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A Characterization of Chen Surfaces in $\mathbb{E}^4$

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Abstract. In the present study, we calculate the coefficients of the second fundamental form and curvature ellipse of Chen surfaces in $\mathbb{E}^4$. Further, we give the necessary and sufficent condition for the origin of N_pM to lie on the curvature ellipse of Vranceanu surfaces, which are special type of Chen surfaces.

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1. Introduction

Let M be a smooth surface embedded by $x(u, v)$ in $\mathbb{E}^4$. Given $p \in M$ consider the unit circle in T_pM parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ, the curve obtained by intersecting M with the hyperplane (3-space) at p composed by the direct sum of the normal plane N_pM and the straight line in tangent direction represented by θ. Such a curve is called *normal section* of M in the direction θ. The curvature vector η_θ of γ_θ in M lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the *curvature ellipse* of M at p. A point p in M is said to be hyperbolic, parabolic or elliptic according to whether p lies outside or inside the curvature ellipse of M at p. When the curvature ellipse at p degenerates to a segment, the point p is said to be semiumbilical center of M. An inflection point is a parabolic semiumbilical center (i.e. a point such that the corresponding curvature ellipse is a line segment and it lies on the line determined by this segment). Inflection point can be imaginary, real or flat according to they lie out inside or at one of the end points of the segment. A direction θ in T_pM for which $\dfrac{\partial \eta}{\partial \theta}$ and $\eta(\theta)$ are parallel is said to be *asymptotic direction.* There are exactly $2, 0$ or 1 asymptotic directions respectively at a hyperbolic, elliptic or parabolic point of M. This ellipse may degenerate on a radial segment of straight line, in which case p is known as an inflection point of the surface. The inflection point is of real type when p belongs to

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the curvature ellipse, and of imaginary type when it does not. An inflection point is flat when p is an end point of the curvature ellipse [9].

The torsion τ_θ of γ_θ at p is called the *normal torsion* of M in the direction of θ at p.

In [2], B. Y. Chen defined the allied vector field $a(v)$ of a normal vector field v. In particular, the allied mean curvature vector field is orthogonal to H. Further, B. Y. Chen defined the $\mathcal{A}$-surface to be these surfaces for which $a(H)$ vanishes identically. Such surfaces are also called Chen surfaces [5]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which dim $N_1 \leq 1$, in particular all hypersurfaces. These Chen surfaces are said to be trivial $\mathcal{A}$-surface [6]. For more details, see also [11], [4] and [7].

In the present study, we calculate the coefficients of the second fundamental form and curvature ellipse of Chen surfaces. Finally, we give the necessary and sufficient condition for the origin of N_pM to lie on the curvature ellipse of Vranceanu surfaces, which are special type of Chen surfaces. For more details see also [8] and [1].

2. Basic Concepts

Let M be a smooth surface immersed in $\mathbb{E}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{E}^4$. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus N_pM$ where N_pM is the orthogonal complement of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given local vector fields e_1, e_2 on M. The induced Riemannian connection on M is defined by $\nabla_{e_1}e_2 = \left(\widetilde{\nabla}_{e_1}e_2\right)^T$.

Let $\chi(M)$ and $N(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map:

$$h : \chi(M) \times \chi(M) \to N(M), \quad h(e_1, e_2) = \widetilde{\nabla}_{e_1}e_2 - \nabla_{e_1}e_2. \tag{2.1}$$

This map is well defined, symmetric and bilinear. Recall the shape operator

$$A_v : T_pM \to T_pM, \; A_v e_1 = -\left(\widetilde{\nabla}_{e_1}e_2\right)^T \tag{2.2}$$

where v is the normal vector field at $p \in M$ and T means the tangent component. This operator is bilinear, self-adjoint and for any $e_1, e_2 \in T_pM$ satisfies the following equation:

$$\langle A_v e_1, e_2 \rangle = \langle h(e_1, e_2), v \rangle .$$

On M we choose a local field of orthonormal frame e_1, e_2, e_3, e_4 such that, restricted to the surface, e_1, e_2 are tangent and e_3, e_4 are normal to M. Let w^1, w^2, w^3, w^4 denote the field of dual frames.

The structure equations of M are then given by

$$dw^A = -\sum_B w^A_B \wedge w^B, \quad w^A_B + w^B_A = 0, \qquad A, B = 1, 2, 3, 4.$$

When we restrict these forms to M we have $w^3 = w^4 = 0$. Thus, it follows from the structure equation that

$$0 = dw^r = -\sum_i w_i^r \wedge w^i.$$

Therefore, after applying Cartan's lemma, we have

$$dw_i^r = \sum_j h_{ij}^r w^j, \quad h_{ij}^r = h_{ji}^r, \quad i,j = 1,2; \quad r = 3,4.$$

It is well-known that the coefficients of the second fundamental form h satisfy

$$h_{ij}^r = \langle h(e_i, e_j), e_r \rangle, \quad i,j = 1,2; \quad r = 3,4. \tag{2.3}$$

Recall that a submanifold of a Riemannian manifold is said to be *minimal* if its *mean curvature vector* H vanishes identically (see, for instance, [2]). In the case under consideration, Imf is *minimal* if and only if $h(e_1, e_1) + h(e_2, e_2) = 0$, where h denotes the *second fundamental form* of f, or equivalently

$$< h(e_1, e_1) + h(e_2, e_2), e_r >= 0, \quad r = 3,4.$$

3. Curvature Ellipse of a Surface

Let M be a smooth surface in $\mathbb{E}^4$. The second fundamental form of M is characterized by two quadratic forms. Their functional coefficients will be denoted by $(h_{11}^3, h_{12}^3, h_{22}^3)$ and $(h_{11}^4, h_{12}^4, h_{22}^4)$ respectively. The shape operator matrices of the surface $M^2 \subseteq \mathbb{E}^4$ become

$$A_{e_3} = \begin{bmatrix} h_{11}^3 & h_{12}^3 \\ h_{21}^3 & h_{22}^3 \end{bmatrix}, \tag{3.1}$$

$$A_{e_4} = \begin{bmatrix} h_{11}^4 & h_{12}^4 \\ h_{21}^4 & h_{22}^4 \end{bmatrix}. \tag{3.2}$$

Let γ_θ *normal section* of M in the direction θ. Denote

$$\gamma_\theta' = X = \cos\theta e_1 + \sin\theta e_2, \tag{3.3}$$

the unit vector of the normal section. Then the curvature ellipse of normal curvature $\|\eta(\theta)\|$ is given by

$$\eta(X) = B\cos 2\theta + C\sin 2\theta + H, \tag{3.4}$$

where H is the mean curvature vector, B and C are normal vectors defined by

$$\begin{aligned} B &= \frac{h(e_1, e_1) - h(e_1, e_2)}{2}, \\ C &= h(e_1, e_2), \end{aligned}$$

in the frame e_3, e_4 (see, [10]).

We have the following functions associated to the coefficients of the second fundamental form:

$$\Delta(p)=\frac{1}{4}\det\begin{bmatrix} h_{11}^3 & 2h_{12}^3 & h_{22}^3 & 0 \\ h_{11}^4 & 2h_{12}^4 & h_{22}^4 & 0 \\ 0 & h_{11}^3 & 2h_{12}^3 & h_{22}^3 \\ 0 & h_{11}^4 & 2h_{12}^4 & h_{22}^4 \end{bmatrix}(p) \tag{3.5}$$

$$K(p)=(h_{11}^3h_{22}^3-(h_{12}^3)^2+h_{11}^4h_{22}^4-(h_{12}^4)^2)(p). \tag{3.6}$$

(Gaussian curvature of M) and the matrix

$$\alpha(p)=\begin{bmatrix} h_{11}^3 & h_{12}^3 & h_{22}^3 \\ h_{11}^4 & h_{12}^4 & h_{22}^4 \end{bmatrix}(p). \tag{3.7}$$

By identifying p with the origin of $N_p(M)$, it can be shown that:

(a) $\Delta(p)<0 \Rightarrow p$ lies outside the curvature ellipse (such a point is said to be a hyperbolic point of M),

(b) $\Delta(p)>0 \Rightarrow p$ lies inside the curvature ellipse (elliptic point),

(c) $\Delta(p)=0 \Rightarrow p$ lies on the curvature ellipse (parabolic point).

More detailed study of this case allows us to distinguish among the following possibilities:

(d) $\Delta(p)=0,\ K(p)>0 \Rightarrow p$ is an inflection point of imaginary type,

(e) $\Delta(p)=0,\ K(p)<0$ and rank$\alpha(p)=2 \Rightarrow$ ellipse is non-degenerate; rank$\alpha(p)=1 \Rightarrow p$ is an inflection point of real type,

(f) $\Delta(p)=0,\ K(p)=0 \Rightarrow p$ is an inflection point of flat type [9].

4. Chen Surfaces

Let M be an n-dimensional smooth submanifold of m-dimensional Riemannian manifold N and ζ be a normal vector field of M. Let ξ_x be $m-n$ mutually orthogonal unit normal vector fields of M such that $\zeta=\|\zeta\|\,\xi_1$. In [2], B. Y. Chen defined the allied vector field $a(\zeta)$ of a normal vector field ζ by the formula

$$a(v)=\frac{\|\zeta\|}{n}\sum_{x=2}^{m-n}\{tr(A_1A_x)\}\,\xi_x,$$

where $A_x=A_{\xi_x}$ is the shape operator. In particular, the allied mean curvature vector field of the mean curvature vector H is a well-defined normal vector field orthogonal to H. If the allied mean vector $a(H)$ vanishes identically, then the submanifold M is called $\mathcal{A}$-submanifold of N. Furthermore, $\mathcal{A}$-submanifolds are also called Chen submanifolds [5].

For the case M is a smooth surface of $\mathbb{E}^4$, the allied vector $a(H)$ becomes

$$a(H) = \frac{\|H\|}{2}\left\{tr(A_{e_3}A_{e_4})\right\}e_4 \tag{4.1}$$

where $\{e_3, e_4\}$ is an orthonormal basis of $N(M)$.

Proposition 4.1. [6] *Let M be a smooth surface immersed in $\mathbb{E}^4$. We choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on M such that e_3, e_4 is a basis for NM with e_3 in the direction of H and such that e_1, e_2 are principal directions of A_{e_3}. Then M is a non-trivial $\mathcal{A}$-surface if and only if*

$$A_{e_3} = \begin{bmatrix} h_{11}^3 & 0 \\ 0 & h_{22}^3 \end{bmatrix}, \tag{4.2}$$

$$A_{e_4} = \begin{bmatrix} 0 & h_{12}^4 \\ h_{21}^4 & 0 \end{bmatrix}. \tag{4.3}$$

whereby $((h_{11}^3)^2 - (h_{22}^3)^2)h_{21}^4$ is nonzero.

We have the following result.

Theorem 4.1. *Let M be a non-trivial $\mathcal{A}$-surface in $\mathbb{E}^4$ with e_3 in the direction of H and e_1, e_2 are principal directions of A_{e_3}.*

(i) *If the coefficients h_{11}^3 and h_{22}^3 are the same sign (resp. different sign) then the origin of N_pM lies outside (resp. inside) of the curvature ellipse of M.*
(ii) *If one of the coefficients h_{11}^3 or h_{22}^3 is identically zero then the origin of N_pM lies on the curvature ellipse of M.*

Proof. Let M be a non-trivial $\mathcal{A}$-surface in $\mathbb{E}^4$. Then, using equations (4.2), (4.3) with equation (3.5), we get

$$\Delta = -\left(h_{11}^3 h_{22}^3 (h_{12}^4)^2\right) \text{, where } h_{12}^4 \neq 0. \tag{4.4}$$

By identifying p with the origin of $N_p(M)$ we have the following cases:

(i) $\Delta(p) < 0$ (resp. $\Delta(p) > 0$) $\Rightarrow p$ lies outside (resp. inside) of the curvature ellipse,
(ii) $\Delta(p) = 0 \Rightarrow p$ lies on the curvature ellipse of M.

Our theorem is thus proved. ∎

Consequently, we get the following:

Corollary 4.1. *Let M be a non-trivial $\mathcal{A}$-surface in $\mathbb{E}^4$ with e_3 in the direction of H and e_1, e_2 are principal directions of A_{e_3}. If M is a flat surface then the origin of N_pM lies outside of the curvature ellipse of M.*

Proof. From equations (4.2), (4.3) and (3.6), we have

$$K = h_{11}^3 h_{22}^3 - (h_{12}^4)^2. \tag{4.5}$$

Now, suppose that M is flat. Then by equation (4.5), we see that $h_{11}^3 h_{22}^3 = (h_{12}^4)^2$. Thus, substituting this into equation (4.4) we get $\Delta = -(h_{12}^4)^4 < 0$. So, the origin of N_pM lies outside of the curvature ellipse of M. ∎

The following Chen surfaces are studied by Vranceanu [14] . So, they are called Vranceanu surfaces which are tensor product of two plane curves. For more details, see also [8] and [1].

Proposition 4.2. *Let $\mathbb{V}^2$ be the Vranceanu surface given by the parametrization*

$$\mathbb{V}^2 = (u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t). \tag{4.6}$$

If $2(u')^2 - uu'' + u^2 = 0$ then the origin of $N_p\mathbb{V}^2$ lies on the curvature ellipse of $\mathbb{V}^2$.

Proof. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to $\mathbb{V}^2$ and e_3, e_4 are normal to $\mathbb{V}^2$ as given the following:

$$\begin{aligned} e_1 &= \frac{1}{u}\frac{\partial}{\partial t} = (-\cos s\sin t, \cos s\cos t, -\sin s\sin t, \sin s\cos t), \\ e_2 &= \frac{1}{A}\frac{\partial}{\partial s} = \frac{1}{A}(B\cos t, B\sin t, C\cos t, C\sin t), \\ e_3 &= \frac{1}{A}(-C\cos t, -C\sin t, B\cos t, B\sin t), \\ e_4 &= (-\sin s\sin t, \sin s\cos t, \cos s\sin t, -\cos s\cos t), \end{aligned}$$

where

$$\begin{aligned} A &= \sqrt{u^2 + (u')^2},\ B = u'\cos s - u\sin s, \text{ and} \\ C &= u'\sin s + u''\cos s. \end{aligned} \tag{4.7}$$

Using equations (2.1) and (2.3), we can get that the coefficients of the second fundamental form h are as following:

$$\begin{aligned} h^3_{11} &= \frac{1}{\sqrt{u^2 + (u')^2}} = \lambda,\ h^3_{22} = \frac{2(u')^2 - uu'' + u^2}{(u^2 + (u')^2)^{3/2}} = \mu,\ h^3_{21} = 0 \\ h^4_{11} &= h^4_{22} = 0,\ h^4_{12} = -\lambda \end{aligned} \tag{4.8}$$

Thus by the use of equation (3.5), we have

$$\Delta(p) = \frac{1}{4}\det\begin{bmatrix} \lambda & 0 & \mu & 0 \\ 0 & -2\lambda & 0 & 0 \\ 0 & \lambda & 0 & \mu \\ 0 & 0 & -2\lambda & 0 \end{bmatrix}(p) = -\lambda^3\mu.$$

So $\mu = \dfrac{2(u')^2 - uu'' + u^2}{(u^2 + (u')^2)^{3/2}} = 0$ implies $\Delta(p) = 0$, which completes the proof. ∎

We prove the following result:

Corollary 4.2. *Let $\mathbb{V}^2$ be the Vranceanu surface given by the parametrization (4.6). If the origin of $N_p\mathbb{V}^2$ lies on the curvature ellipse of $\mathbb{V}^2$, then*

$$u(s) = c_1\sec(s + c_2) \tag{4.9}$$

where c_1 and c_2 are real constants.

Proof. Suppose the origin of $N_p\mathbb{V}^2$ lies on the curvature ellipse of $\mathbb{V}^2$, then by Proposition 4, the following equality holds:

$$2(u')^2 - uu'' + u^2 = 0. \tag{4.10}$$

The second degree of equation (4.9) can be written as

$$\left(\frac{u'}{u}\right)' - \left[\left(\frac{u'}{u}\right)^2 + 1\right] = 0. \tag{4.11}$$

Further, substituting $\dfrac{u'}{u} = z$ and after some computation we get equation (4.9). ∎

Remark 4.1. (i) If the Vranceanu surface $\mathbb{V}^2$ is flat then $u(s) = ce^{bs}$, for some constants b and c.

(ii) If the Vranceanu surface $\mathbb{V}^2$ is minimal then $u(s) = \left(1 + \dfrac{1}{(\tan 2s)^2}\right)^{\frac{1}{5}}$ (see, [8]).

(iii) For the special values $b = 0$ and $c = 1$ the surface $\mathbb{V}^2$ will become the product of two plane circles with the same radius, which is minimal surface of $S^3(1)$.

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