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On the Spectra of Some Non-Normal Operators

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Abstract. In this paper, we prove the following:
(1) If T is invertible ω-hyponormal completely non-normal, then the point spectrum is empty.
(2) If T_1 and T_2 are injective ω-hyponormal and if T and S are quasisimilar, then they have the same spectra and essential spectra.
(3) If T is (p,k)-quasihyponormal operator, then $\sigma_{jp}(T)-\{0\}=\sigma_{ap}(T)-\{0\}$.
(4) If $T^*, S \in \mathbf{B}(\mathcal{H})$ are injective (p,k)-quasihyponormal operator, and if $XT = SX$, where $X \in \mathbf{B}(\mathcal{H})$ is an invertible, then there exists a unitary operator U such that $UT = SU$ and hence T and S are normal operators.

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1. Introduction

Let $\mathcal{H}$ be infinite dimensional complex Hilbert spaces with inner product $\langle .,. \rangle$ and let $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be positive, in symbol $T \geq 0$ if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For a positive operators A and B, write $A \geq B$ if $A - B \geq 0$. If A and B are invertible and positive operators, it is well known that $A \geq B$ implies that $\log A \geq \log B$. However [2], $\log A \geq \log B$ does not necessarily imply $A \geq B$. A result due to Ando [5] states that for invertible positive operators A and B, $\log A \geq \log B$ if and only if $A^r \geq (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{1}{2}}$ for all $r \geq 0$. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and $\ker(T) = \ker(U) = \ker(|T|)$, where $\ker(S)$ denotes the kernel of operator S.

Recall [1, 2, 7, 9] that an operator $T \in \mathbf{B}(\mathcal{H})$ is called p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$ for $p \in (0,1]$. If $p = 1$, T is said to be hyponormal and if $p = \frac{1}{2}$, T is said to be semi-hyponormal. The Löwner-Heinz inequality implies that if T is p-hyponormal then its q-hyponormal for any $0 < q \leq p$ and $U|T|^p$ is hyponormal. An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$.

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Following [20], Xia introduced the class of semi-hyponormal operators and obtained results analogous to those of hyponormal operators. Aluthge [1] studied p-hyponormal operators for $0 < p \leq 1$. In particular he defined the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ which is called the Aluthge transformation and the operator $\widetilde{\widetilde{T}} = |\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$. An operator T is said to be ω-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T^*}|$.

The class of ω-hyponormal operators were introduced, and their properties were studied in [3]. In particular, it was shown in [3] that the class of ω-hyponormal operators contains both the p-hyponormal and log-hyponormal operators. If an operator T is p-hyponormal, then $\ker T \subset \ker T^*$; and if T is log-hyponormal, then $\ker T = \ker T^*$. However, if T is ω-hyponormal, it is not known whether the kernel condition $\ker T \subset \ker T^*$ holds. For $T \in \mathbf{B}(\mathcal{H})$, $\sigma_p(T)$ denotes the point spectrum of T,*i.e.*, the set of its eigenvalues. Let $\sigma_{jp}(T)$ denotes the joint point spectrum of T. We note that $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector x such that $Tx = \lambda x$, $T^*x = \overline{\lambda}x$. It is evident that $\sigma_{jp}(T) \subset \sigma_p(T)$. It is well known that, if T is normal, then $\sigma_{jp}(T) = \sigma_p(T)$.

If $T = U|T|$ is the polar decomposition of T and $\lambda = |\lambda|e^{i\theta}$ be the complex number, $|\lambda| > 0$, $|e^{i\theta}| = 1$. Then $\lambda \in \sigma_{jp}(T)$ if and only if there exist a non-zero vector x such that $Ux = e^{i\theta}$, $|T|x = |\lambda|x$. Let $\sigma_{ap}(T)$ denotes the approximate point spectrum of T,*i.e.*, the set of all complex numbers λ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_n \|(T - \lambda)x_n\| = 0$. It is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$. Let $\sigma_{jap}(T)$ be the joint approximate point spectrum of T, then $\lambda \in \sigma_{jap}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $\lim\limits_{n\longrightarrow\infty} \|(T - \lambda)x_n\| = \lim\limits_{n\longrightarrow\infty} \|(T^* - \overline{\lambda})x_n\| = 0$. It is evident that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$ for all $T \in \mathbf{B}(\mathcal{H})$. It is well known that, for a normal operator T, $\sigma_{jap}(T) = \sigma_{ap}(T) = \sigma(T)$.

Following [15] an operator $X \in \mathbf{B}(\mathcal{H})$ is called a quasiaffinity if X is injective and has dense range. Two operators T, S are said to be quasisimilar if there exists quasiaffinities X and Y such that $TX = XS$ and $YT = SY$. Also an operator T is said to be pure if there exists no non-trivial reducing subspace N of $\mathcal{H}$ such that the restriction of $T|_N$ is normal and is completely hyponormal if it is pure.

2. ω-hyponormal Operators

Following [20], every operator $T \in \mathbf{B}(\mathcal{H})$ has a direct sum decomposition $T = T_1 \oplus T_2$, where T_1 and T_2 are normal and pure parts, respectively.

Theorem 2.1. *Let T be invertible ω-hyponormal and completely non-normal ω-hyponormal operator. Then the point spectrum of T is empty.*

Proof. Suppose $T = U|T|$ is the polar decomposition of T, $\lambda = \rho e^{i\theta}$, $\rho \geq 0$, $|e^{i\theta}| = 1$. It follows from [3, Corollary 3.3] that $\sigma_{jp}(T) = \sigma_p(T)$, it follows there exists $x \neq 0$ such that $Ux = e^{i\theta}$, $|T|x = \rho x$. Thus, the one dimensional vector space $M = \{\lambda x | \lambda \in \mathbb{C}\}$ reduces T, and the restriction of T to M is normal, which contradicts the assumption that T is completely non-normal, hence $\sigma_p(T) = \phi$. ∎

Theorem 2.2. *If $T^* \in \mathbf{B}(\mathcal{H})$ is p-hyponormal, $S \in \mathbf{B}(\mathcal{H})$ is ω-hyponormal and $XT = SX$ for $X \in \mathbf{B}(\mathcal{H})$ is quasiaffinity, then $XT^* = S^*X$.*

Proof. Since by assumption $XT = SX$, we can see that $(\ker X)^{\perp}$and $\overline{ranX}$ are invariant of T^* and S respectively. Then $T^*|_{(\ker X)^{\perp}}$ is p-hyponormal and $S|_{\overline{ranX}}$ is also ω-hyponormal. Now consider the decomposition $\mathcal{H} = (\ker X)^{\perp} \oplus \ker X$ and $\mathcal{H} = \overline{ranX} \oplus (\overline{ranX})^{\perp}$. Then we have the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where T_1^* is p-hyponormal, S_1 is ω-hyponormal and X_1 is injective with dense range. Therefore we have $X_1T_1x = XTx = SXx = S_1X_1x$ for $x \in (\ker X)^{\perp}$. That is, $X_1T_1 = S_1X_1$ and T_1 and S_1 are normal by [3, Theorem 2.4]. By Fuglede-Putnam theorem we have $X_1T_1^* = S_1^*X_1$. Therefore $(\ker X)^{\perp}$ and $(\overline{ranX})$ reduces T^* and S, respectively. Hence we obtain the $XT^* = S^*X$. ∎

Theorem 2.3. *Let $T \in \mathbf{B}(\mathcal{H})$ be ω-hyponormal and $N \in \mathbf{B}(\mathcal{H})$ be a normal operator. If $X \in \mathbf{B}(\mathcal{H})$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof. Since $TX = XN$ and X has dense range, we have $\overline{X\overline{ran(N)}} = \overline{T}$. If we denote the restriction of X to $\overline{ranN}$ by X_1, then $X_1 : \overline{ran(N)} \longrightarrow \overline{ran(T)}$ has dense range and for every $x \in \overline{ran(N)}$

$$X_1Nx = XNx = TXx = TX_1x$$

so that $X_1N = TX_1$. Since T is ω-hyponormal, it follows from [3, theorem 2.4] that $\widetilde{T}$ is semi-hyponormal and $\widetilde{\widetilde{T}}$ is hyponormal and hence there is a quasiaffinity Y such that $\widetilde{\widetilde{T}}Y = YT$. Thus we have

$$\widetilde{\widetilde{T}}YX_1 = YTX_1 = YX_1N.$$

Since YX_1 has dense range, $\widetilde{\widetilde{T}}$ is normal, and so T is normal. ∎

Lemma 2.1. *([3]) Let $T \in \mathbf{B}(\mathcal{H})$ be ω-hyponormal and M be invariant subspace, then the restriction $T|_M$ of T to M is also ω-hyponormal.*

Theorem 2.4. *Let $T_i \in \mathbf{B}(\mathcal{H})$ $(i = 1, 2)$ be injective ω-hyponormal such that T_1 and T_2 are quasisimilar and let $T_i = N_i \oplus V_i$ on $\mathcal{H} \oplus \mathcal{H}$, where N_i and V_i are the normal and pure parts of T_i, respectively. Then N_1 and N_2 are unitarily equivalent and there exists $X^* \in \mathbf{B}(\mathcal{H})$ and $Y^* \in \mathbf{B}(\mathcal{H})$ with dense range such that $V_1X^* = X^*V_2$ and $Y^*V_1 = V_2Y^*$.*

Proof. There exists quasiaffinities $X \in \mathbf{B}(\mathcal{H})$ and $Y \in \mathbf{B}(\mathcal{H})$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$. Let

$$X := \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, Y := \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix},$$

then we show that $X^* = X_4$ and $Y^* = Y_4$. By simple matrix calculation shows that $V_1X_3 = X_3N_2$ and $V_2Y_3 = Y_3N_1$. We claim that $X_3 = Y_3 = 0$. Indeed, letting $M = \overline{ranX_3}$, the subspace M is invariant under V_1. So let $V_1' = V_1|_M$ and let $X_3' : \mathcal{H} \longrightarrow M$ be defined by $X_3'x = X_3x$ for each $x \in \mathcal{H}$. Since V_1' is ω-hyponormal, then by Lemma 2.1, X_3' has dense range, and $V_1'X_3' = X_3'N_2$. Hence V_1' is normal by Theorem 2.3. Thus M reduces V_1. Since, however, V_1 is pure, we have that $M = \{0\}$, and hence $X_3 = 0$. Similarly, we have $Y_3 = 0$. Thus it follows that X_1 and Y_1 are injective. Since $N_1X_1 = X_1N_2$ and $Y_1N_1 = N_2Y_1$, we have that N_1 and N_2 are unitarily equivalent. Also, we can notice that X_4, Y_4 have dense ranges and $V_1X_4 = X_4V_2$ and $Y_4V_1 = V_2Y_4$. Hence the proof is complete. ∎

Theorem 2.5. *Let $T_1 \in \mathbf{B}(\mathcal{H})$ and $T_2 \in \mathbf{B}(\mathcal{H})$ be injective ω-hyponormal operators. If T and S are quasisimilar, then they have same spectra and essential spectra.*

Proof. Let $T_i = N_i \oplus V_i$ on $\mathcal{H} \oplus \mathcal{H}$, where N_i, V_i are the normal and pure parts of T_i $(i = 1, 2)$. Since N_1 and N_2 are unitarily equivalent by Theorem 2.4, we have $\sigma(N_1) = \sigma(N_2)$ and $\sigma_e(N_1) = \sigma_e(N_2)$. Also, since there exists operators $X^* \in \mathbf{B}(\mathcal{H})$ and $Y^* \in \mathbf{B}(\mathcal{H})$ having dense ranges such that $V_1X^* = X^*V_2$ and $Y^*V_1 = V_2Y^*$. Now, since V_1 and V_2 are pure, then V_1 and V_2 are completely hyponormal. So we have $\sigma(V_1) = \sigma(V_2)$ and $\sigma_e(V_1) = \sigma_e(V_2)$. Hence we have

$$\sigma(T_1) = \sigma(T_2), \quad \sigma_e(T_1) = \sigma_e(T_2).$$ ∎

Corollary 2.1. *Let $T_1 \in \mathbf{B}(\mathcal{H})$ be injective ω-hyponormal operator and $T_2 \in \mathbf{B}(\mathcal{H})$ be an isometry. Assume that there exists quasiaffinities X and Y such that $T_1X = XT_2$ and $YT_1 = T_2Y$, if either X or Y is compact, then T_1 and T_2 are unitary operators and unitarily equivalent.*

Proof. Since T_1 and T_2 have the same spectra by Theorem 2.5, we have

$$\sigma(T_1) = \sigma(T_2) \subseteq \overline{D},$$

where $\overline{D}$ is the closed unit disc. Since T_1 is normaliod, we have $\|T_1\| \leq 1$ (*i.e.* T_1 is a contraction). Assume Y is compact, apply [6, Theorem 2] to the equation operator $YT_1 = T_2Y$ to conclude T_2 is unitary and apply [6, Theorem 1] to the equation $T_1X = XT_2$ to obtain T_1 is unitary. Hence the result is proved. ∎

Theorem 2.6. *If T is ω-hyponormal operator, then $\sigma_\omega(T) = \sigma(T) - \pi_{00}(T)$, where $\pi_{00}(T)$ is the isolated eigenvalues of finite multiplicity.*

Proof. Since T is ω-hyponormal operator, then by [3, Theorem 2.4] $\widetilde{T}$ is semi-hyponormal, then by [11, Corollary 11] we have $\sigma_\omega(\widetilde{T}) = \sigma(\widetilde{T}) - \pi_{00}(\widetilde{T})$. Now by [3, Corollary 2.3] we have $\sigma(T) = \sigma(\widetilde{T})$ and since every isolated point of T is an eigenvalue, we have $\sigma_\omega(\widetilde{T}) = \sigma(T) - \pi_{00}(T)$. Put $r = 1$ in the [3, Theorem 2.4], then we have $\sigma_\omega(\widetilde{T}) = \sigma_\omega(T)$. So the result is proved. ∎

3. Class A Operators

Recall [18] that a bounded operator T is said to be class A (or belong to class A) if $|T^2| \geq |T|^2$.

Lemma 3.1. [*Berberian's Techniques*] [20] *Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\widehat{\mathcal{H}} \supset \mathcal{H}$ and a map $\psi : \mathbf{B}(\mathcal{H}) \longrightarrow \mathbf{B}(\widehat{\mathcal{H}})$ such that*

(1) *ψ is an isometric $*$-algebraic isomorphism preserving the order,* i.e., *$\psi(T^*) = (\psi(T))^*$, $\psi(I) = I$, $\psi(\alpha T + \beta S) = \alpha\psi(T) + \beta\psi(S)$, $\psi(TS) = \psi(T)\psi(S)$, $\|\psi(T)\| = \|T\|$ and $\psi(T) \leq \psi(S)$ whenever $T \leq S$ for all $T, S \in \mathbf{B}(\mathcal{H})$, where $\alpha, \beta \in \mathbb{C}$.*

(2) $\sigma(T) = \sigma(\psi(T))$, $\sigma_{ap}(T) = \sigma_{ap}(\psi(T)) = \sigma_p(\psi(T))$, $\sigma_{jp}(T) = \sigma_{jp}(\psi(T))$ *and* $\sigma_{jp}(\psi(T)) = \sigma_{jap}(\psi(T))$.

Theorem 3.1. *If T is class A operator, then $\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\}$*

Proof. It is enough to show that $\sigma_{jp}(T) - \{0\} \supseteq \sigma_p(T) - \{0\}$. Let $\lambda \in \sigma_p(T) - \{0\}$, then it follows from [18] that there exists a non-zero vector x such that $(T - \lambda)x = 0$. But T is class A, by [19, Lemma 8] $(T^* - \overline{\lambda})x = 0$. This implies that $\lambda \in \sigma_{jp}(T)$. ∎

Theorem 3.2. *If T is a class A, then $\sigma_{jap}(T) - \{0\} = \sigma_{ap}(T) - \{0\}$.*

Proof. Let ψ be the representation of Lemma 3.1. First, we show that $\psi(T)$ is class A.

$$\begin{aligned} |\psi(T)|^2 &= |\psi(T)||\psi(T)| \\ &= \psi(|T|)\psi(|T|) \\ &= \psi(|T|^2) \\ &\leq |\psi(T^2)| && \text{(since } T \text{ belong to class } A\text{)} \\ &= |(\psi(T))^2|. \end{aligned}$$

Now by Berberian techniques, we have

$$\begin{aligned} \sigma_{ap}(T) - \{0\} &= \sigma_{ap}(\psi(T)) - \{0\} \\ &= \sigma_p(\psi(T)) - \{0\} \\ &= \sigma_{jp}(\psi(T)) - \{0\} && \text{(by Theorem 3.1)} \\ &= \sigma_{jap}(T) - \{0\}. \end{aligned}$$ ∎

Corollary 3.1. *If T is an invertible class A operator, then $\sigma_{ap}(T) = \sigma_{jap}(T)$.*

Theorem 3.3. *IF T is class A operator, then $\sigma(T) - \{0\} = \{\lambda : \overline{\lambda} \in \sigma_{ap}(T^*)\} - \{0\}$.*

Proof. Note that, for any operator $T \in \mathbf{B}(\mathcal{H})$, the equality $\sigma(T) = \sigma_p(T) \cup \{\lambda : \overline{\lambda} \in \sigma_{ap}(T^*)\} - \{0\}$ holds. If T is class A, then Theorem 3.1 implies $\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\}$. Since $\sigma_p(T) - \{0\} = \sigma_{jp}(T) - \{0\} \subset \{\lambda : \overline{\lambda}(T^*)\} - \{0\}$. Since $\sigma_p(T^*) \subset \sigma_{ap}(T^*)$, the result follows. ∎

Theorem 3.4. *Let T be a class A on a separable Hilbert space $\mathcal{H}$. Then the numerical range of T has at most a countable number of extreme points.*

Proof. To every extreme point there corresponds an eigenvalue, and we know that distinct eigenvalues corresponds orthogonal eigenvectors. Since the space is separable, we are done. ∎

4. (p, k)-quasihyponormal Operators

Definition 4.1. [14] *A bounded operator T is said to be (p, k)-quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, where $p \in (0, 1]$ and k is a positive integer. Especially, when $p = 1$, $k = 1$ and $p = k = 1$, T is called k-quasihyponormal, p-quasihyponormal, quasihyponormal, respectively.*

Lemma 4.1. [14] *Let T be (p,k)-quasihyponormal operator, then T has a matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{ran(T^k)} \oplus \ker T^{*k}$, where T_1 is p-hyponormal on $\overline{ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Lemma 4.2. [17] *If T is (p,k)-quasihyponormal operator and $\lambda \in \mathbb{C} - \{0\}$ and $(T-\lambda)x = 0$ for some $x \in \mathcal{H}$. Then $(T^* - \overline{\lambda})x = 0$.*

Corollary 4.1. *If T is (p,k)-quasihyponormal operator, then $\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\}$.*

Theorem 4.1. *If T is (p,k)-quasihyponormal operator, then $\sigma_{jp}(T) - \{0\} = \sigma_{ap}(T) - \{0\}$.*

Proof. Let ψ be the representation of Berberian. First, we show that $\psi(T)$ is (p,k)-quasihyponormal.

$$\begin{aligned}(\psi(T))^{*k}[(\psi(T)^*\psi(T))^p - (\psi(T)\psi(T)^*)^p](\psi(T))^k &= \psi(T^{*k})[(\psi(T^*)\psi(T))^p \\ &\quad - (\psi(T)\psi(T^*))^p]\psi(T^k) \\ &= \psi(T^{*k})[(\psi(T^*T)^p) \\ &\quad - (\psi(TT^*)^p)]\psi(T^k) \\ &= \psi[T^{*k}((T^*T)^p - (TT^*)^p)T^k].\end{aligned}$$

But T is (p,k)-quasihyponormal, then $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$. So

$$\psi(T^{*k}((T^*T)^p - (TT^*)^p)T^k) \geq 0.$$

Thus $\psi(T)$ is (p,k)-quasihyponormal. Now

$$\begin{aligned}\sigma_a(T) - \{0\} &= \sigma_a(\psi(T)) - \{0\} \\ &= \sigma_p(\psi(T)) - \{0\} \\ &= \sigma_{jp}(\psi(T)) - \{0\} \qquad \text{(by Corollary 4.1)} \\ &= \sigma_{jap}(T) - \{0\}.\end{aligned}$$

∎

Corollary 4.2. *If T is an invertible (p,k)-quasihyponormal, then*

$$\sigma_{jap}(T) = \sigma_{ap}(T).$$

Definition 4.2. [8, exercise 2, page 349] *The compression spectrum of T, denoted by $\sigma_c(T)$, is*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma_p(T^*)\}.$$

Corollary 4.3. *If T is (p,k)-quasihyponormal operator, then*

$$\sigma(T) - \{0\} = \sigma_c(T) - \{0\}.$$

Proof. Note that, for any operator $T \in \mathbf{B}(\mathcal{H})$ the equality $\sigma(T) - \{0\} = \sigma_p(T) \cup \sigma_c(T) - \{0\}$ holds. If T is (p,k)-quasihyponormal, then Corollary 4.1 implies that $\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\} \subseteq \sigma_c(T) - \{0\}$. Since $\sigma_p(T^*) \subset \sigma_{ap}(T^*)$, the result follows. ∎

Theorem 4.2. *If T and $S \in \mathbf{B}(\mathcal{H})$ are quasisimilar (p,k)-quasihyponormal operators, then $\sigma_\omega(T) = \sigma_\omega(S)$.*

Proof. Suppose that $X, Y \in \mathbf{B}(\mathcal{H})$ and are injective operators with dense range such that $XT = SX$ and $TY = YS$. If the range of T^k is dense, then $S^kX = XT^k$ implies that the range of S^k is also dense. Therefore T and S are quasisimilar p-hyponormal operators, and hence the result follows from [22, Corollary 12]. If instead the range of T^k is not dense, the $T^kX = XS^k$ implies that the range of S^k is not dense. Therefore by Lemma 4.1, T and S have the following matrix representations: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\overline{ran(T^k)} \oplus \ker(T^{*k})$ and $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\overline{ran(S^k)} \oplus \ker(S^{*k})$, where T_1 and S_1 are p-hyponormal operators and $T_3^k = 0$ and $S_3^k = 0$. Since quasisimilar p-hyponormal have equal Weyl's spectra. It suffices to show:

(i) T_1 and S_1 are quasisimilar,

(ii) Domain (T_3)=$\{0\}$ if and only if Domain (S_3)=$\{0\}$.

Observe that

$$XT^k = SXT^{k-1} = S^2XT^{k-2} = \cdots = S^kX \tag{4.1}$$

$$YT^k = SYT^{k-1} = S^2YT^{k-2} = \cdots = S^kY \tag{4.2}$$

If we denote the $X_1 : \overline{ran(T^k)} \longrightarrow \overline{ran(S^k)}$ and $Y_1 : \overline{ran(S^k)} \longrightarrow \overline{ran(T^k)}$, then X_1 and Y_1 are injective and have dense range. Now for any $x \in \overline{ran(T^k)}$, $X_1T_1x = XTx = SXx = S_1X_1x$ and for any $y \in \overline{ran(S^k)}$, $Y_1S_1y = YSy = TYy = T_1Y_1x$. Hence T_1 and S_1 are quasisimilar. Assume that $T^{*k}x = 0$ for $x \neq 0$ in $\mathcal{H}$. Then by Equations 4.1, we have that $S^{*k}Y^*x = 0$. Since Y^* is one to one, we have the domain (S_3)=$\{0\}$ implies domain(T_3)=$\{0\}$, and similarly domain(T_3)=$\{0\}$ implies domain (S_3)=$\{0\}$. ∎

In 1976, Stampfli and Wadhwa [16] showed that if $T^* \in \mathbf{B}(\mathcal{H})$ is hyponormal, $S \in \mathbf{B}(\mathcal{H})$ is dominant, $X \in \mathbf{B}(\mathcal{H})$ is injective and has dense range, and if $XT = SX$, then T and S are normal. on the other hand, in 1981, Gupta and Ramanujan [10] showed that if $T \in \mathbf{B}(\mathcal{H})$ is k-quasihyponormal operator and $S \in \mathbf{B}(\mathcal{H})$ is normal operator for which $TY = YS$ where $Y \in \mathbf{B}(\mathcal{H})$ is injective with dense range, then T is normal operator unitarily equivalent to S. In the following theorem, we extend the result of Gupta and Ramanujan to the class of (p,k)-quasihyponormal operators, we need the following lemma:

Lemma 4.3. [12, Corollary 7] *Let $T \in \mathbf{B}(\mathcal{H})$ be p-hyponormal operator and let $S \in \mathbf{B}(\mathcal{H})$ be p-hyponormal operator. If $TX = XS$, where $X \in \mathbf{B}(\mathcal{H})$ is an injective with dense range, then T is normal unitarily equivalent to S.*

Theorem 4.3. *If $T^* \in \mathbf{B}(\mathcal{H})$ is injective (p,k)-quasihyponormal operator, $S \in \mathbf{B}(\mathcal{H})$ is injective (p,k)-quasihyponormal operator, and if $XT = SX$, where $X \in \mathbf{B}(\mathcal{H})$ is an injective with dense range. Then $XT^* = S^*X$.*

Proof. Since by assumption $XT = SX$, we can see that $(\ker X)^\perp$ and $\overline{ranX}$ are invariant subspace of T^* and S, respectively. Therefore we have that $T^*|_{(\ker X)^\perp}$ and $S|_{\overline{ranX}}$ are also (p,k)-quasihyponormal operators. Now consider the decomposition $\mathcal{H} = (\ker X)^\perp \oplus \ker X$. Then we have the matrix representations:

$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

where T_1^* and S_1 are injective p-hyponormal and X_1 is an injective with dense range. Therefore we have

$$X_1T_1x = XTx = SXx = S_1X_1x \quad \text{for} \quad x \in (\ker X)^\perp. \tag{4.4}$$

that is, $X_1T_1 = S_1X_1$ and hence T_1 and S_1 are normal and $X_1T_1^* = S_1^*X_1$ by the Fuglede-Putnam Theorem. Therefore $(\ker X)^\perp$ and $\overline{ranX}$ reduces T^* and S, respectively. Hence, we obtain the $XT^* = S^*X$. ∎

Theorem 4.4. *If $T^* \in \mathbf{B}(\mathcal{H})$ is injective (p,k)-quasihyponormal operator, $S \in \mathbf{B}(\mathcal{H})$ is injective (p,k)-quasihyponormal operator, and if $XT = SX$, where $X \in \mathbf{B}(\mathcal{H})$ is an invertible. Then there exists a unitary operator U such that $UT = SU$ and hence T and S are normal operators.*

Proof. Since $XT = SX$, it follows by Theorem 4.3 that $XT^* = S^*X$ and so $TX^* = X^*S$. Now $TX^*X = X^*SX = X^*XT$. Let $X = UP$ be the polar decomposition of X. Since X is invertible, it follows that P is invertible and U is unitary. Since $TP^2 = P^2T$ and $P \geq 0$, it follows that $TP = PT$. Thus $TUP = UPS$ implies $SUP = UPT = UTP$, since P is invertible, we have $UT = SU$. Now T and S are unitarily equivalent. Hence, T and S are normal operators. ∎

Corollary 4.4. *If $T^*, S \in \mathbf{B}(\mathcal{H})$ are injective (p,k)-quasihyponormal operators, and if $XT = SX$, where $X \in \mathbf{B}(\mathcal{H})$ is positive. Then $T = S$.*

Theorem 4.5. *If $T^* \in \mathbf{B}(\mathcal{H})$ is injective (p,k)-quasihyponormal operator, and let $S \in \mathbf{B}(\mathcal{H})$ be an isometry. Assume there exists quasiaffinities X and Y such that $TX = XS$ and $YT = SY$. If either X or Y is compact, then T and S are unitary.*

Proof. Since T and S have the same spectra by [13, Theorem 2.8] we have $\sigma(T) = \sigma(S) \subseteq \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ is the closed unit disc. Since T is normaliod, we have that $\|T\| \leq 1$ (*i.e.*, T is a contraction). Assume that Y is compact. Applying [6, Theorem 2] to the operator equation $YT = SY$ to conclude S is unitary. Now applying of [6, Theorem 1] to $XT = SX$, it follows that T is unitary. So the result follows. ∎

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