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Convex Bodies of Constant Width and the Apollonian Metric

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Abstract. The study of constant width sets goes at least as far back as the time of Euler. The Apollonian metric, on the other hand, is a relatively new concept. It was introduced by Beardon in 1998 as a generalization of the hyperbolic metric of a ball to arbitrary domains [3]. Close connections between these concepts were established in [20] and [21]. In this paper, we study the Apollonian metric of domains which are the complements of constant width sets. We verify Beardon's conjecture for such domains and show that in such domains the circular arcs which are orthogonal to the boundary and only they are the pseudogeodesic lines.

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1. Introduction

A compact set in $\mathbb{R}^n$ is of *constant width* if the distance between the parallel support hyperplanes in every direction is constant. The Reuleaux triangles are well-known examples of planar sets of constant width. On the other hand, Euclidean balls are trivially of constant width. In fact, sets of constant width can be defined by taking a certain property of a ball as a definition. For example, the width of a ball in every direction is constant. Some non-trivial curves of constant width were already known to Euler [4]. Sets of constant width are well-studied objects in classical Euclidean geometry and functional analysis. Let $\mathbf{W}^n$ be the class of convex bodies of constant width in $\mathbb{R}^n$. Fillmore [12] has shown that there are sets in $\mathbf{W}^n$ with analytic boundaries but no nontrivial symmetry group as well as sets with analytic boundaries and the same symmetry group as the regular n-simplex. On the other hand, there exist extremely singular members of $\mathbf{W}^n$, i.e., convex bodies whose sets of singular points are dense in their boundaries [11, 23]. Moreover, $\mathbf{W}^n$ is closed in the Hausdorff metric, and any member of $\mathbf{W}^n$ can be approximated arbitrarily closely by sets in $\mathbf{W}^n$ both with analytic boundaries and with singular boundaries (see [10, 9, 13, 24] for properties of sets of constant width.)

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The concept of Apollonian metric, not the name, goes back to Barbilian, who generalized the hyperbolic metric of a disk to plane Jordan domains by treating it as a distance-ratio metric [1]. The main objectives of Barbilian's investigations were to see if there are plane Jordan domains other than disks and half-planes on which the Poincaré model of the hyperbolic geometry can be constructed. One of the conclusions of his work states that such a metric space is totally geodesic if and only if the underlying domain is a disk. The complete proof of this fact has recently been given by Gehring and Hag [14]. Although the resulting geometry, called Barbilian geometry, has been an object of many studies over the years (see [5, 6, 7, 8]), the original idea of Barbilian has been forgotten for nearly sixty years, except for one paper by Kelly [22]. Beardon revived Barbilian's original idea by introducing the Apollonian metric for arbitrary domains in $\overline{\mathbb{R}^n}$ [3].

The notion of conformality establishes a connection between the Apollonian metric and sets of constant width. The Apollonian metric of a domain $D \subset \overline{\mathbb{R}^n}$ is conformal at every point (i.e., the infinitesimal balls in the Apollonian metric about every point are similar to Euclidean balls) if and only if D is a ball. This property of balls can be explained by the fact that they are the only convex bodies of constant width which are preserved under all Möbius transformations of $\overline{\mathbb{R}^n}$. The latter is, in fact, a consequence of the following result on the conformality of the Apollonian metric: the Apollonian metric a_D of a domain $D \subset \overline{\mathbb{R}^n}$ is conformal either at every point, or at only one point, or at no point of D [21, Theorem 2]. More precisely, the first case happens if and only if D is a ball. For the second case we have the following result [21, Theorem 3]: The Apollonian metric of a domain D is conformal at one point if and only if D is, up to Möbius transformations, the complement of a convex body of constant width. Hence the convex bodies of constant width arise naturally in the context of the Apollonian metric. The Apollonian metric is a natural extension of the hyperbolic metric of a ball in the same way as the convex bodies of constant width are natural extensions of balls (see [20] and [21] for more on the relations between the Apollonian metric and convex bodies of constant width).

Let W be a convex body of constant width and $D = \overline{\mathbb{R}^n} \setminus W$ be its complementary domain. Although the Apollonian metric space (D, a_D) is not totally geodesic, there are infinitely many geodesic lines in D, and in particular each point of D lies in a geodesic segment [18, Theorem 2]. But instead of geodesic lines, we consider a weaker notion of pseudogeodesic lines introduced in [18]. In particular, each geodesic line is a pseudogeodesic line. Note that if W is a ball, then the circular arcs in D which are orthogonal to ∂D are geodesic lines and hence pseudogeodesic lines. Furthermore, every pair of points in D lies on such a line. The latter is a property that only balls possess. In general, not every pair of points in D lies on a pseudogeodesic line. Nevertheless, there are plenty of pseudogeodesic lines in D so that every point in D belongs to at least one such line [18].

Theorem 1.1. *Suppose that W is a convex body of constant width and suppose that it is regular. Let $a, b \in \partial D$ and γ be a circular arc joining a and b in D. Then γ is a pseudogeodesic line if and only if it is orthogonal to ∂D. Moreover, any point in D can be joined to $\infty \in D$ by a unique pseudogeodesic line.*

Pseudogeodesic lines are invariant under Apollonian isometries. Beardon asked if each Apollonian isometry is the restriction of a Möbius transformation. Partial results confirming Beardon's conjecture have been given in [3], [15] and [21]. The most general result to date, which include these partial results, is given in [18] and [19]. Here we verify Beardon's conjecture for domains which are the complements of convex bodies of constant width.

Theorem 1.2. *Let W be a convex body of constant width and let $D = \overline{\mathbb{R}^n} \setminus W$. Then every Apollonian isometry $f\colon D \to \overline{\mathbb{R}^n}$ is the restriction of a Möbius map.*

Theorem 1.2 can be reformulated as a uniqueness result for convex bodies of constant width. Namely, two convex bodies of constant width are similar if and only if their exterior domains are isometric in the Apollonian metric.

Section 2 contains preliminaries on the convex bodies of constant width and on the Apollonian metric. Section 3 contains auxiliary results on the convex bodies of constant width as well as on the Apollonian metric of their complements. These results are needed in the the proofs of Theorem 1.1 and Theorem 1.2 at the end of Section 3.

2. Preliminaries

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The open and closed balls of radius $r > 0$ and centered at $x \in \mathbb{R}^n$ are denoted by $B^n(x,r)$ and $\overline{B}^n(x,r)$, respectively. $S^{n-1}(x,r)$ is a sphere of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. The closed segment between $x, y \in \mathbb{R}^n$ is denoted by $[x,y]$. Given a vector u and a point $x \in \mathbb{R}^n$, the open ray emanating from x in the direction of u is denoted by $R(x,u)$ and the hyperplane passing through x and orthogonal to u is denoted by $P(x,u)$. For distinct points $a, b \in \mathbb{R}^n$, the open ray emanating from a in the direction of a vector $a-b$ is denoted by $R(a,b)$. The diameter of a set A is denoted by $\operatorname{diam}(A)$. The one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ is denoted by $\overline{\mathbb{R}^n}$. For a set A in $\mathbb{R}^n$ or $\overline{\mathbb{R}^n}$ the topological operations $\overline{A}$ (closure) and ∂A (boundary) are always taken with respect to $\overline{\mathbb{R}^n}$. The *cross-ratio* of a quadruple a, b, c, d of points in $\overline{\mathbb{R}^n}$ with $a \neq b$ and $c \neq d$ is defined by

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|},$$

where we use the convention that $|x-\infty|/|y-\infty| = 1$ for all $x, y \in \mathbb{R}^n$. A homeomorphism $f\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius transformation if and only if

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d| \quad \text{for all} \quad a,b,c,d \in \overline{\mathbb{R}^n}.$$

(See, for instance, [2, Theorem 3.2.7].)

By a sphere (ball) in $\overline{\mathbb{R}^n}$ we mean the image of $S^{n-1}(0,1)$ ($B^n(0,1)$) under a Möbius transformation of $\overline{\mathbb{R}^n}$. The points $x, y \in \overline{\mathbb{R}^n}$ are called *inversive* with respect to a sphere S in $\overline{\mathbb{R}^n}$ if $y = h(x)$ where h is the inversion in S.

Most of the notations and concepts related to convex bodies are taken from [10], [9], [13] and [24]. A *convex body* in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with interior points. Let K be a convex body. A hyperplane H is said to be a support hyperplane of K at a if $a \in K \cap H$ and K is contained in only one of the two closed halfspaces bounded by H. A boundary point $a \in \partial K$ of a convex body K is called *regular* if K has only one supporting hyperplane containing a; otherwise it is called *singular*. We say that K is *regular* if each point of ∂K is regular. A closed segment $[a, b]$ is called a *chord* of K if $a \neq b$ and $a, b \in \partial K$. A chord $[a, b]$ is said to be *diametrical* if there exist two parallel supporting hyperplanes, say H_1 and H_2, of K such that $a \in H_1$ and $b \in H_2$. A chord $[a, b]$ is called *normal* if it is orthogonal to a supporting hyperplane of K at one of its endpoints; it is called *double normal* if it is orthogonal at both endpoints. A convex body K is said to be of *constant width* if the distance between the parallel support hyperplanes of K in every direction is constant.

For each triple $(x, y; k)$, where $x, y \in \mathbb{R}^n$ are distinct points and $k \in (0, +\infty)$, the *Apollonian ball* $B_a(x, y; k)$ about x with respect to y and of Apollonian radius k is defined by

$$B_a(x, y; k) = \left\{ z \in \overline{\mathbb{R}^n} \colon \ \frac{|x - z|}{|y - z|} < k \right\}.$$

The boundary of $B_a(x, y; k)$ is called the *Apollonian sphere* and is denoted by $S_a(x, y; k)$. Observe that

$$S_a(x, y; k) = \begin{cases} S^{n-1}\Big(\frac{x - k^2 y}{1 - k^2}, \frac{k|x - y|}{|1 - k^2|}\Big) & \text{if } k \neq 1, \\ H \cup \{\infty\} & \text{if } k = 1, \end{cases}$$

where H is the hyperplane passing through the midpoint of $[x, y]$ and orthogonal to it. The Apollonian balls about (with respect to) ∞ are defined as

$$B_a(x, \infty; k) = B^n(x, k) \quad \text{and} \quad B_a(\infty, x; k) = \overline{\mathbb{R}^n} \setminus \overline{B^n}(x, k).$$

Hence the Apollonian spheres are just spheres in $\overline{\mathbb{R}^n}$ and the points x and y are inversive with respect to $S_a(x, y; k)$.

Let $D \subset \overline{\mathbb{R}^n}$ be a domain with at least two boundary points. For each pair of distinct points $x, y \in D$ we let k_{xy} be the Apollonian radius of the largest Apollonian ball about x with respect to y which is contained in D. The Apollonian ball $B_a(x, y; k_{xy})$ is called the *maximal Apollonian ball* in D about x with respect to y. The *Apollonian distance* $a_D(x, y)$ between the points x and y is defined as

$$a_D(x, y) = \max_{a, b \in \partial D} \log\Big(\frac{|x - a||y - b|}{|x - b||y - a|}\Big) = \log\Big(\frac{1}{k_{xy}} \cdot \frac{1}{k_{yx}}\Big).$$

In particular, $a_{f(D)}(f(x), f(y)) = a_D(x, y)$ for all Möbius transformations of $\overline{\mathbb{R}^n}$. For each pair of distinct points $x, y \in D$, we define

$$\mathrm{E}(\partial D; x, y) = \partial D \cap S_a(x, y; k_{xy}).$$

When there is no confusion over the domain D, we denote $\mathrm{E}(\partial D; x, y)$ by $\mathrm{E}(x, y)$. An arc γ is said to join the points $x \in \overline{D}$ and $y \in \overline{D}$ in D if $\gamma \subset D$ and x and y are the endpoints of γ. An arc $\gamma \subset D$ is called an *Apollonian geodesic* or simply a *geodesic* if

$$a_D(x, y) = a_D(x, z) + a_D(z, y),$$

for each ordered triple of points $x, z, y \in \gamma$. We say that γ is a geodesic segment (ray or line) if two (one or none) of its endpoints lie in ∂D. We end this section with a definition of a pseudogeodesic line [18, Definition 4.5].

Definition 2.1. *Let $\gamma \subset D$ be a curve joining two boundary points a and b. Suppose that $z \in \gamma$ and that there exists a non-degenerate geodesic subcurve S of γ containing z in its interior, with the additional property that for every $x \in \gamma$ there exists a closed non-degenerate subcurve S' of S with one endpoint at z such that $a_D(x,z) = a_D(x,y) + a_D(y,z)$ for all $y \in S'$. Then γ is called a pseudogeodesic line through z (or simply a pseudogeodesic line).*

3. Main Results

In this section, we study some properties of the Apollonian metric of domains that are complements of convex bodies of constant width. Lemma 3.2 and Lemma 3.3 provide new properties of convex bodies of constant width. We will need the following two properties of convex bodies of constant width (see [13, Lemma 7.1.13] and [9, (VI), p. 54]).

Proposition 3.1. *Let $W \in \boldsymbol{W}^n$. Then every normal of W is a diameter, and hence a double normal. Any two diameters of W lying in the same 2-dimensional plane must intersect. In addition, W is strictly convex.*

Proposition 3.2. *A convex body W is of constant width if and only if every chord $[a,b]$ of W that is a normal of W at a is also a normal of W at b.*

We begin with some properties of convex bodies of constant width.

Lemma 3.1. *Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and let $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{T}$ be distinct points given in this order with $|\xi_1 - \xi_3| = |\xi_2 - \xi_4|$. Then, the 4-gon with vertices at $\xi_1, \xi_2, \xi_3, \xi_4$ is a trapezoid.*

Proof. Let z be the intersection point of the segments $[\xi_1, \xi_3]$ and $[\xi_2, \xi_4]$. If $z = 0$, then the 4-gon in question is a rectangle. Hence we can assume that $z \neq 0$. Since $|\xi_1 - \xi_3| = |\xi_2 - \xi_4|$, we have

$$\operatorname{dist}(0, [\xi_1, \xi_3]) = \operatorname{dist}(0, [\xi_2, \xi_4]) = r,$$

for some $r \in (0,1)$. Without loss of generality, we can assume that $|\xi_1 - z| < |\xi_3 - z|$ and $|\xi_2 - z| < |\xi_4 - z|$. Then it is enough to show that $|\xi_3 - z| = |\xi_4 - z|$. Let $\eta_{13} \in [\xi_1, \xi_3]$ and $\eta_{24} \in [\xi_2, \xi_4]$ be the unique points with $|\eta_{13}| = |\eta_{24}| = r$. Then $|\eta_{13} - \xi_3| = |\eta_{24} - \xi_4|$ from the right triangles with vertices at $\xi_3, \eta_{13}, 0$ and $\xi_4, \eta_{24}, 0$. Similarly, $|\eta_{13} - z| = |\eta_{24} - z|$ from the right triangles with vertices at $z, \eta_{13}, 0$ and $z, \eta_{24}, 0$. Hence $|\xi_3 - z| = |\xi_4 - z|$, as required. ∎

Recall that if a, c, b, d are four points in $\overline{\mathbb{R}^n}$, then $|a,b,c,d| + |b,a,c,d| = 1$ if and only if they lie on a circle in $\overline{\mathbb{R}^n}$ in this order. (For a proof see, for instance, [4, Proposition 10.9.2]). As an immediate consequence of this along with Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Let W be a convex body of constant width and let $[a,b]$ and $[c,d]$ be diametrical chords of W. Then $|a,b,c,d| + |b,a,c,d| = 1$ if and only if the 4-gon $Q(a,c,b,d)$ with vertices at a, c, b and d in this order is either a segment or a triangle or a trapezoid.*

Proof. Let $|a,b,c,d| + |b,a,c,d| = 1$. If $a = c$ and $b = d$ or $a = d$ and $b = c$, then $Q(a,c,b,d)$ is a segment. If $a = c$ and $b \neq d$ or $a = d$ and $b \neq c$, then $Q(a,c,b,d)$ is a triangle. If the points a,b,c,d are distinct, then $Q(a,c,b,d)$ is a trapezoid by Lemma 3.1.

Conversely, if $Q(a,c,b,d)$ is either a segment, or a triangle or a trapezoid, then the points a,c,b,d lie on a circle in this order and hence $|a,b,c,d| + |b,a,c,d| = 1$. ∎

The trapezoids spanned on the diametrical chords of a convex body of constant width, as in Lemma 3.2, have the following additional property, which will be crucial in the proof of Theorem 1.1.

Lemma 3.3. *Let W be a convex body of constant width. Suppose that $[a,b]$ and $[c,d]$ are diametrical chords of W and let $Q(a,c,b,d)$ be a trapezoid with vertices at a, c, b and d in this order. Let C be the circle passing though a and c and tangent to $[a,b]$ and $[c,d]$ at these points. Let $R(a,d) \cap C = \{a'\}$ and $R(c,b) \cap C = \{c'\}$. Then the points a,a',c',c lie on C in this order.*

Proof. If $|a-c| \geq |b-d|$ then the statement is obvious. Suppose that $|a-c| < |b-d|$. Then $R(a,d) \cap R(c,b) \neq \emptyset$. Let e be the intersection point of these rays and let s be the center of C. It is enough to show that $|e-s| \geq |a-s|$. Let $\alpha = \widehat{sce}$ be the angle at the vertex c of a triangle $\Delta(sce)$. Similarly, let $\beta = \widehat{sec}$. Then $\widehat{dcb} = \pi/2 - \alpha$ and $\widehat{dbc} = \pi/2 - \beta$. Since $[c,d]$ is a diametrical chord, we have $|d-c| \geq |d-b|$. Applying the law of sine twice, we obtain $\alpha \geq \beta$ and hence $|e-s| \geq |a-s|$, as required. ∎

Next, we will discuss properties of the Apollonian metric of domains which are the complements of convex bodies of constant width. Throughout the rest of this section we fix a convex body of constant width $W \in \mathbf{W}^n$ and put $D = \overline{\mathbb{R}^n} \setminus W$. Observe that $W \subset \overline{B}^n(a, \operatorname{diam}(W))$ for each $a \in \partial W$. In particular, if $[a,b]$ is a diametrical chord of W, there exists a unique closed ball which is the smallest of all the balls of the form $\overline{B}^n(s, |s-b|)$, $s \in [a,b]$, containing W (see, for instance, [21, Lemma 4.13]). We denote such a ball by $B(a,b;W)$. Next corollary is an easy consequence of Proposition 3.1.

Corollary 3.1. *For each $x \in D \setminus \{\infty\}$, there exist unique points $a,b \in \partial D$ such that*

$$|x-a| = \operatorname{dist}(x, \partial D) \qquad \text{and} \qquad |x-b| = \max\{|x-\zeta| : \ \zeta \in \partial D\}.$$

Moreover, $[a,b]$ is a diametrical chord and $x \in R(a,b)$.

The next lemma is an extension of Corollary 3.1.

Lemma 3.4. *Let $x,y \in D \setminus \{\infty\}$ be distinct points and let $a,b \in \partial D$ be points with $|x-a| = \operatorname{dist}(x,\partial D)$ and $|y-b| = \operatorname{dist}(y,\partial D)$. Then*

$$a_D(x,y) = a_D(x,\infty) + a_D(\infty,y)$$

if and only if $[a,b]$ is a diametrical chord and $x \in R(a,b)$ and $y \in R(b,a)$.

Proof. Let $c,d \in \partial D$ be distinct points such that

$$|x-c| = \max\{|x-\zeta| : \ \zeta \in \partial D\} \quad \text{and} \quad |y-d| = \max\{|y-\zeta| : \ \zeta \in \partial D\}.$$

Then by Corollary 3.1, we have

$$\mathrm{E}(x,\infty) = \{a\}, \quad \mathrm{E}(\infty,x) = \{c\}, \quad \mathrm{E}(y,\infty) = \{b\} \quad \text{and} \quad \mathrm{E}(\infty,y) = \{d\}.$$

Moreover, we have that both $[a,c]$ and $[b,d]$ are diametrical chords of W and that $x \in R(a,c)$ and $y \in R(b,d)$.

Sufficiency. Since $[a,b]$ is a diametrical chord, we have $c = b$ and $d = a$. Let B be the maximal Apollonian ball in D about x with respect to y. Since the points x and y are inversive with respect to ∂B, the center of ∂B lies on $R(x,y) \cup \{\infty\} \cup R(y,x)$. Hence $\partial B \cap \partial W = \{a\}$, i.e., $\mathrm{E}(x,y) = \{a\}$. Similarly, we show that $\mathrm{E}(y,x) = \{b\}$. Then the proof is completed by using Lemma 4.1 [18].

Necessity. By Lemma 4.1 [18], we have

$$\mathrm{E}(x,y) \subset \Big(\mathrm{E}(x,\infty) \cap \mathrm{E}(\infty,y)\Big) \quad \text{and} \quad \mathrm{E}(y,x) \subset \Big(\mathrm{E}(y,\infty) \cap \mathrm{E}(\infty,x)\Big).$$

Since $\mathrm{E}(x,y)$ and $\mathrm{E}(y,x)$ are nonempty subsets of ∂D, so are $\mathrm{E}(x,\infty) \cap \mathrm{E}(\infty,y)$ and $\mathrm{E}(y,\infty) \cap \mathrm{E}(\infty,x)$. Hence we obtain $d = a$ and $c = b$, as required. ∎

We have the following corollary to Lemma 3.4.

Corollary 3.2. *For each $x \in D \setminus \{\infty\}$, there exists a unique point $y \in D \setminus \{\infty\}$ such that*

$$a_D(x,\infty) = a_D(y,\infty) \quad \textit{and} \quad a_D(x,y) = a_D(x,\infty) + a_D(\infty,y).$$

Lemma 3.5. *Suppose that W is regular. Let $x,y,z \in D$ be distinct points and let C be the unique circle in $\overline{\mathbb{R}^n}$ containing these points. If*

$$a_D(x,y) = a_D(x,z) + a_D(z,y)$$

then C intersects W orthogonally. Moreover,

$$C \cap \partial W = \mathrm{E}(x,y) \cup \mathrm{E}(y,x).$$

Proof. By Lemma 4.4 [18], there exists an Apollonian geodesic $\gamma \subset C$ containing z in its (relative) interior. Furthermore, for all $x_1, y_1 \in \gamma$ such that the points x, x_1, y_1, y lie on C in this order, we have

$$\mathrm{E}(x,y) \subset \mathrm{E}(x_1,y_1) \quad \text{and} \quad \mathrm{E}(y,x) \subset \mathrm{E}(y_1,x_1).$$

So, in proving the lemma, we can assume that the points x, z, y lie in γ in this order. Let h be a preliminary Möbius transformation of $\overline{\mathbb{R}^n}$ such that $h(z) = \infty$. Fix $a \in \mathrm{E}(h(x),h(y))$. Then $a \in \mathrm{E}(h(x),\infty)$ by Lemma 4.1 [18]. Since

$$a_{h(D)}(h(x),x') + a_{h(D)}(x',\infty) = a_{h(D)}(h(x),\infty)$$

for all $x' \in h(\gamma)$ lying between $h(x)$ and ∞, we have $a \in \mathrm{E}(x',\infty)$. Hence for each such x', we have

$$a \in \partial B, \qquad \text{where} \qquad B = B_a(x',\infty;k).$$

By considering the balls B and using Corollary 3.1 as well as the regularity of W, we conclude that $a \in h(C)$. Since a is arbitrary, we obtain that $\mathrm{E}(h(x),h(y)) = \{a\}$. By letting x' tend to ∞ along $h(C)$, we conclude that the hyperplane containing a and orthogonal to $h(C)$ is a support hyperplane of $h(W)$ at a. Hence $h(C)$ intersects $h(W)$ at a orthogonally. Similarly, we show that $\mathrm{E}(h(y),h(x)) = \{b\}$ and $h(C)$ intersects $h(W)$ at b orthogonally. In particular,

$$h(C) \cap h(W) = \{a,b\} = \mathrm{E}(h(x),h(y)) \cup \mathrm{E}(h(y),h(x)).$$

Finally, Corollary 2.3.15 in [20] implies that

$$\mathrm{E}(x,y)=\{h^{-1}(a)\} \qquad \text{and} \qquad \mathrm{E}(y,x)=\{h^{-1}(b)\}.$$

Thus, C intersects W orthogonally and

$$C\cap\partial W=\mathrm{E}(x,y)\cup\mathrm{E}(y,x),$$

thus completing the proof. ∎

A homeomorphism $f\colon\, D\to\overline{\mathbb{R}^n}$ is called an *Apollonian isometry* if

$$a_{f(D)}(f(x),f(y))=a_D(x,y) \qquad \text{for all} \qquad x,y\in D.$$

Clearly, every Möbius mapping is an Apollonian isometry. The following theorem is the same as Theorem 3.4.1 in [20], but here we give an alternative proof of it using Lemma 3.4, which is significantly shorter than the original proof in [20].

Theorem 3.1. *Let W be a convex body of constant width, $D=\overline{\mathbb{R}^n}\setminus W$, $f\colon\, D\to\overline{\mathbb{R}^n}$ be an Apollonian isometry with $f(\infty)=\infty$ and $D'=f(D)$. Then $W'=\overline{\mathbb{R}^n}\setminus D'$ is a convex body of constant width.*

Proof. Let $[a,b]$ be any diametrical chord of W. First, we will show that there exist $a',b'\in\partial W'$ such that

$$f(R(a,b))=R(a',b') \qquad \text{and} \qquad f(R(b,a))=R(b',a')$$

and the hyperplanes $P(a',u)$ and $P(b',u)$ are support hyperplanes of W', where $u=a'-b'$.

Let $x\in R(a,b)$ and $y\in R(b,a)$ be arbitrary points. By Lemma 3.4 we have

$$a_D(x,y)=a_D(x,\infty)+a_D(\infty,y)$$

and since f is an isometry, we have

$$a_{D'}(x',y')=a_{D'}(x',\infty)+a_{D'}(\infty,y'),$$

where $x'=f(x)$ and $y'=f(y)$. By Lemma 4.4 [18] there is a closed non-degenerate geodesic subarc γ' of $R(x',y')\cup R(y',x')$ containing ∞ in its (relative) interior. Let x_1' and y_1' be the endpoints of γ' so that the points x',x_1',∞ lie on γ' in this order. Put $x_1=f^{-1}(x_1')$ and $y_1=f^{-1}(y_1')$ and let $a_1,b_1\in\partial W$ be such that $|x_1-a_1|=\operatorname{dist}(x_1,\partial W)$ and $|y_1-b_1|=\operatorname{dist}(y_1,\partial W)$. Since

$$a_D(x_1,y_1)=a_D(x_1,\infty)+a_D(\infty,y_1),$$

Lemma 3.4 implies that $[a_1,b_1]$ is a diametrical chord of W and that $x_1\in R(a_1,b_1)$ and $y_1\in R(b_1,a_1)$. Since $\gamma=f^{-1}(\gamma')$ is a geodesic, for any $z\in\gamma$ lying between y_1 and ∞ we have

$$a_D(x_1,z)=a_D(x_1,\infty)+a_D(\infty,z).$$

Then by applying Lemma 3.4 to the pair (x_1,z) we conclude that $z\in R(b_1,a_1)$ and hence $f^{-1}(R(y_1',x_1'))\subset R(y_1,x_1)$. Similarly, we obtain $f^{-1}(R(x_1',y_1'))\subset R(x_1,y_1)$. Thus,

$$\gamma=R(x_1,y_1)\cup\{\infty\}\cup R(y_1,x_1).$$

Let now x_2' be an arbitrary point on $R(x',y')$ lying between x' and x_1'. By Lemma 4.4 [18] there exists a closed non-degenerate subarc γ_2' of γ' having ∞ as one of its endpoints and such that

$$a_{D'}(x_2',\infty) = a_{D'}(x_2',z') + a_{D'}(z',\infty) \quad \text{for all} \quad z' \in \gamma_2'.$$

Hence

$$a_D(x_2,\infty) = a_D(x_2,z) + a_D(z,\infty) \quad \text{for all} \quad z \in \gamma_2 = f^{-1}(\gamma_2'),$$

where $x_2 = f^{-1}(x_2')$. Lemma 4.1 [18] implies that

$$\mathrm{E}(x_2,\infty) \subset \mathrm{E}(z,\infty) = \{a_1\} \quad \text{and} \quad \mathrm{E}(\infty,x_2) \subset \mathrm{E}(\infty,z) = \{b_1\}.$$

Then Corollary 3.1 implies that $x_2 \in R(a_1,b_1)$. Similarly, $f^{-1}(y_2') \in R(b_1,a_1)$ for all $y_2' \in R(y',x')$ lying between y' and y_1'. Since x_2' and y_2' are arbitrary points, we conclude that

$$a_1 = a,\ b_1 = b,\ f^{-1}(R(x',y')) = R(x,y) \quad \text{and} \quad f^{-1}(R(y',x')) = R(y,x).$$

Since x and y can be taken arbitrarily close to a and b, respectively, we conclude that

$$f(R(a,b)) = R(a',b') \qquad \text{and} \qquad f(R(b,a)) = R(b',a')$$

for some $a',b' \in \partial W'$.

Next, we will show that $P^+(a',u) \cap W' = \emptyset$, where $P^+(a',u)$ is the open half-space determined by $P(x,u)$ and having u as an inner normal vector. It is enough to show that for each $x' \in R(a',b')$, the point a' is a unique point with $|x'-a'| = \mathrm{dist}(x',\partial W')$. Let $d' \in \partial W'$ be any point with $|x'-d'| = \mathrm{dist}(x',\partial W')$. Assume first that $|x'-d'| < |x'-a'|$. Let $x_0' \in R(a',b')$ be the unique point with $|x_0'-a'| = (|x'-a'| - |x'-d'|)/2$. Then the point $x_0 = f^{-1}(x_0')$ belongs to the ray $R(a,b)$ and hence by Lemma 4.4 [18] there exists $z \in R(a,b)$ such that for $x = f^{-1}(x')$ we have

$$a_D(x_0,\infty) = a_D(x_0,z) + a_D(z,\infty)$$

and

$$a_D(x,\infty) = a_D(x,z) + a_D(z,\infty).$$

Therefore for $z' = f(z)$ we have

$$a_{D'}(x_0',\infty) = a_{D'}(x_0',z') + a_{D'}(z',\infty)$$

and

$$a_{D'}(x',\infty) = a_{D'}(x',z') + a_{D'}(z',\infty).$$

Let $c' \in \partial W'$ be a point with $|x_0'-c'| = \mathrm{dist}(x_0',\partial W')$. Then by our choice of x_0', we have $c' \neq d'$. Since $c' \in \mathrm{E}(x_0',\infty)$ and $d' \in \mathrm{E}(x',\infty)$, by Lemma 4.1 [18] we have $d',c' \in \mathrm{E}(z',\infty)$, i.e., $|z'-d'| = |z'-c'|$. But since $z' \in R(a',b')$ and $|x'-d'| < |x'-a'|$, we obtain that $|z'-d'| < |z'-c'|$, a contradiction. Hence we conclude that $|x'-d'| = |x'-a'|$. Then there exists $z' \in R(a',b)$ such that

$$a_{D'}(x',\infty) = a_{D'}(x',z') + a_{D'}(z',\infty).$$

Again, by Lemma 4.1 [18], we have $a',d' \in \mathrm{E}(x',\infty) \subset \mathrm{E}(z',\infty)$, i.e., $|z'-d'| = |z'-a'|$. Since $z' \in R(a',b')$, we conclude that $d' = a'$. Similarly, we obtain that $P^-(b',u) \cap W' = \emptyset$, where $P^-(a',u)$ is the open half-space determined by $P(x,u)$ and having u as an outer normal vector. Thus, the hyperplanes $P(a',u)$ and $P(b',u)$ are support hyperplanes of W' at a' and b', respectively. Moreover, for each $x' \in R(a', b')$ ($y' \in R(b', a')$) the point a' (b') is the unique point in W' with

$$|x' - a'| = \operatorname{dist}(x', \partial W') \qquad (|y' - b'| = \operatorname{dist}(y', \partial W')).$$

In particular, for each $x' \in D'$ there exists a unique point $a' \in W'$ such that $|x' - a'| = \operatorname{dist}(x', \partial W')$. Then Theorem 11.1.7.3 [4] implies that W' is a convex set. Since α_D (and hence $a_{D'}$) is a metric, we conclude by Theorem 1.1[3] that W' does not lie in any hyperplane in $\mathbb{R}^n$ and hence is a convex body. Next we will show that W' is of constant width. Let $[a', b']$ be a chord of W' that is normal at a'. Let $x' \in R(a', b')$ be any point and $x = f^{-1}(x')$. By Corollary 3.1 there exists a unique diametrical chord $[a, b]$ of W with $x \in R(a, b)$. Then

$$f(R(b, a)) = R(b', a') \qquad \text{and} \qquad f(R(a, b)) = R(a', b')$$

as we have shown above. Since $P(b', u)$ with $u = a' - b'$ is a support hyperplane of W' at b', we obtain that the chord $[a', b']$ is normal of W' at b' and by Property 3.2 we conclude that W' is a convex body of constant width. ∎

We end this section with the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Suppose that γ is a pseudogeodesic line. Then there exist distinct points $x, z, y \in \gamma$ such that

$$a_D(x, y) = a_D(x, z) + a_D(z, y)$$

and hence γ intersects ∂D orthogonally by Lemma 3.5.

Suppose now that γ intersects ∂D orthogonally at a and b. Let $[a, c]$ and $[b, d]$ be diametrical chords of W. Since W is regular, we have $c \neq d$. Assume first that $c \neq b$. Then $d \neq a$ and hence the points a, b, c, d are distinct. Due to regularity of W the diametrical chords $[a, c]$ and $[b, d]$ are tangent to γ. In particular, they lie in the 2-dimensional plane containing γ. By Proposition 3.1, the chords $[a, c]$ and $[b, d]$ intersect each other, say at p. Then $|a-p| = |b-p|$ and since $|a-c| = |b-d|$, we obtain that $[a, b]$ is parallel to $[c, d]$ and hence the 4-gon $Q(a, b, c, d)$ is a trapezoid. Let a' and b' be the intersection points of γ with the rays $R(a, d)$ and $R(b, c)$, respectively. By Lemma 3.3 the points a, a', b', b lie on γ in this order. Let $z \in D \cap \gamma$ be such that $|z - a| = |z - b|$. We will show that γ is a pseudogeodesic line through z. Write $B_1 = B(c, a; W)$ and $B_2 = B(d, b; W)$. Let z_1 (z_2) be inversive to z with respect to ∂B_1 (∂B_2). Let C be the circle containing γ. Since C is orthogonal to B_1 and B_2 and since the points a, a', b', b lie on γ in this order, $\{z_1, z_2\} = C \cap W$ and such that the points a, z_2, z_1, b lie on C in this order. Let h be an inversion in the sphere $S^{n-1}(z, |z - a|)$. Then $h(C)$ is a straight line, the points $h(a) = a$, $h(z_2)$, $h(z_1)$, $h(b) = b$ lie on $h(C)$ in this order,

$$h(B_1) = \overline{B}^n(h(z_1), |h(z_1) - a|) \quad \text{and} \quad h(B_2) = \overline{B}^n(h(z_2), |h(z_2) - b|).$$

Moreover,

$$h(W) \subset h(B_1) \cap h(B_2).$$

We need to show that $h(\gamma)$ is a pseudogeodesic line through ∞. Let

$$S = h(\gamma) \setminus \big(h(B_1) \cup h(B_2)\big).$$

Clearly, S contains ∞ in its interior and is non-degenerate. By Corollary 2.4.7 [20], S is an Apollonian geodesic. Now let $x \in h(\gamma)$ be arbitrary. If $x \in S$ then we let S' be the subarc of S from x to ∞. Otherwise, we let S' be the subarc of S from $x^\star$ to ∞. Here $x^\star$ is the image of x under the inversion in the sphere $\partial(h(B_2))$ if $x \in R(a,b)$ and the image of x under the inversion in the sphere $\partial(h(B_1))$ if $x \in R(b,a)$. It remains to show that

$$a_{h(D)}(x,\infty) = a_{h(D)}(x,y) + a_{h(D)}(y,\infty) \qquad \text{for all} \qquad y \in S'. \tag{3.1}$$

There is nothing to prove if $x \in S$. Otherwise, we can assume, due to symmetry, that $x \in R(b,a) \setminus S$. It is easy to see that

$$\mathrm{E}(x,\infty) = \mathrm{E}(x,y) = \mathrm{E}(y,\infty) = \{b\} \quad \text{and} \quad \mathrm{E}(\infty,x) = \mathrm{E}(\infty,y) = \{a\}.$$

By the choice of $x^\star$ the center of the maximal Apollonian ball about y with respect to x lies in $[h(z_1), x]$. Since $h(W) \subset h(B_1)$, we obtain that $\mathrm{E}(y,x) = \{a\}$. Hence (3.1) holds by Lemma 4.1 [18], as required.

Assume now that $c = b$. Then $d = a$ and hence $[a,b]$ is a diametrical chord and $\gamma = R(a,b) \cup \{\infty\} \cup R(b,a)$. Since $W \subset \overline{B^n}(a, |a-b|) \cap \overline{B^n}(b, |a-b|)$, the same reasoning as above applied to $h(C)$ implies that γ is a pseudogeodesic line. ∎

Proof of Theorem 1.2. For disks this result was proved by Gehring and Hag ([16, Theorem 3.29]) and was generalized for balls in $\overline{\mathbb{R}^n}$ by Hästö ([17, Theorem 1.9]). Hence we can assume that W is not a ball. Since ∂W does not lie on any hyperplane in $\overline{\mathbb{R}^n}$, by Theorem 5.7 [18] we can assume, without loss of generality, that f extends continuously to $\overline{D}$, $f(D) = D$ and that $f|_{\partial D} = \mathrm{id}_{\partial D}$. First, we will show that $f(\infty) = \infty$. Assume that $f(\infty) \neq \infty$ and let $x = f(\infty)$. Let h be the inversion in $S^{n-1}(x,1)$. Then the map $h \circ f$ satisfies the assumptions of Theorem 3.1 and hence the set

$$W' = \overline{\mathbb{R}^n} \setminus (h \circ f)(D) = \overline{\mathbb{R}^n} \setminus h(D) = h(W)$$

is a convex body of constant width. By Theorem 3 [21], the Apollonian metric a_D is conformal at x. Since a_D is also conformal at ∞, Theorem 2 [21] implies that W is a ball, which is a required contradiction. Thus, $f(\infty) = \infty$. Let $x \in D \setminus \{\infty\}$ be an arbitrary point. By Corollary 3.1 there exists a unique diametrical chord $[a,b]$ of W such that $x \in R(a,b)$. From the proof of Theorem 3.1 we obtain that $f(R(a,b)) = R(a',b')$ and $f(R(b,a)) = R(b',a')$. In particular, $a' \in \partial D$ is a unique point with $|f(x) - a'| = \mathrm{dist}(f(x), \partial D)$. Since $f(a) = a$ and $f(b) = b$, we obtain that $a' = a$ and $b' = b$. Hence $f(x) \in R(a,b)$. Since $a_D(x,\infty) = a_D(f(x),\infty)$, we obtain that $f(x) = x$. Thus, $f|_D = \mathrm{id}|_D$, which completes the proof. ∎

Remark 3.1. We believe that the regularity assumption in Theorem 1.1 can be omitted. For sets of constant width in $\mathbb{R}^2$, Theorem 1.2 is the same as Theorem 3.5.1 in ([20]).

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