BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **31***(2) (2008), 217–221*

On Quasi $\widetilde{g}s$-open and Quasi $\widetilde{g}s$-closed Functions

[1]N. RAJESH AND [2]E. EKICI
[1]Department of Mathematics, Kongu Engineering College
Perundurai, Erode-638 052 Tamilnadu, India,
[2]Department of Mathematics, Canakkale Onsekiz Mart University
Terzioglu Campus 17020 Canakkale, Turkey.
[1]nrajesh_topology@yahoo.co.in [2]eekici@comu.edu.tr

Abstract. The purpose of this paper is to give a new type of open function called quasi $\widetilde{g}s$-open function. Also, we obtain its characterizations and its basic properties.

2000 Mathematics Subject Classification: 54C10, 54C08, 54C05

Key words and phrases: Topological spaces, $\widetilde{g}s$-open set, $\widetilde{g}s$-closed set, $\widetilde{g}s$-interior, $\widetilde{g}s$-closure, quasi $\widetilde{g}s$-open function.

1. Introduction and Preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences.

Recently, as a generalization of closed sets, the notion of $\widetilde{g}s$-closed sets were introduced and studied by Sundaram *et al.* [3]. In this paper, we will continue the study of related functions by involving $\widetilde{g}s$-open sets. We introduce and characterize the concept of quasi-$\widetilde{g}s$-open functions.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1.1. *A subset A of a space (X, τ) is called semi-open* [4] *if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called semi-closed.*

The semi-closure [1] of a subset A of X, denoted by $s\,\mathrm{Cl}(A)$, is defined to be the intersection of all semi-closed sets containing A in X.

Received: August 2, 2007; *Revised:* January 2, 2008.

Definition 1.2. *A subset A of a space X is called:*

(i) *$\widehat{g}$-closed* [8] *if* $\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* U *is semi-open in* X*. The complement of $\widehat{g}$-closed set is called $\widehat{g}$-open.*

(ii) *${}^*g$-closed* [7] *if* $\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* U *is $\widehat{g}$-open in* X*. The complement of ${}^*g$-closed set is called ${}^*g$-open.*

(iii) *${}^\#g$-semi-closed* [9] *if* $s\,\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* U *is ${}^*g$-open in* X*. The complement of ${}^\#g$-semi-closed is called ${}^\#g$-semi-open.*

(iv) *$\widetilde{g}$-semiclosed (briefly $\widetilde{g}s$-closed)* [3] *if* $\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* U *is ${}^\#g$-semi-open in* X*. The complement of $\widetilde{g}s$-closed set is called $\widetilde{g}s$-open.*

The union (resp. intersection) of all $\widetilde{g}s$-open (resp. $\widetilde{g}s$-closed) sets, each contained in (resp. containing) a set A in a space X is called the $\widetilde{g}s$-interior (resp. $\widetilde{g}s$-closure) of A and is denoted by $\widetilde{g}s\text{-Int}(A)$ (resp. $\widetilde{g}s\text{-Cl}(A)$) [6].

Definition 1.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called:*

(i) *$\widetilde{g}s$-irresolute* [5] *($\widetilde{g}s$-continuous* [5]*) if $f^{-1}(V)$ is $\widetilde{g}s$-closed in X for every $\widetilde{g}s$-closed (resp. closed) subset V of Y;*

(ii) *$\widetilde{g}s$-open* [2] *(resp. $\widetilde{g}s$-closed* [2]*) if $f(V)$ is $\widetilde{g}s$-open (resp. $\widetilde{g}s$-closed) in Y for every open (resp. closed) subset of X.*

2. Quasi $\widetilde{g}s$-open Functions

We have introduced the following definition:

Definition 2.1. *A function $f : X \to Y$ is said to be quasi $\widetilde{g}s$-open if the image of every $\widetilde{g}s$-open set in X is open in Y.*

It is evident that, the concepts quasi $\widetilde{g}s$-openness and $\widetilde{g}s$-continuity coincide if the function is a bijection.

Theorem 2.1. *A function $f : X \to Y$ is quasi-$\widetilde{g}s$-open if and only if for every subset U of X, $f(\widetilde{g}s\text{-Int}(U)) \subset \mathrm{Int}(f(U))$.*

Proof. Let f be a quasi $\widetilde{g}s$-open function. Now, we have $\mathrm{Int}(U) \subset U$ and $\widetilde{g}s\text{-Int}(U)$ is a $\widetilde{g}s$-open set. Hence, we obtain that $f(\widetilde{g}s\text{-Int}(U)) \subset f(U)$. As $f(\widetilde{g}s\text{-Int}(U))$ is open, $f(\widetilde{g}s\text{-Int}(U)) \subset \mathrm{Int}(f(U))$.
Conversely, assume that U is a $\widetilde{g}s$-open set in X. Then, $f(U) = f(\widetilde{g}s\text{-Int}(U)) \subset \mathrm{Int}(f(U))$ but $\mathrm{Int}(f(U)) \subset f(U)$. Consequently, $f(U) = \mathrm{Int}(f(U))$ and hence f is quasi $\widetilde{g}s$-open. ∎

Lemma 2.1. *If a function $f : X \to Y$ is quasi $\widetilde{g}s$-open, then $\widetilde{g}s\text{-Int}(f^{-1}(G)) \subset f^{-1}(\mathrm{Int}(G))$ for every subset G of Y.*

Proof. Let G be any arbitrary subset of Y. Then, $\widetilde{g}s\text{-Int}(f^{-1}(G))$ is a $\widetilde{g}s$-open set in X and f is quasi $\widetilde{g}s$-open, then $f(\widetilde{g}s\text{-Int}(f^{-1}(G))) \subset \mathrm{Int}(f(f^{-1}(G))) \subset \mathrm{Int}(G)$. Thus, $\widetilde{g}s\text{-Int}(f^{-1}(G)) \subset f^{-1}(\mathrm{Int}(G))$. ∎

Recall that a subset S is called a $\widetilde{g}s$-neighbourhood [5] of a point x of X if there exists a $\widetilde{g}s$-open set U such that $x \in U \subset S$.

Theorem 2.2. *For a function $f : X \to Y$, the following are equivalent:*

(i) *f is quasi $\widetilde{g}s$-open;*

(ii) *For each subset* U *of* X, $f(\widetilde{g}s\text{-Int}(U)) \subset \text{Int}(f(U))$;

(iii) *For each* $x \in X$ *and each* $\widetilde{g}s$*-neighbourhood* U *of* x *in* X, *there exists a neighbourhood* V *of* $f(x)$ *in* Y *such that* $V \subset f(U)$.

Proof. (i)⇒(ii): It follows from Theorem 2.1.
(ii)⇒(iii): Let $x \in X$ and U be an arbitrary $\widetilde{g}s$-neighbourhood of x in X. Then there exists a $\widetilde{g}s$-open set V in X such that $x \in V \subset U$. Then by (ii), we have $f(V) = f(\widetilde{g}s\text{-Int}(V)) \subset \text{Int}(f(V))$ and hence $f(V) = \text{Int}(f(V))$. Therefore, it follows that $f(V)$ is open in Y such that $f(x) \in f(V) \subset f(U)$.
(iii)⇒(i): Let U be an arbitrary $\widetilde{g}s$-open set in X. Then for each $y \in f(U)$, by (iii) there exists a neighbourhood V_y of y in Y such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exists an open set W_y in Y such that $y \in W_y \subset V_y$. Thus, $f(U) = \bigcup\{W_y : y \in f(U)\}$ which is an open set in Y. This implies that f is quasi $\widetilde{g}s$-open function. ∎

Theorem 2.3. *A function* $f : X \to Y$ *is quasi* $\widetilde{g}s$*-open if and only if for any subset* B *of* Y *and for any* $\widetilde{g}s$*-closed set* F *of* X *containing* $f^{-1}(B)$, *there exists a closed set* G *of* Y *containing* B *such that* $f^{-1}(G) \subset F$.

Proof. Suppose f is quasi $\widetilde{g}s$-open. Let $B \subset Y$ and F be a $\widetilde{g}s$-closed set of X containing $f^{-1}(B)$. Now, put $G = Y - f(X - F)$. It is clear that $f^{-1}(B) \subset F$ implies $B \subset G$. Since f is quasi $\widetilde{g}s$-open, we obtain G as a closed set of Y. Moreover, we have $f^{-1}(G) \subset F$.
Conversely, let U be a $\widetilde{g}s$-open set of X and put $B = Y\backslash f(U)$. Then $X\backslash U$ is a $\widetilde{g}s$-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X\backslash U$. Hence, we obtain $f(U) \subset Y\backslash F$. On the other hand, it follows that $B \subset F$, $Y\backslash F \subset Y\backslash B = f(U)$. Thus, we obtain $f(U) = Y\backslash F$ which is open and hence f is a quasi $\widetilde{g}s$-open function. ∎

Theorem 2.4. *A function* $f : X \to Y$ *is quasi* $\widetilde{g}s$*-open if and only if* $f^{-1}(\text{Cl}(B)) \subset \widetilde{g}s\text{-Cl}(f^{-1}(B))$ *for every subset* B *of* Y.

Proof. Suppose that f is quasi $\widetilde{g}s$-open. For any subset B of Y, $f^{-1}(B) \subset \widetilde{g}s\text{-Cl}(f^{-1}(B))$. Therefore by Theorem 2.3, there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset \widetilde{g}s\text{-Cl}(f^{-1}(B))$. Therefore, we obtain $f^{-1}(\text{Cl}(B)) \subset f^{-1}(F) \subset \widetilde{g}s\text{-Cl}(f^{-1}(B))$.
Conversely, let $B \subset Y$ and F be a $\widetilde{g}s$-closed set of X containing $f^{-1}(B)$. Put $W = \text{Cl}_Y(B)$, then we have $B \subset W$ and W is closed and $f^{-1}(W) \subset \widetilde{g}s\text{-Cl}(f^{-1}(B)) \subset F$. Then by Theorem 2.3, f is quasi $\widetilde{g}s$-open. ∎

Lemma 2.2. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions and* $g \circ f : X \to Z$ *is quasi* $\widetilde{g}s$*-open. If* g *is continuous injective, then* f *is quasi* $\widetilde{g}s$*-open.*

Proof. Let U be a $\widetilde{g}s$-open set in X, then $(g \circ f)(U)$ is open in Z since $g \circ f$ is quasi $\widetilde{g}s$-open. Again g is an injective continuous function, $f(U) = g^{-1}(g \circ f(U))$ is open in Y. This shows that f is quasi $\widetilde{g}s$-open. ∎

3. Quasi $\widetilde{g}s$-closed Functions

Definition 3.1. *A function* $f : X \to Y$ *is said to be quasi* $\widetilde{g}s$*-closed if the image of each* $\widetilde{g}s$*-closed set in* X *is closed in* Y.

Clearly, every quasi $\widetilde{g}s$-closed function is closed as well as $\widetilde{g}s$-closed.

Remark 3.1. Every $\widetilde{g}s$-closed (resp. closed) function need not be quasi $\widetilde{g}s$-closed as shown by the following example.

Example 3.1. Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{b, c\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then clearly f is $\widetilde{g}s$-closed as well as closed but not quasi $\widetilde{g}s$-closed.

Lemma 3.1. *If a function $f : X \to Y$ is quasi $\widetilde{g}s$-closed, then $f^{-1}(\mathrm{Int}(B)) \subset \widetilde{g}s\text{-}\mathrm{Int}(f^{-1}(B))$ for every subset B of Y.*

Proof. This proof is similar to the proof of Lemma 2.1. ∎

Theorem 3.1. *A function $f : X \to Y$ is quasi $\widetilde{g}s$-closed if and only if for any subset B of Y and for any $\widetilde{g}s$-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.*

Proof. This proof is similar to that of Theorem 2.3. ∎

Definition 3.2. *A function $f : X \to Y$ is called $\widetilde{g}s^*$-closed* [2] *if the image of every $\widetilde{g}s$-closed subset of X is $\widetilde{g}s$-closed in Y.*

Theorem 3.2. *If $f : X \to Y$ and $g : Y \to Z$ are two quasi $\widetilde{g}s$-closed function, then $g \circ f$: $X \to Z$ is a quasi $\widetilde{g}s$-closed function.*

Proof. Obvious. ∎

Furthermore, we have the following theorem:

Theorem 3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then:*

(i) *If f is $\widetilde{g}s$-closed and g is quasi $\widetilde{g}s$-closed, then $g \circ f$ is closed;*
(ii) *If f is quasi $\widetilde{g}s$-closed and g is $\widetilde{g}s$-closed, then $g \circ f$ is $\widetilde{g}s^*$-closed;*
(iii) *If f is $\widetilde{g}s^*$-closed and g is quasi $\widetilde{g}s$-closed, then $g \circ f$ is quasi $\widetilde{g}s$-closed.*

Proof. Obvious. ∎

Theorem 3.4. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi $\widetilde{g}s$-closed.*

(i) *If f is $\widetilde{g}s$-irresolute surjective, then g is closed.*
(ii) *If g is $\widetilde{g}s$-continuous injective, then f is $\widetilde{g}s^*$-closed.*

Proof. (i) Suppose that F is an arbitrary closed set in Y. As f is $\widetilde{g}s$-irresolute, $f^{-1}(F)$ is $\widetilde{g}s$-closed in X. Since $g \circ f$ is quasi $\widetilde{g}s$-closed and f is surjective, $(g \circ f(f^{-1}(F))) = g(F)$, which is closed in Z. This implies that g is a closed function. *(ii)* Suppose F is any $\widetilde{g}s$-closed set in X. Since $g \circ f$ is quasi $\widetilde{g}s$-closed, $(g \circ f)(F)$ is closed in Z. Again g is a $\widetilde{g}s$-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is $\widetilde{g}s$-closed in Y. This shows that f is $\widetilde{g}s^*$-closed. ∎

Theorem 3.5. *Let X and Y be topological spaces. Then the function $g : X \to Y$ is a quasi $\widetilde{g}s$-closed if and only if $g(X)$ is closed in Y and $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ whenever V is $\widetilde{g}s$-open in X.*

Proof. Necessity: Suppose $g : X \to Y$ is a quasi $\widetilde{g}s$-closed function. Since X is $\widetilde{g}s$-closed, $g(X)$ is closed in Y and $g(V) \setminus g(X\backslash V) = g(V) \cap g(X) \setminus g(X\backslash V)$ is open in $g(X)$ when V is $\widetilde{g}s$-open in X.
Sufficiency: Suppose $g(X)$ is closed in Y, $g(V) \setminus g(X\backslash V)$ is open in $g(X)$ when V is $\widetilde{g}s$-open in X, and let C be closed in X. Then $g(C) = g(X) \setminus (g(X\backslash C) \setminus g(C))$ is closed in $g(X)$ and hence, closed in Y. ∎

Corollary 3.1. *Let X and Y be topological spaces. Then a surjective function $g : X \to Y$ is quasi $\widetilde{g}s$-closed if and only if $g(V) \setminus g(X\backslash V)$ is open in Y whenever U is $\widetilde{g}s$-open in X.*

Proof. Obvious. ∎

Corollary 3.2. *Let X and Y be topological spaces and let $g : X \to Y$ be a $\widetilde{g}s$-continuous quasi $\widetilde{g}s$-closed surjective function. Then the topology on Y is $\{g(V) \setminus g(X\backslash V) : V$ is $\widetilde{g}s$-open in $X\}$.*

Proof. Let W be open in Y. Then $g^{-1}(W)$ is $\widetilde{g}s$-open in X, and $g(g^{-1}(W)) \setminus g(X\backslash g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V) \setminus g(X\backslash V)$, V is $\widetilde{g}s$-open in X. On the other hand, all sets of the form $g(V) \setminus g(X\backslash V)$, V is $\widetilde{g}s$-open in X, are open in Y from Corollary 3.1. ∎

Definition 3.3. *A topological space (X, τ) is said to be $\widetilde{g}s$-normal if for any pair of disjoint $\widetilde{g}s$-closed subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.*

Theorem 3.6. *Let X and Y be topological spaces with X is $\widetilde{g}s$-normal. If $g : X \to Y$ is a $\widetilde{g}s$-continuous quasi $\widetilde{g}s$-closed surjective function. Then Y is normal.*

Proof. Let K and M be disjoint closed subsets of Y. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint $\widetilde{g}s$-closed subsets of X. Since X is $\widetilde{g}s$-normal, there exist disjoint open sets V and W such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V)\backslash g(X\backslash V)$ and $M \subset g(W)\backslash g(X\backslash W)$. Further by Corollary 3.1, $g(V)\backslash g(X\backslash V)$ and $g(W)\backslash g(X\backslash W)$ are open sets in Y and clearly $(g(V)\backslash g(X\backslash V)) \bigcap (g(W) \setminus g(X\backslash W)) = \varnothing$. This shows that Y is normal. ∎

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