Some Common Fixed Point Theorems in Complete \( L \)-Fuzzy Metric Spaces

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Abstract. In this paper, we prove a common fixed point theorem in \( L \)-fuzzy metric space which is a generalization of some results in Adibi et. al. [1].

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1. Introduction and preliminaries

The notion of fuzzy sets was introduced by Zadeh [23]. Various concepts of fuzzy metric spaces were considered in [7, 8, 13, 14]. Many authors have studied fixed point theory in fuzzy metric spaces; see for example [3, 4, 11, 12, 16, 17]. In the sequel, we shall adopt the usual terminology, notation and conventions of \( L \)-fuzzy metric spaces introduced by Saadati et al. [19] which are a generalization of fuzzy metric spaces [10] and intuitionistic fuzzy metric spaces [18, 20].

Definition 1.1. [11] Let \( \mathcal{L} = (L, \leq_L) \) be a complete lattice, and \( U \) a non-empty set called a universe. An \( L \)-fuzzy set \( A \) on \( U \) is defined as a mapping \( A : U \rightarrow L \). For each \( u \) in \( U \), \( A(u) \) represents the degree (in \( L \)) to which \( u \) satisfies \( A \).

Lemma 1.1. [5, 6] Consider the set \( L^* \) and the operation \( \leq_{L^*} \) defined by:

\[
L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},
\]

\[
(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^*.
\]

Then \( (L^*, \leq_{L^*}) \) is a complete lattice.

Classically, a triangular norm \( T \) on \(([0, 1], \leq)\) is defined as an increasing, commutative, associative mapping \( T : [0, 1]^2 \rightarrow [0, 1] \) satisfying \( T(1, x) = x \), for all \( x \in [0, 1] \).

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These definitions can be straightforwardly extended to any lattice $L = (L, \leq_L)$. Define first $0_L = \inf L$ and $1_L = \sup L$.

**Definition 1.2.** A triangular norm (t-norm) on $L$ is a mapping $T : L^2 \to L$ satisfying the following conditions:

1. $(\forall x \in L)(T(x, 1_L) = x)$; (boundary condition)
2. $(\forall (x, y) \in L^2)(T(x, y) = T(y, x))$; (commutativity)
3. $(\forall (x, y, z) \in L^3)(T(x, T(y, z)) = T(T(x, y), z))$; (associativity)
4. $(\forall (x, x', y, y') \in L^4)(x \leq_L x'$ and $y \leq_L y'$ $\Rightarrow T(x, y) \leq_L T(x', y')$).

A $t$–norm $T$ on $L$ is said to be continuous if for any $x, y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to $x$ and $y$ we have

$$\lim_{n} T(x_n, y_n) = T(x, y)$$

For example, $T(x, y) = \min(x, y)$ and $T(x, y) = xy$ are two continuous $t$–norms on $[0, 1]$. A $t$-norm can also be defined recursively as an $(n + 1)$-ary operation $(n \in \mathbb{N})$ by

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

**Definition 1.3.** A negation on $L$ is any decreasing mapping $N : L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$. If $N(N(x)) = x$, for all $x \in L$, then $N$ is called an involutive negation.

**Definition 1.4.** The 3-tuple $(X, M, T)$ is said to be an $L$-fuzzy metric space if $X$ is an arbitrary (non-empty) set, $T$ is a continuous $t$–norm on $L$ and $M$ is an $L$-fuzzy set on $X^2 \times [0, +\infty[)$ satisfying the following conditions for every $x, y, z$ in $X$ and $t, s$ in $[0, +\infty[$:

1. $M(x, y, t) >_L 0_L$;
2. $M(x, y, t) = 1_L$ for all $t > 0$ if and only if $x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $T(M(x, y, t), M(y, z, s)) \leq_L M(x, z, t + s)$;
5. $M(x, y, \cdot) : [0, +\infty[ \to L$ is continuous and $\lim_{t \to +\infty} M(x, y, t) = 1_L$.

Let $(X, M, T)$ be an $L$-fuzzy metric space. For $t \in [0, +\infty[$, we define the open ball $B(x, r, t)$ with center $x \in X$ and a fixed radius $r \in L \setminus \{0_L, 1_L\}$, as

$$B(x, r, t) = \{y \in X : M(x, y, t) >_L N(r)\}.$$ 

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_L, 1_L\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_M$ denote the family of all open subsets of $X$. Then $\tau_M$ is called the topology induced by the $L$-fuzzy metric $M$.

**Example 1.1.** [21] Let $(X, d)$ be a metric space. Denote $T(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $L^*$ and let $M$ and $N$ be fuzzy sets on $X^2 \times [0, +\infty[$ be defined as follows:

$$M_{M, N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right).$$
Then \((X, \mathcal{M}_{M,N}, T)\) is an intuitionistic fuzzy metric space.

**Example 1.2.** [1] Let \((X, d)\) be a metric space. Denote \(T(a, b) = (a_1b_1, \min(a_2 + b_2, 1))\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\) in \(L^+\) and let \(M\) and \(N\) be fuzzy sets on \(X^2 \times (0, \infty)\) defined as follows:

\[
\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)}\right),
\]

for all \(t, h, m, n \in \mathbb{R}^+\). Then \((X, \mathcal{M}_{M,N}, T)\) is an intuitionistic fuzzy metric space.

**Lemma 1.2.** [10] Let \((X, \mathcal{M}, T)\) be an \(\mathcal{L}\)-fuzzy metric space. Then, \(\mathcal{M}(x, y, t)\) is nondecreasing with respect to \(t\), for all \(x, y\) in \(X\).

**Definition 1.5.** A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in an \(\mathcal{L}\)-fuzzy metric space \((X, \mathcal{M}, T)\) is called a Cauchy sequence, if for each \(\varepsilon \in \mathcal{L} \setminus \{0_\mathcal{L}\}\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(m \geq n \geq n_0\) (\(n \geq m \geq n_0\)),

\[
\mathcal{M}(x_m, x_n, t) >_\mathcal{L} N(\varepsilon).
\]

The sequence \(\{x_n\}_{n \in \mathbb{N}}\) is said to be convergent to \(x \in X\) in the \(\mathcal{L}\)-fuzzy metric space \((X, \mathcal{M}, T)\) (denoted by \(x_n \rightarrow_\mathcal{M} x\)) if \(\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \rightarrow 1_\mathcal{L}\) whenever \(n \rightarrow +\infty\) for every \(t > 0\). A \(\mathcal{L}\)-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Henceforth, we assume that \(T\) is a continuous \(t\)-norm on the lattice \(\mathcal{L}\) such that for every \(\mu \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}\), there is a \(\lambda \in \mathcal{L} \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}\) such that

\[
T^{n-1}(N(\lambda), ..., N(\lambda)) >_\mathcal{L} N(\mu).
\]

For more information see [19].

**Definition 1.6.** Let \((X, \mathcal{M}, T)\) be an \(\mathcal{L}\)-fuzzy metric space. \(\mathcal{M}\) is said to be continuous on \(X \times X \times [0, \infty]\) if

\[
\lim_{n \rightarrow \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)
\]

whenever a sequence \(\{(x_n, y_n, t_n)\}\) in \(X \times X \times [0, \infty]\) converges to a point \((x, y, t) \in X \times X \times [0, \infty]\) i.e., \(\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_\mathcal{L}\) and \(\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)\).

**Lemma 1.3.** Let \((X, \mathcal{M}, T)\) be an \(\mathcal{L}\)-fuzzy metric space. Then \(\mathcal{M}\) is a continuous function on \(X \times X \times [0, \infty]\).

**Proof.** The proof is the same as that for fuzzy spaces (see Proposition 1 of [15]).

**Definition 1.7.** Let \(A\) and \(S\) be mappings from an \(\mathcal{L}\)-fuzzy metric space \((X, \mathcal{M}, T)\) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, \(Ax = Sx\) implies that \(ASx = SAx\).

**Definition 1.8.** Let \(A\) and \(S\) be mappings from an \(\mathcal{L}\)-fuzzy metric space \((X, \mathcal{M}, T)\) into itself. Then the mappings are said to be compatible if

\[
\lim_{n \rightarrow \infty} \mathcal{M}(ASx_n, SAx_n, t) = 1_\mathcal{L}, \forall t > 0
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.
\]

**Proposition 1.1.** [22] If self-mappings \( A \) and \( S \) of an \( L \)-fuzzy metric space \((X, \mathcal{M}, T)\) are compatible, then they are weak compatible.

**Lemma 1.4.** [1, 19] Let \((X, \mathcal{M}, T)\) be an \( L \)-fuzzy metric space. Define \( E_{\lambda, \mathcal{M}} : X^2 \to \mathbb{R^+} \cup \{0\} \) by
\[
E_{\lambda, \mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(x, y, t) > L \mathcal{N}(\lambda)\}
\]
for each \( \lambda \in L \setminus \{0_L, 1_L\} \) and \( x, y \in X \). Then we have
(i) For any \( \mu \in L \setminus \{0_L, 1_L\} \) there exists \( \lambda \in L \setminus \{0_L, 1_L\} \) such that
\[
E_{\mu, \mathcal{M}}(x_1, x_n) \leq E_{\lambda, \mathcal{M}}(x_1, x_2) + E_{\lambda, \mathcal{M}}(x_2, x_3) + \cdots + E_{\lambda, \mathcal{M}}(x_{n-1}, x_n)
\]
for any \( x_1, \ldots, x_n \in X \); (ii) The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is convergent to \( x \) w.r.t. \( L \)-fuzzy metric \( \mathcal{M} \) if and only if \( E_{\lambda, \mathcal{M}}(x_n, x) \to 0 \).

Also the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is Cauchy w.r.t. \( L \)-fuzzy metric \( \mathcal{M} \) if and only if it is Cauchy with \( E_{\lambda, \mathcal{M}} \).

**Lemma 1.5.** Let \((X, \mathcal{M}, T)\) be an \( L \)-fuzzy metric space. If
\[
\mathcal{M}(x_n, x_{n+1}, t) \geq L \mathcal{M}(x_0, x_1, k^n t)
\]
for some \( k > 1 \) and \( n \in \mathbb{N} \). Then \( \{x_n\} \) is a Cauchy sequence.

**Proof.** For every \( \lambda \in L \setminus \{0_L, 1_L\} \) and \( x_n \in X \), we have
\[
E_{\lambda, \mathcal{M}}(x_{n+1}, x_n) = \inf\{t > 0 : \mathcal{M}(x_{n+1}, x_n, t) > L \mathcal{N}(\lambda)\} \leq \inf\{t > 0 : \mathcal{M}(x_0, x_1, k^n t) > L \mathcal{N}(\lambda)\} = \inf\{\frac{t}{k^n} : \mathcal{M}(x_0, x_1, t) > L \mathcal{N}(\lambda)\}
\]
\[
= \frac{1}{k^n} \inf\{t > 0 : \mathcal{M}(x_0, x_1, t) > L \mathcal{N}(\lambda)\} = \frac{1}{k^n} E_{\lambda, \mathcal{M}}(x_0, x_1).
\]
From Lemma 1.4, for every \( \mu \in L \setminus \{0_L, 1_L\} \) there exists \( \lambda \in L \setminus \{0_L, 1_L\} \), such that
\[
E_{\mu, \mathcal{M}}(x_n, x_m) \leq E_{\lambda, \mathcal{M}}(x_n, x_{n+1}) + E_{\lambda, \mathcal{M}}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda, \mathcal{M}}(x_{m-1}, x_m)
\]
\[
\leq \frac{1}{k^n} E_{\lambda, \mathcal{M}}(x_0, x_1) + \frac{1}{k^{n+1}} E_{\lambda, \mathcal{M}}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}} E_{\lambda, \mathcal{M}}(x_0, x_1)
\]
\[
= E_{\lambda, \mathcal{M}}(x_0, x_1) \sum_{j=n}^{m-1} \frac{1}{k^j} \to 0.
\]
Hence sequence \( \{x_n\} \) is a Cauchy sequence.

**Definition 1.9.** [9] We say that the \( L \)-fuzzy metric space \((X, \mathcal{M}, T)\) has property \((C)\), if it satisfies the following condition:
\[
\mathcal{M}(x, y, t) = C, \text{ for all } t > 0 \text{ implies } C = 1_L.
\]
2. Main Result

**Theorem 2.1.** Let $A, B, S$ and $T$ be self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, T)$, which has property (C), satisfying:

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $T(X), S(X)$ are two closed subsets of $X$;

(ii) the pairs $(A, S)$ and $(B, T)$ are weak compatible;

(iii) $\mathcal{M}(Ax, By, t) \geq L \mathcal{M}(Sx, Ty, kt)$, for every $x, y$ in $X$ and some $k > 1$.

Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$ be an arbitrary point in $X$. By (i), there exists $x_1, x_2 \in X$ such that $y_0 = Ax_0 = Tx_1$, $y_1 = Bx_1 = Sx_2$. Inductively, construct sequences $\{y_n\}$ and $\{x_n\}$ in $X$ such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \cdots$. Now, we prove that $\{y_n\}$ is a Cauchy sequence. Let $d_n(t) = \mathcal{M}(y_n, y_{n+1}, t)$, $t > 0$. Then, we have

$$d_{2n}(t) = \mathcal{M}(y_{2n}, y_{2n+1}, t) = \mathcal{M}(Ax_{2n}, Bx_{2n+1}, t) \geq \mathcal{M}(Sx_{2n}, Tx_{2n+1}, kt) = \mathcal{M}(y_{2n-1}, y_{2n}, kt) = d_{2n-1}(kt).$$

Thus $d_{2n}(t) \geq d_{2n-1}(kt)$ for every $m = 2n \in \mathbb{N}$ and $\forall t > 0$. Similarly for an odd integer $m = 2n+1$, we have $d_{2n+1}(t) \geq d_{2n}(kt)$. Hence, for every $n \in \mathbb{N}$, we have $d_n(t) \geq d_{n-1}(kt)$. That is,

$$\mathcal{M}(y_n, y_{n+1}, t) \geq \mathcal{M}(y_{n-1}, y_n, kt) \geq \mathcal{M}(y_0, y_1, k^n t).$$

So, by Lemma 1.5, $\{y_n\}$ is Cauchy and the completeness of $X$ implies $\{y_n\}$ converges to $y$ in $X$. That is, $\lim_{n \to \infty} y_n = y$

$$\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Sx_{2n+2} = y.$$

As $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$. By (iii), we have

$$\mathcal{M}(A(u, Bx_{2n+1}, t) \geq L \mathcal{M}(Su, Tx_{2n+1}, kt).$$

Since $\mathcal{M}$ is continuous, we get (whenever $n \to \infty$ in the above inequality):

$$\mathcal{M}(Au, y, t) \geq L \mathcal{M}(y, y, kt) = 1_\mathcal{L}.$$

Thus $\mathcal{M}(Au, y, t) = 1_\mathcal{L}$, i.e. $Au = y$. Therefore, $Au = Su = y$.

Since $A(X) \subseteq T(X)$, there exists $v \in X$, such that $Tv = y$. Thus,

$$\mathcal{M}(y, Bv, t) = \mathcal{M}(Au, Bv, t) \geq L \mathcal{M}(Su, Tv, kt) = 1_\mathcal{L}.$$

Hence $Tv = Bv = Au = Su = y$. Since $(A, S)$ is weak compatible, we conclude that $ASu = SAu$, that is $Ay = Sy$. Also, since $(B, T)$ is weak compatible then, $TBv = BTv$, that is $Ty = By$. We now prove that $Ay = y$. By (iii), we have

$$\mathcal{M}(Ay, y, t) = \mathcal{M}(Ay, Bv, t) \geq L \mathcal{M}(Sy, Tv, kt) = \mathcal{M}(Ay, y, kt) \geq L \mathcal{M}(Ay, y, k^n t).$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(Ay, y, t) \leq L \mathcal{M}(Ay, y, k^n t).$$
Hence, $\mathcal{M}(Ay, y, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, T)$ has property (C), it follows that $C = 1_\mathcal{L}$, i.e., $Ay = y$, therefore $Ay = Sy = y$.

Similarly we prove that $By = y$. By (iii), we have

$$\mathcal{M}(y, By, t) = \mathcal{M}(Ay, By, t) \geq_L \mathcal{M}(Ty, kt) = \mathcal{M}(y, By, kt)$$

$$\vdots$$

$$\geq_L \mathcal{M}(y, By, k^n t).$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(y, By, t) \leq_L \mathcal{M}(y, By, k^n t).$$

Hence, $\mathcal{M}(y, By, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, T)$ has property (C), it follows that $C = 1_\mathcal{L}$, i.e., $By = y$. Therefore, $Ay = By = Sy = Ty = y$, that is, $y$ is a common fixed of $A, B, S$ and $T$. For uniqueness, let $x$ be another common fixed point of $A, B, S$ and $T$ i.e., $x = Ax = Bx = Sx = Tx$. Hence

$$\mathcal{M}(y, x, t) = \mathcal{M}(Ay, Bx, t) \geq_L \mathcal{M}(Sy, Tx, kt) = \mathcal{M}(y, x, kt)$$

$$\vdots$$

$$\geq_L \mathcal{M}(y, x, k^n t)$$

On the other hand, from Lemma 1.2 we have that

$$\mathcal{M}(y, x, t) \leq_L \mathcal{M}(y, x, k^n t).$$

Hence, $\mathcal{M}(y, x, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, T)$ has property (C), it follows that $C = 1_\mathcal{L}$, i.e., $y = x$. Therefore, $y$ is the unique common fixed point of self-maps $A, B, S$ and $T$.

**Theorem 2.2.** Let $\{A_i\}$ and $\{B_j\}$ be two sequences of self-mappings of a complete $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, T)$, which has property (C), such that,

(i) there exists $i_0, j_0 \in \mathbb{N}$ such that $A_{i_0}^m(X) \subseteq T(X)$, $B_{j_0}^n(X) \subseteq S(X)$ and $T(X), S(X)$ are two closed subsets of $X$,

(ii) $A_iB_j = B_jA_i$, $TA_i = A_iT$, $SA_i = A_iS$, $TB_i = B_iT$ and $SB_i = B_iS$, for all $i, j \in \mathbb{N}$,

(iii) $\mathcal{M}(A_i^m x, B_j^n y, t) \geq_L \mathcal{M}(Sx, Ty, k_{ij} t)$, for every $x, y$ in $X$, for some $n, m \in \mathbb{N}$; here $k_{ij} > k > 1$ for $i, j = 1, 2, \cdots$, and $t > 0$.

Then, $A_i, B_j, S$ and $T$ have a unique common fixed point in $X$ for all $i, j = 1, 2, \cdots$.

**Proof.** By Theorem 2.1, $S, T$ and $A = A_{i_0}^m$ and $B = B_{j_0}^n$ for some $i_0, j_0 \in \mathbb{N}$, have a unique common fixed point in $X$. That is, there exists a unique $x \in X$ such that $S(x) = T(x) = A_{i_0}^m(x) = B_{j_0}^n(x) = x$.

Suppose there exists $j \in \mathbb{N}$ such that $j \neq j_0$. Then we have

$$\mathcal{M}(x, B_j^n x, t) = \mathcal{M}(A_i^m x, B_j^n x, t) \geq \mathcal{M}(Sx, Tx, k_{ij} t) = 1_\mathcal{L}.$$ 

Hence, for every $j \in \mathbb{N}$ we have $B_j^n x = x$. Similarly for every $i \in \mathbb{N}$, we get $A_i^m x = x$. Therefore for every $i, j \in \mathbb{N}$, we have

$A_i^m x = B_j^n x = Sx = Tx = x.$
To show uniqueness, assume that \( y \) is another fixed point of \( A_i^m, B_j^n, S \) and \( T \). Then we have
\[
\mathcal{M}(x, y, t) = \mathcal{M}(A_i^m x, B_i^n y, t) \geq_L \mathcal{M}(S x, T y, k_{ij} t) = \mathcal{M}(x, y, k_{ij} t) \geq_L \mathcal{M}(x, y, k_{ij}^2 t) \geq_L \mathcal{M}(x, y, k_{ij}^n t).
\]
On the other hand, by Lemma 1.2 we have
\[
\mathcal{M}(x, y, t) \leq \mathcal{M}(x, y, k_{ij}^n t).
\]
Hence, \( \mathcal{M}(x, y, t) = C \) for all \( t > 0 \). Since \((X, \mathcal{M}, T)\), has property \((C)\), it follows that \( C = 1_L \), i.e., \( x = y \). Now, we prove that \( A_i x = x \). Since
\[
A_i x = A_i(B_i^n x) = B_i^n(A_i x) \text{ and } A_i x = A_i(A_i^m x) = A_i^m(A_i x),
\]
also
\[
A_i x = A_i S x = S A_i x \text{ and } A_i x = A_i T x = T A_i x.
\]
That is \( A_i x \) is also a common fixed point of \( A_i^m, B_i^n, S \) and \( T \). Therefore \( A_i x = x \). Similarly, one can show that \( B_j x = x \), i.e., \( x \) is a unique common fixed point of the mappings \( A_i, B_j, S \) and \( T \), for \( i, j = 1, 2, \ldots \). This completes the proof.

**Corollary 2.1.** Let \((X, \mathcal{M}, T)\) be a complete \( \mathcal{L} \)-fuzzy metric space. Let \( \{A_n\} \) be a sequence of mappings \( A_i \) of a complete \( \mathcal{L} \)-fuzzy metric space \((X, \mathcal{M}, T)\) which has property \((C)\), into itself such that, for any two mappings \( A_i, A_j \),
\[
\mathcal{M}(A_i^m x, A_j^n y, \alpha_{ij} t) \geq_L \mathcal{M}(x, y, t)
\]
for some \( m \); here \( 0 < \alpha_{ij} < k < 1 \) for \( i, j = 1, 2, \ldots, x, y \in X \) and \( t > 0 \). Then the sequence \( \{A_n\} \) has a unique common fixed point in \( X \).

**Proof.** By Theorem 2.2, it is enough to set \( S = T = I \), where \( I \) is the identity map and \( B_j^n = A_j^m \). Also we replace \( \alpha_{ij} \) by \( \frac{1}{k_{ij}} \).

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