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Fuzzy Pre-semi-closed Sets

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Abstract. In this paper, we introduce fuzzy pre-semi-closed sets in fuzzy topological spaces and investigate their properties. Using fuzzy pre-semi-closed sets, equivalences of fuzzy regular open sets are established. As applications to fuzzy pre-semi-closed sets, we introduce fuzzy spaces with new kinds of separation axioms, namely, fuzzy pre-semi-$T_{\frac{1}{2}}$ spaces, fuzzy pre-semi-$T_{\frac{3}{4}}$ spaces and fuzzy semi-pre-$T_{\frac{1}{3}}$ spaces and characterize them.

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1. Introduction and Preliminaries

Recently, Veera Kumar [13] introduced pre-semi-closed sets in crisp topological spaces. Extending this concept to fuzzy topological spaces(*fts*), we define a new class of fuzzy generalized sets namely, fuzzy pre-semi-closed sets and investigate their properties. In this section, we list out the definitions and the results which are needed in sequel. Throughout this paper, fts X denote a fuzzy topological space (X, τ), in the sense of Chang [6]. Fuzzy sets in X will be denoted by λ, μ, ν, η. For a fuzzy set λ, the operators fuzzy closure and fuzzy interior are denoted and defined by $\mathrm{Cl}\,\lambda = \wedge\{\mu : \mu \geq \lambda, 1 - \mu \in \tau\}$ and $\mathrm{Int}\,\lambda = \vee\{\mu : \mu \leq \lambda, \mu \in \tau\}$ if X is a finite set and $X = \{a_i : i = 1, 2, \ldots, k\}$, where k is a positive integer. Fuzzy sets in X may represented as

$$\lambda = \frac{\alpha_1}{a_1} + \frac{\alpha_2}{a_2} + \cdots + \frac{\alpha_k}{a_k}$$

and $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_k \leq 1$. We do not assume any meaning to symbols used unless defined here. The following lemmas are well-known.

Lemma 1.1. [2, 4, 12] *In a fuzzy topological space* (X, τ),

(i) *every fuzzy regular open set is fuzzy open.*

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(ii) *every fuzzy open set is fuzzy α-open.*
(iii) *every fuzzy α-open set is both fuzzy semiopen and fuzzy preopen.*
(iv) *every fuzzy semiopen set is fuzzy semi-preopen.*
(v) *every fuzzy preopen set is fuzzy semi-preopen.*

Lemma 1.2. *A fuzzy set λ in a fuzzy topological space (X, τ) is*

(i) *fuzzy regular closed* [2] *(simply Fr-closed) if* $\lambda = \operatorname{Cl}\operatorname{Int}\lambda$.
(ii) *fuzzy α-closed* [4] *(simply Fα-closed) if* $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\lambda \leq \lambda$.
(iii) *fuzzy pre-closed* [4] *(simply Fp-closed) if* $\operatorname{Cl}\operatorname{Int}\lambda \leq \lambda$.
(iv) *fuzzy semi-closed* [2] *(simply Fs-closed) if* $\operatorname{Int}\operatorname{Cl}\lambda \leq \lambda$.
(v) *fuzzy semi-pre-closed* [12] *(simply Fsp-closed) if* $\operatorname{Int}\operatorname{Cl}\operatorname{Int}\lambda \leq \lambda$.

Notation 1.1. Let (X, τ) be a fuzzy topological space. Then the family of fuzzy regular closed (resp. fuzzy semi-closed, fuzzy pre-closed, fuzzy semi-pre-closed) sets in X, may be denoted by its adjective as r(resp. α, s, p, sp).

Definition 1.1. [2, 4, 12] *Let λ be a fuzzy set in a fuzzy topological space (X, τ). Then fuzzy β-closure of λ is denoted and defined by $\beta\operatorname{Cl}\lambda = \wedge\{\mu : \lambda \leq \mu, \mu \in \beta\}$ where $\beta \in \{r, \alpha, s, p, sp\}$.*

The following three lemmas are well-known.

Lemma 1.3. *Let λ be a fuzzy set in a fuzzy topological space (X, τ). Then*

(i) $sp\operatorname{Cl}\lambda \leq s\operatorname{Cl}\lambda \leq \alpha\operatorname{Cl}\lambda \leq \operatorname{Cl}\lambda \leq r\operatorname{Cl}\lambda$,
(ii) $sp\operatorname{Cl}\lambda \leq p\operatorname{Cl}\lambda \leq \alpha\operatorname{Cl}\lambda$.

Lemma 1.4. *Let λ be a fuzzy set in a fuzzy topological space (X, τ). Then*

(i) $\alpha\operatorname{Cl}\lambda = \lambda \vee \operatorname{Cl}\operatorname{Int}\operatorname{Cl}\lambda$.
(ii) $s\operatorname{Cl}\lambda = \lambda \vee \operatorname{Int}\operatorname{Cl}\lambda$.
(iii) $p\operatorname{Cl}\lambda \geq \lambda \vee \operatorname{Cl}\operatorname{Int}\lambda$.
(iv) $sp\operatorname{Cl}\lambda \geq \lambda \vee \operatorname{Int}\operatorname{Cl}\operatorname{Int}\lambda$.

Lemma 1.5. *Let λ be a fuzzy set in a fuzzy topological space (X, τ). Then*

(i) $1 - sp\operatorname{Int}(\mu) = sp\operatorname{Cl}(1-\mu)$ *and* $1 - sp\operatorname{Cl}\mu = sp\operatorname{Int}(1-\mu)$.
(ii) $sp\operatorname{Cl}(sp\operatorname{Cl}\mu) = sp\operatorname{Cl}\mu$.

Definition 1.2. *A fuzzy set λ in a fuzzy topological space (X, τ) is called:*

(i) [3] *Fuzzy generalized closed set if $\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open. We briefly denote it as Fg-closed set.*
(ii) [8] *Fuzzy semi-generalized closed set if $s\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy semiopen. We briefly denote it as Fsg-closed set.*
(iii) [9] *Fuzzy generalized strongly closed set if $\alpha\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open. We briefly denote it as $Fg\alpha$-closed set.*
(iv) [9] *Fuzzy generalized almost strongly semi-closed set if $\alpha\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy α-open. We briefly denote it as $F\alpha g$-closed set.*
(v) [11] *Fuzzy semi-pre-generalized closed set if $sp\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy semi-pre-open. We briefly denote it as $Fspg$-closed set.*
(vi) [10] *Regular generalized fuzzy closed set if $\operatorname{Cl}\lambda \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy regular open. We briefly denote it as rFg-closed set.*

(vii) [5] *Fuzzy pre-generalized closed set if* $p\,\mathrm{Cl}\,\lambda \leq \mu$ *whenever* $\lambda \leq \mu$ *and* μ *is fuzzy pre-open. We briefly denote it as* Fpg*-closed set. Similarly, we may define:*

(viii) *Fuzzy generalized semi-closed set if* $s\,\mathrm{Cl}\,\lambda \leq \mu$ *whenever* $\lambda \leq \mu$ *and* μ *is fuzzy open. We briefly denote it as* Fgs*-closed set.*

(ix) *Fuzzy generalized pre-closed set if* $p\,\mathrm{Cl}\,\lambda \leq \mu$ *whenever* $\lambda \leq \mu$ *and* μ *is fuzzy open. We briefly denote it as* Fgp*-closed set.*

(x) *Fuzzy generalized semi-pre-closed set if* $sp\,\mathrm{Cl}\,\lambda \leq \mu$ *whenever* $\lambda \leq \mu$ *and* μ *is fuzzy open. We briefly denote it as* $Fgsp$*-closed set.*

Definition 1.3. *A fuzzy set* λ *in a fuzzy topological space* (X, τ) *is called fuzzy generalized open set (briefly,* Fg*-open) if* $1-\lambda$ *is fuzzy generalized closed. Similarly, fuzzy generalized semi-open, fuzzy semi-generalized open, fuzzy generalized strongly open, fuzzy generalized almost strongly semi-open, fuzzy generalized semi-pre-open, fuzzy semi-pre-generalized open, fuzzy generalized pre-open, fuzzy pre-generalized open sets are defined.*

Proposition 1.1. *In a fuzzy topological space* (X, τ)*, the following hold and the converse of each statement is not true:*

(i) [3] *Every fuzzy closed set is* Fg*-closed.*

(ii) [8] *Every fuzzy semi-closed set is* Fsg*-closed.*

(iii) [9] *Every fuzzy* α*-closed set is* $F\alpha g$*-closed.*

(iv) [5] *Every fuzzy pre-closed set is* Fpg*-closed.*

(v) [11] *Every fuzzy semi-pre-closed set is* $Fspg$*-closed.*

Definition 1.4. *A fuzzy topological space* (X, τ) *is said to be a*

(i) [3] *fuzzy-*$T_{\frac{1}{2}}$ *space if every* Fg*-closed set is fuzzy closed.*

(ii) [8] *fuzzy semi-*$T_{\frac{1}{2}}$ *space if every* Fsg*-closed set is fuzzy semi-closed.*

(iii) [5] *fuzzy pre-*$T_{\frac{1}{2}}$ *space if every* Fpg*-closed set is fuzzy pre-closed.*

2. Fuzzy Pre-semi-closed Sets

In this section, we define a new class of fuzzy generalized closed sets called a fuzzy pre-semi-closed sets and study its properties.

Definition 2.1. *Let* μ *be a fuzzy set in a* fts (X, τ)*. Then* λ *is called a fuzzy pre-semi-closed set* X *if* $sp\,\mathrm{Cl}\,\mu \leq \lambda$*, whenever* $\mu \leq \lambda$ *and* λ *is a* Fg*-open set in* X*.*

The following proposition asserts that the class of fuzzy pre-semi-closed set contains the class of fuzzy semi-pre-closed sets.

Proposition 2.1. *Every fuzzy semi-pre-closed set in a* fts (X, τ) *is fuzzy pre-semi-closed.*

Proof. Let μ be a fuzzy semi-pre-closed set in a fts (X, τ). Suppose that $\mu \leq \lambda$ and λ is a Fg-open set in X. Since $sp\,\mathrm{Cl}\,\mu = \mu$, it follows that $sp\,\mathrm{Cl}\,\mu = \mu \leq \lambda$, and hence μ is fuzzy pre-semi-closed in X. The reverse implication in the above proposition is not true as seen in the following example. ∎

Example 2.1. Consider the fts (X,τ), where $X=\{a,b,c\}$ and

$$\tau=\left\{0,1,\mu=\frac{0.7}{a}+\frac{0.3}{b}+\frac{1}{c},\quad \lambda=\frac{0.7}{a}+\frac{0}{b}+\frac{0}{c}\right\}.$$

Fuzzy closed sets in X are:

$$0,1,\mu'=\frac{0.3}{a}+\frac{0.7}{b}+\frac{0}{c},\quad \lambda'=\frac{0.3}{a}+\frac{1}{b}+\frac{1}{c}.$$

So the family of Fg-closed sets is

$$\left\{0,1,\mu',\lambda',\frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}\text{ either }\alpha_1>0.7\text{ or }\alpha_2>0.3\right\}.$$

Hence the family of Fg-open sets is

$$\left\{0,1,\mu,\lambda,\frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}\text{ either }\alpha_1<0.3\text{ or }\alpha_2<0.7\right\}.$$

Now

$$\nu=\frac{1}{a}+\frac{0.3}{b}+\frac{0}{c}$$

is not a fuzzy semi-pre-closed set in X, for Int $\nu=\lambda$ and so, Int Cl Int $\nu=$ Int $\mathrm{Cl}\,\lambda=1>\nu$.

Moreover, ν is fuzzy pre-semi-closed. Indeed, let $\nu\le\eta$ and η be Fg-open in X. Then $\eta=1$ and $sp\,\mathrm{Cl}\,\nu\le\eta$.

From the succeeding two examples, it can be seen that fuzzy pre-semi-closedness is independent from Fg-closedness, Fgs-closedness and $Fg\alpha$-closedness.

Example 2.2. Consider the fts (X,τ), where $X=\{a,b,c\}$ and

$$\tau=\left\{0,1,\lambda=\frac{0.9}{a}+\frac{0.2}{b}+\frac{0}{c}\right\}.$$

Fuzzy closed sets are:

$$0,1,\lambda'=\frac{0.1}{a}+\frac{0.8}{b}+\frac{1}{c}.$$

So the family of Fg-closed sets in X is

$$\left\{0,1,\lambda',\text{and }\frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}\text{ either }0.9<\alpha_1\text{ or }0.2<\alpha_2\right\}.$$

Hence the family of Fg-open sets is

$$\left\{0,1,\lambda,\frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}\text{ either }\alpha_1<0.1\text{ or }\alpha_2<0.8\right\}.$$

Now

$$\nu=\frac{0.9}{a}+\frac{0.1}{b}+\frac{0}{c}$$

is a fuzzy pre-semi-closed set in X, for if $\nu\le\eta$ and η is Fg-open set in X, then $\eta=1$ and hence $sp\,\mathrm{Cl}\,\nu\le\eta$. But, Int $\mathrm{Cl}\,\nu=1$, $\nu\le\lambda$ and Int $\mathrm{Cl}\,\nu=1>\lambda$, so ν is not a Fgs-closed set in X and hence by Lemma 1.3, it is neither Fg-closed nor $Fg\alpha$-closed.

Example 2.3. Consider the fts $(X, \tau) : X = \{a, b, c\}$ and

$$\tau = \left\{0, 1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}\right\}.$$

Fuzzy closed sets are:

$$0, 1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}.$$

So the family of Fg-closed sets in X is

$$\left\{0, 1, \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c} \text{ either } \alpha_1 > 0 \text{ or } \alpha_2 > 0\right\}.$$

Hence the family of Fg-open sets in X is

$$\left\{0, 1, \lambda, \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c} \text{ either } \alpha_1 < 1 \text{ or } \alpha_2 < 1\right\}.$$

Now

$$\nu = \frac{1}{a} + \frac{1}{b} + \frac{0}{c}$$

is not a fuzzy pre-semi-closed set in X; for if $\nu \leq \nu$ and ν is Fg-open set, but, Int Cl Int $\nu = 1$, and hence $sp\,\mathrm{Cl}\,\nu = 1 > \nu$. But, ν is Fg-closed set in X and hence it is $Fg\alpha$-closed and Fgs-closed.

Proposition 2.2. *Every fuzzy pre-semi-closed set in a fts (X, τ) is $Fgsp$-closed.*

Proof. Let ν be a fuzzy pre-semi-closed set in a fts (X, τ). Suppose that $\nu \leq \lambda$ and λ is a fuzzy open set in X. Then, $sp\,\mathrm{Cl}\,\nu \leq \lambda$ and λ is Fg-open in X and hence ν is $Fgsp$-closed in X. ∎

Remark 2.1. In Example 2.3, fuzzy set ν is also a $Fgsp$-closed set but not a fuzzy pre-semi-closed set. Thus, the converse of the above proposition is not true.

Proposition 2.3. *Union of two fuzzy pre-semi-closed sets in a fts need not be fuzzy pre-semi-closed.*

Proof. Consider the fts $(X, \tau) : X = \{a, b, c\}$ and

$$\tau = \left\{0, 1, \lambda = \frac{1}{a} + \frac{0.5}{b} + \frac{0}{c}\right\}.$$

Fuzzy closed sets are:

$$0, 1, \lambda' = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c}.$$

So the family of Fg-closed sets in X is

$$\left\{0, 1, \lambda', \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c} \text{ where } \alpha_3 \neq 0\right\}.$$

Hence the family of Fg-open sets in X is

$$\left\{0, 1, \lambda, \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c} \text{ where } 1 \neq \alpha_3\right\}.$$

Let

$$\mu = \frac{1}{a} + \frac{0}{b} + \frac{0}{c} \quad \text{and} \quad \nu = \frac{0}{a} + \frac{0.5}{b} + \frac{0}{c}$$

be fuzzy sets in X. Now, since Int Cl Int $\mu = 0 \leq \mu$ and Int Cl Int $\nu = 0 \leq \nu, \mu$ and ν are fuzzy semi-pre-closed sets in X and hence by Proposition 2.1, μ and ν

are fuzzy pre-semi-closed sets in X. But $\mu \vee \nu$ is not fuzzy pre-semi-closed set in X. Indeed, $\mu \vee \nu = \lambda$ and λ is Fg-open, but $sp\,\mathrm{Cl}\,\lambda = 1 > \lambda$. ∎

Proposition 2.4. *Let μ be a fuzzy set in a fts (X, τ). If μ is Fg-open and fuzzy pre-semi-closed, then μ is fuzzy semi-closed.*

Proof. Since $\mu \leq \mu$ is Fg-open, it follows that

$$\mu \vee \ \mathrm{Int} \ \ \mathrm{Cl}\ \mathrm{Int} \ \ \mu \leq \ sp\,\mathrm{Cl}\,\mu \leq \mu.$$

Hence, Int Cl Int $\mu \leq \mu$ and μ is fuzzy semi-closed. ∎

As a consequence of Lemma 1.1, Lemma 1.3, Proposition 2.2 and Proposition 2.1, we get Figure 1.

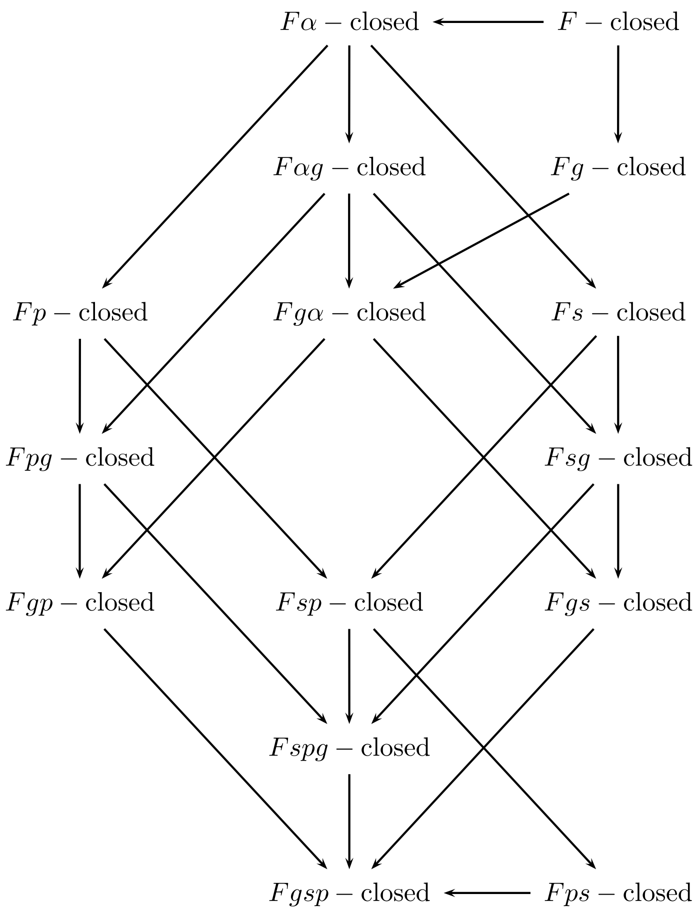

Figure 1

The following proposition characterizes fuzzy regular open sets.

Proposition 2.5. *Let μ be a fuzzy set in a fts (X, τ). Then the following are equivalent:*

(i) *μ is fuzzy regular open.*
(ii) *μ is fuzzy open and fuzzy pre-semi-closed.*
(iii) *μ is fuzzy open and $Fgsp$-closed.*

Proof. (i) $\Rightarrow$ (ii): Let μ be fuzzy regular open in a fts (X, τ). Then it is both fuzzy open and fuzzy semi-closed and hence by Lemma 1.1 and Proposition 2.1, it is fuzzy pre-semi-closed.
(ii) $\Rightarrow$ (iii): Let μ be fuzzy open and fuzzy pre-semi-closed. Then, by Proposition 2.2, it is $Fgsp$-closed.
(iii) $\Rightarrow$ (i): Let μ be fuzzy open and $Fgsp$-closed. Then, $\mu \leq \mu$ and μ is fuzzy open and so, $\mu \vee$ Int Cl Int $\mu \leq sp\,\mathrm{Cl}\,\mu \leq \mu$. Hence, Int Cl Int $\mu \leq \mu$. Since μ is fuzzy open it follows that Int Cl $\mu \leq \mu =$ Int $\mu \leq$ Int Cl μ. Hence μ is fuzzy regular open. ∎

Proposition 2.6. *Let μ be a fuzzy pre-semi-closed set in a fts (X, τ). If μ is a fuzzy set in X such that $\mu \leq \lambda \leq sp\,\mathrm{Cl}\,\mu$, then λ is also fuzzy pre-semi-closed.*

Proof. Let $\lambda \leq \eta$ and η be Fg-open in X. Then $\mu \leq \eta$ and since μ is fuzzy pre-semi-closed, it follows from Lemma 1.5(ii) that $sp\,\mathrm{Cl}\,\lambda \leq sp\,\mathrm{Cl}(sp\,\mathrm{Cl}\,\mu) = sp\,\mathrm{Cl}\,\mu \leq \eta$. Hence, λ is fuzzy pre-semi-closed. ∎

Definition 2.2. *A fuzzy set μ in a fts (X, τ) is said to be fuzzy pre-semi open if $1 - \mu$ is fuzzy pre-semi-closed.*

Proposition 2.7. *Let μ be a fuzzy pre-semi-closed set in a fts (X, τ). If λ is a fuzzy set in X such that $sp\,\mathrm{Int}\,\mu \leq \lambda \leq \mu$, then λ is also a fuzzy pre-semi open.*

Proof. Now, $1 - \mu$ is a fuzzy pre-semi-closed set in X and λ is a fuzzy set in X such that $1 - \mu \leq 1 - \lambda \leq 1 -$ $sp\,\mathrm{Int}\,\mu$. By Lemma 1.3, $1 - \mu \leq 1 - sp\,\mathrm{Int}\,\mu = sp\,\mathrm{Cl}(1 - \mu)$. Hence, by Proposition 2.6, $1 - \lambda$ is also fuzzy pre-semi-closed and λ is fuzzy pre-semi open. ∎

3. Fuzzy Pre-semi-$T_{\frac{1}{2}}$ Spaces

As applications of fuzzy pre-semi-closed sets, three fuzzy spaces namely, fuzzy pre-semi-$T_{\frac{1}{2}}$ spaces, fuzzy pre-semi-$T_{\frac{3}{4}}$ spaces and fuzzy semi-pre-$T_{\frac{1}{3}}$ space are introduced.

Definition 3.1. *A fuzzy topological space (X, τ) is called a*

(i) *fuzzy pre-semi-$T_{\frac{1}{2}}$ space if every fuzzy pre-semi-closed set in it is fuzzy semi-pre-closed,*
(ii) *fuzzy semi-pre-$T_{\frac{1}{2}}$ space if every $Fgsp$-closed set in it is fuzzy semi-pre-closed.*

Proposition 3.1. *Every fuzzy semi-pre-$T_{\frac{1}{2}}$ space is a fuzzy pre-semi-$T_{\frac{1}{2}}$ space.*

Proof. Let (X, τ) be a fuzzy semi-pre-$T_{\frac{1}{2}}$ space and λ be a fuzzy pre-semi-closed set in X. By Proposition 2.2, λ is $Fgsp$-closed set in X and hence it is fuzzy semi-pre-closed set in X. Thus, (X, τ) is fuzzy pre-semi-$T_{\frac{1}{2}}$ space. ∎

The converse of the above proposition is not valid as seen in the following example.

Example 3.1. Consider the fts $(X, \tau) : X = \{a, b, c\}$ and

$$\tau = \left\{0, 1, \lambda = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}\right\}.$$

Fuzzy closed sets are:

$$0, 1, \lambda' = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}.$$

So the family of Fg-closed sets in X is

$$\left\{0, 1, \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c}, \text{ either } \alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0\right\}.$$

Hence the family of Fg-open sets in X is

$$\left\{0, 1, \lambda, \frac{\alpha_1}{a} + \frac{\alpha_2}{b} + \frac{\alpha_3}{c}, \text{ either } \alpha_2 \neq 1 \text{ or } \alpha_3 \neq 1\right\}$$

Now

$$\nu = \frac{1}{a} + \frac{0}{b} + \frac{1}{c}$$

is not a fuzzy pre-semi-closed sets in X. Indeed, $\nu \leq \eta$ and η is Fg-open implies that $\eta = 1$ and hence, $sp\,\mathrm{Cl}\,\nu \leq \eta$. But, ν is not fuzzy semi-pre-closed; for $sp\,\mathrm{Cl}\,\nu = 1 > \nu$. Thus, X is not a fuzzy semi-pre-$T_{\frac{1}{2}}$ space. However, X is a fuzzy pre-semi-$T_{\frac{1}{2}}$ space. Indeed, every fuzzy pre-semi-closed set in X is fuzzy semi-pre-closed. For:
Case 1. Suppose that μ is Fg-open. Then μ is fuzzy pre-semi-closed implies that $sp\,\mathrm{Cl}\,\mu \leq \mu$ and hence $sp\,\mathrm{Cl}\,\mu = \mu$. So, μ is fuzzy semi-pre-closed.

Case 2. Suppose that μ is not Fg-open. Then

$$\mu = \frac{\alpha_1}{a} + \frac{1}{b} + \frac{1}{c} \quad \text{where} \quad 0 \leq \alpha_1 < 1,$$

and 1 is the only Fg-open set containing μ and hence μ is fuzzy pre-semi-closed. Moreover, μ is fuzzy semi-pre-closed. Indeed, Int Cl Int $\mu = 0 < \mu$.

Definition 3.2. *Fuzzy topological space (X, τ) is called a fuzzy semi-pre-$T_{\frac{1}{3}}$ space if every $Fgsp$-closed set in it is fuzzy pre-semi-closed.*

Proposition 3.2. *Every fuzzy semi-pre-$T_{\frac{1}{2}}$ space is a fuzzy semi-pre-$T_{\frac{1}{3}}$ space.*

Proof. Let (X, τ) be a fuzzy semi-pre-$T_{\frac{1}{2}}$ space and μ be a $Fgsp$-closed set in (X, τ). Then μ is fuzzy semi-pre-closed and hence by Proposition 2.1, is fuzzy pre-semi-closed. Thus, (X, τ) is a fuzzy semi-pre-$T_{\frac{1}{3}}$ space. It will be an interesting exercise to prove that the converse of the above proposition is not valid. ∎

Definition 3.3. *A fuzzy topological space (X, τ) is called a fuzzy pre-semi-$T_{\frac{3}{4}}$ space if every fuzzy pre-semi-closed set in it is fuzzy pre-closed.*

Proposition 3.3. *Every fuzzy pre-semi-$T_{\frac{3}{4}}$ space is a fuzzy pre-semi-$T_{\frac{1}{2}}$ space.*

Proof. Let (X, τ) be a fuzzy pre-semi-$T_{\frac{3}{4}}$ space and λ be a fuzzy pre-semi-closed set in (X, τ). Then, λ is fuzzy pre-closed. Thus, (X, τ) is a fuzzy pre-semi-$T_{\frac{1}{2}}$ space. Converse of the above proposition is not true as seen in the following example. ∎

Example 3.2. Let (X,τ) be a *fts*, where $X=\{a,b,c\}$ and

$$\tau=\left\{0,1,\lambda_1=\frac{1}{a}+\frac{0}{b}+\frac{0}{c},\quad \lambda_2=\frac{0}{a}+\frac{1}{b}+\frac{0}{c},\quad \lambda_3=\frac{1}{a}+\frac{1}{b}+\frac{0}{c}\right\}.$$

Fuzzy closed sets in (X,τ) are

$$0,1,\lambda_1'=\frac{0}{a}+\frac{1}{b}+\frac{1}{c},\quad \lambda_2'=\frac{1}{a}+\frac{0}{b}+\frac{1}{c},\quad \lambda_3'=\frac{0}{a}+\frac{0}{b}+\frac{1}{c}.$$

If μ is Fg-closed, then $\mu\le\lambda$ implies $\operatorname{Cl}\mu\le\lambda$ whenever λ is fuzzy open. Thus, Fg-closed sets in (X,τ) are:

$$0,1,\lambda_1'=\frac{0}{a}+\frac{1}{b}+\frac{1}{c},\quad \lambda_2'=\frac{1}{a}+\frac{0}{b}+\frac{1}{c},\quad \lambda_3'=\frac{0}{a}+\frac{0}{b}+\frac{1}{c}\quad \text{and}\quad \frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}$$

where $\alpha_3\neq 0$. So, the family of Fg-open sets in (X,τ) is

$$\left\{0,1,\frac{\alpha_1}{a}+\frac{\alpha_2}{b}+\frac{\alpha_3}{c}\quad \text{where}\quad \alpha_3\neq 1\right\}.$$

It is enough to prove that, if μ is not fuzzy semi-pre-closed then it is not fuzzy pre-semi-closed and there is a fuzzy pre-semi-closed set which is not pre-closed. Let $\mu\neq 0$ be a fuzzy set in X. Then,

$$\operatorname{Int}\mu=\begin{cases}0\\ \lambda_1\\ \lambda_2\\ \lambda_3\\ 1\end{cases},\quad \operatorname{Cl}\operatorname{Int}\ \mu=\begin{cases}0\\ \lambda_2'\\ \lambda_3'\\ 1\\ 1\end{cases}\quad \text{and}\quad \operatorname{Int}\operatorname{Cl}\operatorname{Int}\mu=\begin{cases}0\\ \lambda_1\\ \lambda_2\\ 1\\ 1\end{cases}$$

So, μ is not fuzzy semi-pre-closed if $\lambda_3\le\mu$. In that case, μ is also not fuzzy pre-semi-closed. For μ is Fg-open and $\mu\le\mu$. But $sp\operatorname{Cl}\mu\ge\mu\vee$ Int Cl Int $\mu=1>\mu$. Thus X is a pre-semi-$T_{\frac{1}{2}}$ space. But it is not a fuzzy pre-semi-$T_{\frac{3}{4}}$ space. Indeed,

$$\mu=\frac{1}{a}+\frac{0.3}{b}+\frac{0}{c}$$

is a fuzzy semi-pre-closed and hence it is fuzzy pre-semi-closed but it is not fuzzy pre-closed, as $\operatorname{Cl}\operatorname{Int}\ \mu\neq\mu$.

4. Conclusion

In this paper, by the introduction of fuzzy pre-semi-closed sets, we have equivalences of fuzzy regular open sets and fuzzy spaces with new separation axioms, namely, fuzzy semi-pre-$T_{\frac{1}{3}}$ space, pre-semi-$T_{\frac{3}{4}}$ space and fuzzy pre-semi-$T_{\frac{1}{2}}$ space. In crisp topology construction of counter examples is easy, but here that will be a rewarding exercises.

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