$

For more information see [19].

Definition 1.6. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if*

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0, \infty[$ converges to a point $(x, y, t) \in X \times X \times]0, \infty[$ i.e., $\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_\mathcal{L}$ and $\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 1.3. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X \times X \times]0, \infty[$.*

Proof. The proof is the same as that for fuzzy spaces (see Proposition 1 of [15]). ∎

Definition 1.7. *Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.*

Definition 1.8. *Let A and S be mappings from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. Then the mappings are said to be compatible if*

$$\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, t) = 1_\mathcal{L}, \forall t > 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Proposition 1.1. [22] *If self-mappings* A *and* S *of an* $\mathcal{L}$*-fuzzy metric space* $(X, \mathcal{M}, \mathcal{T})$ *are compatible, then they are weak compatible.*

Lemma 1.4. [1, 19] *Let* $(X, \mathcal{M}, \mathcal{T})$ *be an* $\mathcal{L}$*-fuzzy metric space. Define* $E_{\lambda,\mathcal{M}} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ *by*

$$E_{\lambda,\mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(x, y, t) >_L \mathcal{N}(\lambda)\}$$

for each $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *and* $x, y \in X$*. Then we have*

(i) *For any* $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *there exists* $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ *such that*

$$E_{\mu,\mathcal{M}}(x_1, x_n) \le E_{\lambda,\mathcal{M}}(x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1}, x_n)$$

for any $x_1, ..., x_n \in X$*;*

(ii) *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is convergent to* x *w.r.t.* $\mathcal{L}$*-fuzzy metric* $\mathcal{M}$ *if and only if* $E_{\lambda,\mathcal{M}}(x_n, x) \to 0$*.*

Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ *is Cauchy w.r.t.* $\mathcal{L}$*-fuzzy metric* $\mathcal{M}$ *if and only if it is Cauchy with* $E_{\lambda,\mathcal{M}}$*.*

Lemma 1.5. *Let* $(X, \mathcal{M}, \mathcal{T})$ *be an* $\mathcal{L}$*-fuzzy metric space. If*

$$\mathcal{M}(x_n, x_{n+1}, t) \ge_L \mathcal{M}(x_0, x_1, k^n t)$$

for some $k > 1$ *and* $n \in \mathbb{N}$*. Then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For every $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $x_n \in X$, we have

$$\begin{aligned} E_{\lambda,\mathcal{M}}(x_{n+1}, x_n) &= \inf\{t > 0 \;:\; \mathcal{M}(x_{n+1}, x_n, t) >_L \mathcal{N}(\lambda)\} \\ &\le \inf\{t > 0 \;:\; \mathcal{M}(x_0, x_1, k^n t) >_L \mathcal{N}(\lambda)\} \\ &= \inf\{\frac{t}{k^n} \;:\; \mathcal{M}(x_0, x_1, t) >_L \mathcal{N}(\lambda)\} \\ &= \frac{1}{k^n} \inf\{t > 0 \;:\; \mathcal{M}(x_0, x_1, t) >_L \mathcal{N}(\lambda)\} \\ &= \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_1). \end{aligned}$$

From Lemma 1.4, for every $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, such that

$$\begin{aligned} E_{\mu,\mathcal{M}}(x_n, x_m) &\le E_{\lambda,\mathcal{M}}(x_n, x_{n+1}) + E_{\lambda,\mathcal{M}}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,\mathcal{M}}(x_{m-1}, x_m) \\ &\le \frac{1}{k^n} E_{\lambda,\mathcal{M}}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(x_0, x_1) \\ &= E_{\lambda,\mathcal{M}}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \longrightarrow 0. \end{aligned}$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. ∎

Definition 1.9. [9] *We say that the* $\mathcal{L}$*-fuzzy metric space* $(X, \mathcal{M}, \mathcal{T})$ *has property (C), if it satisfies the following condition:*

$$\mathcal{M}(x, y, t) = C, \quad \text{for all } t > 0 \quad \text{implies} \quad C = 1_{\mathcal{L}}.$$

2. Main Result

Theorem 2.1. *Let A, B, S and T be self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$, which has property (C), satisfying:*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ *and* $T(X), S(X)$ *are two closed subsets of* X*;*
(ii) *the pairs* (A, S) *and* (B, T) *are weak compatible;*
(iii) $\mathcal{M}(Ax, By, t) \geq_L \mathcal{M}(Sx, Ty, kt)$, *for every* x, y *in* X *and some* $k > 1$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (i), there exists $x_1, x_2 \in X$ such that $y_0 = Ax_0 = Tx_1$, $y_1 = Bx_1 = Sx_2$. Inductively, construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \cdots$. Now, we prove that $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = \mathcal{M}(y_m, y_{m+1}, t)$, $t > 0$. Then, we have

$$\begin{aligned} d_{2n}(t) &= \mathcal{M}(y_{2n}, y_{2n+1}, t) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq_L \mathcal{M}(Sx_{2n}, Tx_{2n+1}, kt) = \mathcal{M}(y_{2n-1}, y_{2n}, kt) = d_{2n-1}(kt). \end{aligned}$$

Thus $d_{2n}(t) \geq_L d_{2n-1}(kt)$ for every $m = 2n \in \mathbb{N}$ and $\forall t > 0$. Similarly for an odd integer $m = 2n + 1$, we have $d_{2n+1}(t) \geq_L d_{2n}(kt)$. Hence, for every $n \in \mathbb{N}$, we have $d_n(t) \geq_L d_{n-1}(kt)$. That is,

$$\mathcal{M}(y_n, y_{n+1}, t) \geq_L \mathcal{M}(y_{n-1}, y_n, kt) \geq_L \cdots \geq_L \mathcal{M}(y_0, y_1, k^n t).$$

So, by Lemma 1.5, $\{y_n\}$ is Cauchy and the completeness of X implies $\{y_n\}$ converges to y in X. That is, $\lim_{n\to\infty} y_n = y$

$$\begin{aligned} \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$. By (iii), we have

$$\mathcal{M}(Au, Bx_{2n+1}, t) \geq_L \mathcal{M}(Su, Tx_{2n+1}, kt).$$

Since $\mathcal{M}$ is continuous, we get (whenever $n \longrightarrow \infty$ in the above inequality):

$$\mathcal{M}(Au, y, t) \geq_L \mathcal{M}(y, y, kt) = 1_{\mathcal{L}}.$$

Thus $\mathcal{M}(Au, y, t) = 1_{\mathcal{L}}$, i.e. $Au = y$. Therefore, $Au = Su = y$.
Since $A(X) \subseteq T(X)$, there exists $v \in X$, such that $Tv = y$. Thus,

$$\mathcal{M}(y, Bv, t) = \mathcal{M}(Au, Bv, t) \geq_L \mathcal{M}(Su, Tv, kt) = 1_{\mathcal{L}}.$$

Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we conclude that $ASu = SAu$, that is $Ay = Sy$. Also, since (B, T) is weak compatible then, $TBv = BTv$, that is $Ty = By$. We now prove that $Ay = y$. By (iii), we have

$$\begin{aligned} \mathcal{M}(Ay, y, t) = \mathcal{M}(Ay, Bv, t) &\geq_L \mathcal{M}(Sy, Tv, kt) = \mathcal{M}(Ay, y, kt) \\ &\vdots \\ &\geq_L \mathcal{M}(Ay, y, k^n t). \end{aligned}$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(Ay, y, t) \leq_L \mathcal{M}(Ay, y, k^n t).$$

Hence, $\mathcal{M}(Ay, y, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, \mathcal{T})$ has property (C), it follows that $C = 1_{\mathcal{L}}$, i.e., $Ay = y$, therefore $Ay = Sy = y$.
Similarly we prove that $By = y$. By (iii), we have

$$\begin{aligned} \mathcal{M}(y, By, t) = \mathcal{M}(Ay, By, t) &\geq_L \mathcal{M}(Sy, Ty, kt) = \mathcal{M}(y, By, kt) \\ &\vdots \\ &\geq_L \mathcal{M}(y, By, k^n t). \end{aligned}$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(y, By, t) \leq_L \mathcal{M}(y, By, k^n t).$$

Hence, $\mathcal{M}(y, By, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, \mathcal{T})$ has property (C), it follows that $C = 1_{\mathcal{L}}$, i.e., $By = y$. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed of A, B, S and T. For uniqueness, let x be another common fixed point of A, B, S and T i.e., $x = Ax = Bx = Sx = Tx$. Hence

$$\begin{aligned} \mathcal{M}(y, x, t) = \mathcal{M}(Ay, Bx, t) &\geq_L \mathcal{M}(Sy, Tx, kt) = \mathcal{M}(y, x, kt) \\ &\vdots \\ &\geq_L \mathcal{M}(y, x, k^n t) \end{aligned}$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(y, x, t) \leq_L \mathcal{M}(y, x, k^n t).$$

Hence, $\mathcal{M}(y, x, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, \mathcal{T})$ has property (C), it follows that $C = 1_{\mathcal{L}}$, i.e., $y = x$. Therefore, y is the unique common fixed point of self-maps A, B, S and T. ∎

Theorem 2.2. *Let $\{A_i\}$ and $\{B_j\}$ be two sequences of self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$, which has property (C), such that,*

(i) *there exists $i_0, j_0 \in \mathbb{N}$ such that $A_{i_0}^m(X) \subseteq T(X)$, $B_{j_0}^n(X) \subseteq S(X)$ and $T(X)$, $S(X)$ are two closed subsets of X,*
(ii) *$A_i B_i = B_i A_i$, $TA_i = A_i T$, $SA_i = A_i S$, $TB_i = B_i T$ and $SB_i = B_i S$, for all $i \in \mathbb{N}$,*
(iii) *$\mathcal{M}(A_i^m x, B_j^n y, t) \geq_L \mathcal{M}(Sx, Ty, k_{ij} t)$, for every x, y in X, for some $n, m \in \mathbb{N}$; here $k_{ij} > k > 1$ for $i, j = 1, 2, \cdots$, and $t > 0$.*

Then, A_i, B_j, S and T have a unique common fixed point in X for all $i, j = 1, 2, \cdots$.

Proof. By Theorem 2.1, S, T and $A = A_{i_0}^m$ and $B = B_{j_0}^n$ for some i_0, $j_0 \in \mathbb{N}$, have a unique common fixed point in X. That is, there exists a unique $x \in X$ such that

$$S(x) = T(x) = A_{i_0}^m(x) = B_{j_0}^n(x) = x.$$

Suppose there exists $j \in \mathbb{N}$ such that $j \neq j_0$. Then we have

$$\mathcal{M}(x, B_j^n x, t) = \mathcal{M}(A_{i_0}^m x, B_j^n x, t) \quad \geq \quad \mathcal{M}(Sx, Tx, k_{ij} t) = 1_{\mathcal{L}}.$$

Hence, for every $j \in \mathbb{N}$ we have $B_j^n x = x$. Similarly for every $i \in \mathbb{N}$, we get $A_i^m x = x$. Therefore for every $i, j \in \mathbb{N}$, we have

$$A_i^m x = B_j^n x = Sx = Tx = x.$$

To show uniqueness, assume that y is another fixed point of A_i^m, B_j^n, S and T. Then we have

$$\begin{aligned}\mathcal{M}(x,y,t) = \mathcal{M}(A_i^m x, B_i^n y, t) &\geq_L \mathcal{M}(Sx, Ty, k_{ij}t) = \mathcal{M}(x,y,k_{ij}t) = \mathcal{M}(A_i^m x, B_i^n y, k_{ij}t) \\ &\geq_L \mathcal{M}(x,y,k_{ij}^2 t) \\ &\vdots \\ &\geq_L \mathcal{M}(x,y,k_{ij}^n t) \\ &\geq_L \mathcal{M}(x,y,k^n t).\end{aligned}$$

On the other hand, by Lemma 1.2 we have

$$\mathcal{M}(x,y,t) \leq \mathcal{M}(x,y,k^n t).$$

Hence, $\mathcal{M}(x,y,t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, \mathcal{T})$, has property (C), it follows that $C = 1_L$, i.e., $x = y$. Now, we prove that $A_i x = x$. Since

$$A_i x = A_i(B_i^n x) = B_i^n(A_i x) \textit{ and } A_i x = A_i(A_i^m x) = A_i^m(A_i x),$$

also

$$A_i x = A_i S x = S A_i x \textit{ and } A_i x = A_i T x = T A_i x.$$

That is $A_i x$ is also a common fixed point of A_i^m, B_i^n, S and T. Therefore $A_i x = x$. Similarly, one can show that $B_j x = x$, i.e., x is a unique common fixed point of the mappings A_i, B_j, S and T, for $i, j = 1, 2, \ldots$. This completes the proof. ∎

Corollary 2.1. *Let $(X, \mathcal{M}, \mathcal{T})$ be a complete $\mathcal{L}$-fuzzy metric space. Let $\{A_n\}$ be a sequence of mappings A_i of a complete $\mathcal{L}$-fuzzy metric space $X, \mathcal{M}, \mathcal{T})$ which has property (C), into itself such that, for any two mappings A_i, A_j,*

$$\mathcal{M}(A_i^m x, A_j^m y, \alpha_{ij} t) \geq_L \mathcal{M}(x,y,t)$$

for some m; here $0 < \alpha_{ij} < k < 1$ for $i, j = 1, 2, \ldots, x, y \in X$ and $t > 0$. Then the sequence $\{A_n\}$ has a unique common fixed point in X.

Proof. By Theorem 2.2, it is enough to set $S = T = I$, where I is the identity map and $B_j^n = A_J^m$. Also we replace α_{ij} by $\dfrac{1}{k_{ij}}$. ∎

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