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On the Positive Integral Solutions of the Diophantine Equation $x^3 + by + 1 - xyz = 0$

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Abstract. In this paper, we prove a weaker form of a conjecture of Mohanty and Ramasamy [6] concerning the number of positive integer solutions (x, y, z) of the title equation.

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1. Introduction

The Diophantine equation

$$ax^3 + by + c - xyz = 0, \tag{1.1}$$

was studied by Mohanty, Mohanty-Ramasamy and Utz. In 1979, Mohanty [5] considered the case $a = b = c = 1$ and found all 9 positive integer solutions of this equation. In 1982, Utz [9] used Mohanty's method to determine exactly 13 positive integer solutions to this equation when $a = c = 1$, $b = 2$. In 1984, Mohanty-Ramasamy [6] studied equation (1.1) when $a = c = 1$ and b is a fixed positive integer. They proved that equation (1.1) has a finite number of positive integer solutions. In particular, if $b > 3$ is prime then this number is odd, and this number is exactly 28 when $b = 3$. Moreover, they showed (see [8]) that the more general Diophantine equation

$$ax^3 + by + c - xyz = 0 \tag{1.2}$$

has only finitely many positive integral solutions if $\gcd(ab, c) = 1$ and c is squarefree.

In this note, we consider the case $a = c = 1$ and b a fixed positive integer; i.e., the Diophantine equation

$$x^3 + by + 1 - xyz = 0. \tag{1.3}$$

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Let $N(b)$ be the number of positive integral solutions (x, y, z). At the end of [6], the authors suggest the following conjecture.

Conjecture 1.1. *The inequality* $N(b) \le 8b + 15$ *holds for all positive integers* b. *If* b *is even, then the better inequality* $N(b) \le 4b + 15$ *holds.*

The aim of this note is to confirm a weaker form of Conjecture 1.1 for large b. Namely, we have the following result.

Theorem 1.1. *The estimate*

$$N(b) \le 11b + o(b)$$

holds as $b \to \infty$.

Remark 1.1. In [6], the authors gave the value of $N(b)$, for $1 \le b \le 12$. In fact, we have

$$N(1) = 9,\quad N(2) = 13,\quad N(3) = 28,\quad N(4) = 20,\quad N(5) = 55,\quad N(6) = 21,$$
$$N(7) = 61,\quad N(8) = 45,\quad N(9) = 43,\quad N(10) = 39,\quad N(11) = 97,\quad N(12) = 43.$$

In Section 2, we give the proof of the above theorem by considering separately four cases, namely x small, $\gcd(b, z)$ large, z small, and none of the above, respectively.

2. The Proof

The following description of all positive integer solutions of equation (1.3) appears in Theorem 2 in [6].

Lemma 2.1. *If* $b > 7$ *and* (x, y, z) *is a positive integer solution of equation* (1.3), *then* $z < b$ *and*

$$xz - b \,\Big|\, \frac{b^3 + z^3}{\gcd(b, z)^2}$$

except for at most three solutions (x, y, z). *Two of these exceptional solutions are* $(x, y, z) = (1, 1, b + 2),\ (1, 2, b + 1)$. *The third one appears only when* $b = x^2 + x - 1$ *and in that case* $z = x + 1$.

For $z = 1, \ldots, b - 1$, we let

$$f_b(z) = \#\left\{ d \,\Big|\, \frac{b^3 + z^3}{\gcd(b, z)^2} \ :\ d \equiv -b \pmod{z} \right\}.$$

Lemma 2.1 above shows that

$$N(b) \le 3 + \sum_{z=1}^{b-1} f_b(z). \tag{2.1}$$

Our proof uses known results concerning the average value of the number of divisors of a given polynomial together with a result of Lenstra [4] concerning the number of divisors in residue classes with a large modulus.

Case 1. *x is small.*

Put $B = b/(\log b)^{10}$. Assume that $x \le B$. From equation (1.3), it follows that $y \mid x^3 + 1$, and that if x and y are determined, then z is uniquely determined. Hence, denoting by $\tau(m)$ the number of divisors of the positive integer m, we get that the number of solutions in this category does not exceed

$$N_1(b) = \sum_{x \le B} \tau(x^3+1). \tag{2.2}$$

It is well-known that if $f(X) \in \mathbb{Z}[X]$ is a fixed primitive polynomial having nonzero discriminant (i.e., no double roots) and exactly k irreducible factors (all of degree ≥ 1 and integer coefficients), then

$$\sum_{n \le T} \tau(|f(n)|) \ll T(\log T)^k, \tag{2.3}$$

where the implied constant above depends on the polynomial $f(X)$. Here, we make the convention that $\tau(0) = 0$. Indeed, such a result was first proved by Erdős in [2] when $f(X)$ is irreducible (see also [3]); i.e., when $k = 1$, and in the general form stated in the bound (2.3) above by Ennola in [1]. Applying estimate (2.3) to the sum appearing in the right hand side of (2.2) and using the fact that $k = 2$ when $F(X) = X^3 + 1$, we get

$$N_1(b) \ll B(\log B)^2 \ll \frac{b}{(\log b)^8} = o(b) \tag{2.4}$$

as $b \to \infty$.

Case 2. $\gcd(b, z)$ *is large.*

Let $d = \gcd(b, z)$ and write $z = dz_1,\ b = db_1$. Assume that $d > b^{1/10}$. Then, by Lemma 2.1, we have $xz_1 - b_1 \mid z_1^3 + b_1^3$. If z_1 is fixed, then the number of values of $xz_1 - b_1$ does not exceed $\tau(z_1^3 + b_1^3) \le (2b^3)^{o(1)} = b^{o(1)}$ as $b \to \infty$. Thus, the number of solutions in this category does not exceed

$$\begin{aligned} N_2(b) &= \sum_{\substack{d \mid b \\ d > b^{1/10}}} \sum_{\substack{z_1 \le b/d \\ \gcd(z_1, b/d)=1}} \tau(z_1^3 + b_1^3) \le \sum_{\substack{d \mid b \\ d > b^{1/10}}} \sum_{z_1 \le b/d} b^{o(1)} \\ &\le \sum_{\substack{d \mid b \\ d > b^{1/10}}} \frac{b^{1+o(1)}}{d} \le \frac{b^{1+o(1)}\tau(b)}{b^{1/10}} = b^{9/10+o(1)} = o(b) \end{aligned} \tag{2.5}$$

as $b \to \infty$.

Case 3. z *is small.*

We assume that $z \le B$. We keep the notations from the previous cases. We assume that $d \le b^{1/10}$. Fix $d \mid b$. Then $xz_1 - b_1$ is a divisor of $z_1^3 + b_1^3$. Thus, the number of numbers in this category does not exceed

$$N_3(b) \le \sum_{\substack{d \mid b \\ d \le b^{1/10}}} \sum_{\substack{z_1 \le B/d \\ \gcd(z_1, b/d)=1}} \tau(z_1^3 + b_1^3). \tag{2.6}$$

We split the inner sum at $z_1 = b^{1/2}$. The contribution from the solutions with $z_1 \le b^{1/2}$ is

$$\begin{aligned} N_3(b)' &\le \sum_{\substack{d\,|\,b\\ d\le b^{1/10}}}\sum_{z_1\le b^{1/2}}\tau(z_1^3+b_1^3)\le \sum_{\substack{d\,|\,b\\ d\le b^{1/10}}}\sum_{z_1\le b^{1/2}} b^{o(1)}\\ &\le b^{o(1)}\sum_{d\le b^{1/10}}\sum_{z_1\le b^{1/2}}1\le b^{1/2+1/10+o(1)}=o(b) \qquad (2.7)\end{aligned}$$

as $b\to\infty$. We now assume that $z_1 \ge b^{1/2} \ge b_1^{1/2}$. Fix d. Let $\tau_{b_1}(m)$ be the function whose value at m is $\tau(m)$ if m is coprime to b_1 and 0 otherwise. Clearly, $\tau_{b_1}(m)$ is multiplicative. Write

$$\rho_{b_1}(m) = \#\{n \pmod m : n^3 + b_1^3 \equiv 0 \pmod m\},$$

when m and b_1 are coprime, and $\rho_{b_1}(m) = 0$ otherwise. A result of Nair from [7], shows that since $z_1 > b^{1/2}$, we have

$$\begin{aligned} M_d(b) &= \sum_{\substack{z_1\le B/d\\ \gcd(z_1,b/d)=1}}\tau(z_1^3+b_1^3)=\sum_{z_1\le B/d}\tau_{b_1}(z_1^3+b_1^3)\\ &\ll \frac{B}{d}\prod_{p\le B/d}\left(1-\frac{\rho_{b_1}(p)}{p}\right)\sum_{m\le B/d}\frac{\rho_{b_1}(m)\tau_{b_1}(m)}{m}. \qquad (2.8)\end{aligned}$$

Clearly, $1 \le \rho_{b_1}(p^\alpha) \le 3$ for all prime powers p^α when p does not divide b_1. Thus,

$$\begin{aligned} \prod_{p\le B/d}\left(1-\frac{\rho_{b_1}(p)}{p}\right) &\le \prod_{\substack{p\le b^{9/10}/(\log b)^{10}\\ p\nmid b}}\left(1-\frac{1}{p}\right)\\ &\le \frac{b}{\phi(b)}\prod_{p\le b^{9/10}/(\log b)^{10}}\left(1-\frac{1}{p}\right)\\ &\ll \frac{\log\log b}{\log b}, \qquad (2.9)\end{aligned}$$

where the last estimate follows by Mertens's formula and by the minimal order of the Euler function in the interval $[1,b]$. Furthermore, since both ρ_{b_1} and τ_{b_1} are multiplicative and nonnegative, we have that

$$\begin{aligned} \sum_{z\le B/d}\frac{\rho_{b_1}(m)\tau_{b_1}(m)}{m} &\le \prod_{\substack{p\le b/d\\ p\nmid b_1}}\left(\sum_{\alpha\ge 0}\frac{\rho_{b_1}(p^\alpha)\tau(p^\alpha)}{p^\alpha}\right)\\ &\le \prod_{p\le b}\left(1+\frac{6}{p}+\sum_{\alpha\ge 2}\frac{3(\alpha+1)}{p^\alpha}\right)\\ &\le \exp\left(6\sum_{p\le b}\frac{1}{p}+O\left(\sum_{p\ge 2}\frac{1}{p^2}\right)\right)\\ &= \exp\left(6\log\log b+O(1)\right)\ll(\log b)^6. \qquad (2.10)\end{aligned}$$

Inserting estimates (2.9) and (2.10) into estimate (2.8), we get

$$M_d(b) \ll \frac{B}{d}\left(\frac{\log\log b}{\log b}\right)(\log b)^6 \ll \frac{B(\log b)^5 \log\log b}{d}.$$

Summing the above estimate up over all $d \mid b$ with $d < b^{1/10}$, we get that

$$\begin{aligned} N_3(b)'' &\leq \sum_{\substack{d \mid d \\ d \leq b^{1/10}}} M_d(b) \ll B(\log b)^5 \log\log b \sum_{d \mid b} \frac{1}{d} \\ &\ll B(\log b)^5 (\log\log b)\, \frac{\sigma(b)}{b} \ll B(\log b)^5 (\log\log b)^2, \end{aligned}$$

where we used the maximal order of the sum of divisors function $\sigma(b)$ on the interval $[1, b]$. Recalling the definition of B, we get that

$$N_3(b)'' \ll \frac{b(\log\log b)^2}{(\log b)^5} = o(b) \tag{2.11}$$

as $b \to \infty$. Thus, estimates (2.6) together with estimates (2.7) and (2.11) lead to

$$N_3(b) = N_3(b)' + N_3(b)'' = o(b) \tag{2.12}$$

as $b \to \infty$.

Case 4. *The remaining solutions.*

For the remaining solutions, we fix again d. We have $d \leq b^{1/10}$, $z \geq B$, therefore $z_1 \geq B/d$. Now fix $z_1 \leq b/d$ such that z_1 is coprime to b/d. Then $xz_1 - b_1$ is a divisor of $z_1^3 + b_1^3$ which is in the fixed arithmetical progression $-b_1$ modulo z_1. Note that $z_1^3 + b_1^3 \leq 2(b/d)^3$. Since $z_1 \geq B/d$, it follows that for all $\varepsilon > 0$, there exists b_ε such that if $b > b_\varepsilon$, then $z_1 > (b/d)^{1/(3+\varepsilon)}$, uniformly in the divisor d. In [4], Lenstra showed that for each $\alpha > 1/4$, there exists a constant $c(\alpha)$ such that whenever $1 \leq r < s \leq n$ are such that r and s are coprime and $s > n^\alpha$, then n has at most $c(\alpha)$ divisors $d \equiv r \pmod{s}$. Using this with $\alpha(\varepsilon) = 1/(3+\varepsilon)$, we get that $xz_1 - b_1$ (hence, x) can be chosen it at most $c(\alpha(\varepsilon))$ ways once b is sufficiently large. Lenstra also showed that $c(1/3)$ can be chosen to be 11. This shows that if b is sufficiently large, then there are at most 11 divisors of $z_1^3 + b_1^3$ of the form $xz_1 - b_1$ in this range of the solutions. Since z_1 can be chosen in at most $\phi(b/d)$ ways, we get that the number of solutions in this case does not exceed

$$N_4(b) \leq \sum_{\substack{d \mid b \\ d \leq b^{1/10}}} 11\phi(b/d) \leq 11 \sum_{d \mid b} \phi(b/d) = 11b.$$

This and the previous estimates (2.4), (2.5) and (2.12) complete the proof of the theorem.

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