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A Note on Spectra of Weighted Composition Operators on Weighted Banach Spaces of Holomorphic Functions

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Abstract. We characterize the spectra of bounded weighted composition operators acting on the weighted Banach spaces H_v^∞ of analytic functions on the unit disc defined for a radial weight v, when the symbol of the operator has a fixed point in the open unit disc.

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1. Introduction

Let v be a strictly positive continuous function (weight) on the open unit disc D. We consider

$$H_v^\infty := \{f \in H(D);\ \|f\|_v := \sup_{z \in D} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 := \{f \in H_v^\infty;\ \lim_{|z| \to 1} v(z)|f(z)| = 0\}$$

endowed with norm $\|.\|_v$ where $H(D)$ denotes the space of holomorphic functions on D. For more information on this type of spaces we refer the reader to [2], [4] and [11].

Let $\varphi : D \to D$ be an analytic self map of the unit disc. Each such map induces through composition a linear composition operator $C_\varphi(f) = f \circ \varphi$ acting on this type of spaces. If we choose $\psi \in H(D)$ we obtain a weighted composition operator $C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$.

Composition operators have been studied on various spaces of analytic functions. We refer the reader to the monographs [7] and [14]. The case of operators defined on weighted Banach spaces of the type defined above was treated, for example in [4], [3] and [6]. The spectrum of a composition operator has recently been investigated in [8], [15] and [12] as well as in [1] and [5]. More precisely, Aron and Lindström

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determined the spectrum of a bounded weighted composition operator acting on weighted Banach spaces $H^\infty_{v_p}$ where $v_p(z) = (1 - |z|^2)^p$, $p > 0$ (see [1]). This was generalized by Bonet, Galindo and Lindström in the sense that they were able to characterize the spectrum of a bounded composition operator defined on a weighted Banach space H^∞_v, where v is a typical weight. The aim of this article is to generalize the results of [1] and [5], i.e. to give the spectrum of a bounded weighted composition operator acting on weighted Banach spaces H^∞_v of holomorphic functions using methods of [1] and [5].

2. Notations and Definitions

We refer the reader to [7] and [14] for notation on composition operators. The closed unit ball of H^∞_v (resp. H^0_v) is denoted by B^∞_v (resp. B^0_v). Many results on weighted spaces of analytic functions and on operators between them have to be given in terms of the so-called *associated weights* (see [2]) and in terms of the weights. For a weight v the associated weight $\tilde{v}$ is defined as follows

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H^\infty_v, \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H^\infty_v{}'}}, \quad z \in D,$$

where δ_z denotes the point evaluation of z. The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [2]). Furthermore, for each $z \in D$ there is $f_z \in H^\infty_v$, $||f_z||_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. A weight v is called *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for every } z \in D.$$

For examples of essential weights and conditions when weights are essential, see [2], [4] and [3]. Especially interesting are *radial weights* v, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Every radial weight v which is non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1} v(z) = 0$ is called a *typical weight.* In the sequel every radial weight is assumed to be non-increasing. For typical weights v the polynomials lie dense in H^0_v.

The *essential spectrum* $\sigma_{e,X}(T)$ of a bounded operator T on the Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Fredholm. It is known that $\sigma_{e,X}(T) = \sigma_{e,X^*}(T^*)$, see [9]. The essential spectral radius is given by

$$r_{e,X}(T) = \sup\{|\lambda|;\ \lambda \in \sigma_{e,X}(T)\}.$$

Another way of expressing the essential spectral radius is

$$r_{e,X}(T) = \lim_n \|T^n\|^{\frac{1}{n}}_{e,X}$$

where $\|T\|_{e,X}$ denotes the essential norm of T, i.e. the distance from the compact operators on X,

$$\|T\|_{e,X} = \inf\{\|T - K\|; K \text{ is a compact operator on } X\}.$$

Let $\lambda \in \sigma_X(T)$ be such that $|\lambda| > r_{e,X}(T)$. Then λ lies in the unbounded component of $\mathbb{C}\backslash\sigma_{e,X}(T)$. Now the Fredholm theory gives that λ is an isolated point of $\sigma_X(T)$ which also is an eigenvalue of finite multiplicity. We recall the following well-known result (see [9]):

Lemma 2.1. *Let $T : X \to X$ be a bounded operator. If $\lambda \in \sigma_X(T)$ is such that $|\lambda| > r_{e,X}(T)$, then λ is an isolated eigenvalue of finite multiplicity.*

By [6] Proposition 3.1 weighted composition operators $C_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ are continuous if and only if

$$\sup_{z\in D} |\psi(z)| \frac{v(z)}{\tilde{v}(\varphi(z))} < \infty.$$

For a similar result see also [13]. If $\|C_{\varphi,\psi}\|_{e,v}$ denotes the essential norm of $C_{\varphi,\psi}$, then we get

$$\|C_{\varphi,\psi}\|_{e,v} = \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\psi(z)| \frac{v(z)}{\tilde{v}(\varphi(z))}.$$

(See [6] and [13].)

3. Results

In the case of composition operators the following lemma was given by Bonet, Galindo and Lindström in [5].

Lemma 3.1. *Let v be a typical weight and $\varphi(0) = 0$ as well as $0 < |\varphi'(0)| < 1$. Moreover, we assume that $C_{\varphi,\psi}$ is bounded. Then $\sigma_{H_v^\infty}(C_{\varphi,\psi})$ contains $\psi(0)\varphi'(0)^n$ for non-negative integers n.*

Proof. If $\psi(0) = 0$, then $\psi(0)\varphi'(0)^n \in \sigma_{H_v^\infty}$. Thus we can assume that $\psi(0) \neq 0$. We use an argument of [10] to show that $z^n \in H_v^\infty$, $n \in \mathbb{N}$, is not in the range of $C_{\varphi,\psi} - \psi(0)\varphi'(0)^n I$ on H_v^∞. Let us assume that $f \in H_v^\infty$ and

$$f(\varphi(z))\psi(z) - \varphi'(0)\psi(0)f(z) = z.$$

Then, we obtain $f(0)\psi(0) - \varphi'(0)\psi(0)f(0) = 0$. Since $\psi(0) \neq 0$ and $0 < |\varphi'(0)| < 1$ we obtain $f(0) = 0$. Now differentiation on both sides yields

$$\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z)) - \varphi'(0)\psi(0)f'(z) = 1.$$

With $z = 0$ we obtain $0 = \psi(0)f'(0)\varphi'(0) - \varphi'(0)\psi(0)f'(0) = 1$ which is a contradiction. For $n > 1$, suppose $f \in H_v^\infty$ and $\psi(z)f(\varphi(z)) - \psi(0)\varphi'(0)^n f(z) = z^n$. By repeated differentiation on both sides we get for all $k < n$ that $f^{(k)}(0) = 0$. Then for $k = n$ and $z = 0$ we obtain the contradiction $0 = n!$. This means that $\psi(0)\varphi'(0)^n \in \sigma_{H_v^\infty}(C_{\varphi,\psi})$ for all $n > 0$. If $n = 0$, then $\psi(0) \in \sigma_{H_v^\infty}(C_{\varphi,\psi})$, since $C_{\varphi,\psi} - \psi(0)I$ is not onto. ∎

For a positive integer m and an arbitrary weight v, let we consider the closed subspace $H_{v,m}^\infty$ of H_v^∞ defined by

$$H_{v,m}^\infty := \{f \in H_v^\infty; f \text{ has a zero of at least order } m \text{ at } 0\}.$$

Let $\|.\|_{m,v}$ be the induced norm on $H_{v,m}^\infty$. The following result can be found in [5]. For standard weights, the result was obtained by Aron and Lindström in [1].

Proposition 3.1. *Let $m \in \mathbb{N}$ and v be an arbitrary weight. Then there exists a constant $M_m > 0$ such that*

$$|f(w)| \leq M_m \frac{1}{\tilde{v}(w)} \|f\|_v |w|^m,$$

for all $f \in H_{v,m}^\infty$ and $w \in D$.

Lemma 3.2. *Let v be a typical weight. Suppose that $\varphi(0) = 0$ and that $C_{\varphi,\psi}$ is bounded on H_v^∞. If $\lambda \neq 0$ is an eigenvalue of $C_{\varphi,\psi}$, then $\lambda \in \{\psi(0)\varphi'(0)^n\}_{n=0}^\infty$.*

Proof. Suppose that $\lambda \neq 0$ is an eigenvalue of $C_{\varphi,\psi}$ with the corresponding eigenvector $f \neq 0$. Then for all $z \in \mathbb{D}$, we have

$$C_{\varphi,\psi} f(z) = \psi(z) f(\varphi(z)) = \lambda f(z).$$

We obviously get $\psi(0)f(0) = \lambda f(0)$. If $f(0) \neq 0$, $\lambda = \psi(0)$ follows. If $f(0) = 0$ by differentiation on both sides of the equation we obtain:

$$\psi'(z) f(\varphi(z)) + \psi(z)\varphi'(z) f'(\varphi(z)) = \lambda f'(z).$$

Since $f(0) = 0$, we obtain $\psi(0)\varphi'(0)f'(0) = \lambda f'(0)$. If $f'(0) \neq 0$, then $\lambda = \psi(0)\varphi'(0)$. If $f'(0) = 0$ we proceed by induction until we have found a k such that $f^{(k)}(0) \neq 0$ which must exist. In that case we obtain $\lambda = \psi(0)\varphi'(0)^k$. Thus, the claim follows. ∎

We need the following lemma which is taken from [7].

Lemma 3.3. *If φ is not an automorphism and $\varphi(0) = 0$, then given $0 < r < 1$, there exists $1 \leq M < \infty$ such that if $(z_k)_{k=-K}^\infty$ is an iteration sequence with $|z_n| \geq r$ for some non-negative integer n and $(w_k)_{k=-K}^n$ are arbitrary numbers, then there exists $f \in H^\infty$ with $f(z_k) = w_k$, $-K \leq k \leq n$ and $\|f\|_\infty \leq M \sup\{|w_k| : -K \leq k \leq n\}$. Further there exists $b < 1$ such that for any iteration sequence (z_k), i.e. for any sequence (z_k) with $z_{k+1} = \varphi(z_k)$ we have $|z_{k+1}|/|z_k| \leq b$ whenever $|z_k| \leq 1/2$.*

The proof of the following theorem was inspired by [1] and [5].

Theorem 3.1. *Let v be a typical weight. Suppose φ, not an automorphism has fixed point $a \in D$ and that $C_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is bounded. Then*

$$\sigma_{H_v^\infty}(C_{\varphi,\psi}) = \{\lambda \in \mathbb{C};\ |\lambda| \leq r_{e,H_v^\infty}(C_{\varphi,\psi})\} \cup \{\psi(a)\varphi'(a)^n\}_{n=0}^\infty.$$

Proof. We can assume that $a = 0$. If $a \neq 0$, we consider

$$\varphi_a(z) = \frac{(a - z)}{1 - \overline{a}z}, \quad \psi_1(z) = \psi \circ \varphi_a, \quad \text{and} \quad \varphi_1 = \varphi_a \circ \varphi \circ \varphi_a.$$

Then $\varphi_1(0) = 0$, $\psi_1(0) = \psi(a)$, $\varphi_1'(0) = \varphi'(a)$ and

$$C_{\varphi_a} \circ C_{\varphi_1,\psi_1} \circ C_{\varphi_a}^{-1} = C_{\varphi,\psi}.$$

Hence, $C_{\varphi,\psi}$ and C_{φ_1,ψ_1} are similar. Therefore, they have the same spectrum and the same essential spectral radius.
By Lemma 3.1, we have that $\{\psi(0)\varphi'(0)^n\}_{n=0}^\infty \subset \sigma_{H_v^\infty}(C_{\varphi,\psi})$. If $\lambda \in \sigma_{H_v^\infty}(C_{\varphi,\psi})$ and $|\lambda| > r_{e,H_v^\infty}$, then Lemma 2.1 yields, that λ is an eigenvalue. If $\lambda \neq 0$ is an eigenvalue of $C_{\varphi,\psi}$, then by Lemma 3.2, we can find n such that $\lambda = \psi(0)\varphi'(0)^n$. It remains to show that

$$\{\lambda \in \mathbb{C};\ |\lambda| \leq r_{e,H_v^\infty}(C_{\varphi,\psi})\} \subset \sigma_{H_v^\infty}(C_{\varphi,\psi}).$$

If $r_{e,H_v^\infty}(C_{\varphi,\psi}) = 0$, then we have $0 \in \sigma_{e,H_v^\infty}(C_{\varphi,\psi}) \subset \sigma_{H_v^\infty}(C_{\varphi,\psi})$. So, we assume that $\rho := r_{e,H_v^\infty}(C_{\varphi,\psi}) > 0$. Since $\varphi(0) = 0$ we have that $\varphi(z) = z\phi(z)$ with $\phi \in H^\infty$. Hence $H_{v,m}^\infty$ is an invariant subspace under $C_{\varphi,\psi}$. Further, $H_{v,m}^\infty$ has finite codimension in H_v^∞. Now Lemma 7.17 in [7] which is also valid for Banach spaces gives that $\sigma_{H_{v,m}^\infty}(C_{\varphi,\psi}) \subset \sigma_{H_v^\infty}(C_{\varphi,\psi})$. So it is enough to show that any λ with $0 < |\lambda| < \rho$ belongs to $\sigma_{H_{v,m}^\infty}(C_{\varphi,\psi})$ for some m to be found. Let C_m denote the restriction of $C_{\varphi,\psi}$ to the invariant closed subspace $H^\infty_{v,m}$. Since $C_m - \lambda I$ is not invertible if $(C_m - \lambda I)^*$ is not bounded from below, we just need to find m with $(C_m - \lambda I)^*$ not bounded from below. Let $1 \leq M < \infty$ be the constant in Lemma 3.3 for $r = 1/4$. We will denote iteration sequences by $\xi = (z_k)_{k=-K}^\infty$ with $K > 0$ and $|z_0| \geq 1/2$. Let $n := \max\{k; |z_k| \geq 1/4\}$. Then $n \geq 0$ and $|z_k| < 1/4$ for $k > n$. By Lemma 3.3 there is $b < 1$ with $|z_{k+1}/z_k| \leq b$ for all $k \geq n$. We may assume that $1/2 < b < 1$. This implies

$$|z_k| \leq b^{k-n}|z_n| \text{ for } k \geq n. \tag{3.1}$$

Since $\psi \in H(D)$ is continuous, $0 \leq C := \max\{\sup_{|z|\leq 1/4} |\psi(z)|, |\psi(z_n)|\} < \infty$. We now choose m so large that

$$\frac{b^m C}{|\lambda|} < \frac{1}{2}. \tag{3.2}$$

For every such iteration sequence $\xi = (z_k)_{k=-K}^\infty$ we define the linear functional $L_{\xi,\psi}$ of $H^\infty_{v,m}$ by

$$L_{\xi,\psi}(f) = \sum_{k=-K}^{\infty} \lambda^{-k}\psi(z_{-K})\cdots\psi(z_{k-1})f(z_k),$$

where we agree that $\psi(z_{-K})\cdots\psi(z_{K-1}) = 1$ in the first term of the sum. Indeed $L_{\xi,\psi}$ is bounded. Proposition 3.1 yields

$$\begin{aligned}
&\left|\sum_{k=-K}^{\infty} \lambda^{-k}\psi(z_{-K})\ldots\psi(z_{k-1})f(z_k)\right| \\
&\leq M_m\|f\|_v\Bigg\{\sum_{k=-K}^{n} |\lambda|^{-k}|\psi(z_{-K})|\cdots|\psi(z_{k-1})||z_k|^m\tilde{v}(z_k)^{-1} \\
&\quad + |\psi(z_{-K})|\cdots|\psi(z_{n-1})|\sum_{k=n+1}^{\infty} |\lambda|^{-k}|\psi(z_n)|\cdots|\psi(z_{k-1})||z_k|^m\tilde{v}(z_k)^{-1}\Bigg\}.
\end{aligned}$$

Since $|z_k| < \frac{1}{4}$ for $k > n$ and $\tilde{v}$ is continuous, there is a constant $c > 0$ such that $\tilde{v}(z_k)^{-1} \leq c$ for $k > n$. Further applying (3.1) and (3.2) we get

$$\begin{aligned}
\sum_{k=n+1}^{\infty} |\lambda|^{-k}|\psi(z_n)|\cdots|\psi(z_{k-1})||z_k|^m\tilde{v}(z_k)^{-1} &\leq \sum_{k=n+1}^{\infty} |\lambda|^{-k}C^{k-n}|z_k|^m c \\
&\leq c\frac{|z_n|^m}{|\lambda|^n}\sum_{k=n+1}^{\infty}\left(\frac{b^m C}{|\lambda|}\right)^{k-n} < \infty.
\end{aligned}$$

Thus $\psi(z_{-K})|\cdots|\psi(z_{n-1})|\sum_{k=n+1}^{\infty} |\lambda|^{-k}|\psi(z_n)|\cdots|\psi(z_{k-1})||z_k|^m\tilde{v}(z_k)^{-1} < \infty$, and $L_{\xi,\psi}$ is bounded. Let us find next a lower bound for $\|L_{\xi,\psi}\|_{v,m}$. There exists $f_{z_0} \in H^\infty_v$ with $\|f_{z_0}\|_v \leq 1$, so that $|f_{z_0}(z_0)| = 1/\tilde{v}(z_0)$. By Lemma 3.3, there is $f_1 \in H^\infty$ with $\|f_1\|_\infty \leq M$, satisfying $|f_1(z_0)| = 1$, $z_0^m f_1(z_0)f_{z_0}(z_0) > 0$ and $f_1(z_k) = 0$ for $-K \leq k \leq n$, $k \neq 0$. Now, the function $g(z) := z^m f_1(z)f_{z_0}(z)$

belongs to $H_{v,m}^{\infty}$ and $\|g\|_v \leq M$. Further

$$\begin{aligned} L_{\xi,\psi}(g) &= \psi(z_{-K})\cdots\psi(z_{-1})z_0^m f_1(z_0)f_{z_0}(z_0) \\ &\quad + \sum_{k=n+1}^{\infty} \lambda^{-k}\psi(z_{-K})\cdots\psi(z_{k-1})z_k^m f_1(z_k)f_{z_0}(z_k). \end{aligned} \tag{3.3}$$

Since $\tilde{v}$ is not increasing, we get using again (3.1) and (3.2)

$$\begin{aligned} &\left|\sum_{k=n+1}^{\infty} \lambda^{-k}\psi(z_{-K})\cdots\psi(z_{k-1})z_k^m f_1(z_k)f_{z_0}(z_k)\right| \\ &\leq M|\psi(z_{-K})|\cdots|\psi(z_{n-1})|\frac{|z_n|^m}{|\lambda|^n}\sum_{k=n+1}^{\infty}\left(\frac{b^m C}{|\lambda|}\right)^{k-n}\frac{1}{\tilde{v}(z_n)} \\ &\leq M|\psi(z_{-K})|\cdots|\psi(z_{n-1})|\frac{|z_n|^m}{|\lambda|^n}\frac{1}{\tilde{v}(z_n)}\frac{b^m C}{|\lambda|-b^m C}. \end{aligned}$$

If, in addition we choose m so that

$$M|\psi(z_{-K})|\cdots|\psi(z_{n-1})|\frac{1}{|\lambda|^n}\frac{1}{\tilde{v}(z_n)}\frac{b^m C}{|\lambda|-b^m C} < |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{1}{2\tilde{v}(z_0)},$$

then

$$\begin{aligned} \left|\sum_{k=n+1}^{\infty} \lambda^{-k}\psi(z_{-K})\cdots\psi(z_{k-1})g(z_k)\right| &\leq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{|z_n|^m}{2\tilde{v}(z_0)} \\ &\leq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{|z_0|^m}{2\tilde{v}(z_0)}. \end{aligned}$$

Hence by (3.3)

$$\begin{aligned} |L_{\xi,\psi}(g)| &\geq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{|z_0|^m}{\tilde{v}(z_0)} - \left|\sum_{k=n+1}^{\infty} \lambda^{-k}\psi(z_{-K})\cdots\psi(z_{k-1})g(z_k)\right| \\ &\geq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{|z_0|^m}{2\tilde{v}(z_0)}. \end{aligned} \tag{3.4}$$

Therefore using Proposition 3.1, we obtain the desired lower bound

$$\begin{aligned} \|L_{\xi,\psi}\|_{v,m} &\geq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{|z_0|^m}{2M\tilde{v}(z_0)} \\ &\geq |\psi(z_{-K})|\cdots|\psi(z_{-1})|\frac{1}{2MM_m}\|\delta_{z_0}\|_{v,m}. \end{aligned} \tag{3.5}$$

The final step is to estimate $\|(C_m^* - \overline{\lambda}I)L_{\xi,\psi}\|_{v,m}$ for a suitable iteration sequence ξ. First observe that

$$(C_m^* - \overline{\lambda}I)L_{\xi,\psi} = -\lambda^{K+1}\delta_{z_{-K}}.$$

Recall that

$$\rho = r_{e,H_v^{\infty}}(C_{\varphi,\psi}) = \lim_n \|(C_{\varphi,\psi})^n\|_{e,v}^{\frac{1}{n}}$$

and

$$(C_{\varphi,\psi})^n f(w) = \psi(w)\cdots\psi(\varphi_{n-1}(w))C_{\varphi_n}f(w), \quad w \in D, f \in H_v^{\infty},$$

where $\varphi_n = \varphi \circ \cdots \circ \varphi$ (n times). Hence $(C_{\varphi,\psi})^n : H_v^\infty \to H_v^\infty$ is a bounded weighted composition operator and we get by Theorem 4.2 in [6] that

$$\|(C_{\varphi,\psi})^n\|_{e,v} = \lim_{r\to 1} \sup_{|\varphi(w)|>r} |\psi(w)| \cdots |\psi(\varphi_{n-1}(w))| \frac{v(w)}{\tilde{v}(\varphi_n(w))}.$$

Pick μ so that $|\lambda| < \mu < \rho$. Thus there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\|(C_{\varphi,\psi})^n\|_{e,v} > \mu^n > 0.$$

Hence for any $l \geq n_0$ we can find a $w \in D$ so that

$$|\psi(w)| \cdots |\psi(\varphi_{l-1}(w))| \frac{v(w)}{\tilde{v}(\varphi_l(w))} \geq \mu^l > 0 \text{ and } |\varphi_l(w)| \geq \frac{1}{2}.$$

This means that for every $K \geq n_0$ with the above choice of $w \in D$ we can form an iteration sequence $(z_k)_{k=-K}^\infty$ by letting $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k \geq -K$. Then $|z_0| = |\varphi_K(w)| \geq 1/2$. By (3.4) and (3.5) we obtain

$$\|L_{\xi,\psi}\|_{v,m} \geq |\psi(z_{-K})| \cdots |\psi(z_{-1})| \frac{1}{2MM_m} \frac{|z_0|^m}{\tilde{v}(z_0)}.$$

Moreover

$$\|\delta_{z_{-K}}\|_{v,m} \leq \|\delta_{z_{-K}}\|_v = \frac{1}{\tilde{v}(z_{-K})} \leq \frac{1}{v(z_{-K})}.$$

Finally,

$$\begin{aligned} \frac{\|(C_m^* - \overline{\lambda} I)L_{\xi,\psi}\|_{v,m}}{\|L_{\xi,\psi}\|_{v,m}} &\leq 2\frac{MM_m|\lambda|^{K+1}\tilde{v}(z_0)}{|\psi(z_{-K})| \cdots |\psi(z_{-1})||z_0|^m v(z_{-K})} \\ &\leq |\lambda| MM_m 2^{m+1} \left(\frac{|\lambda|}{\mu}\right)^K. \end{aligned}$$

By choosing $K \geq n_0$ large enough we see that $(C_m^* - \overline{\lambda} I)$ is not bounded from below. ∎

The proof of the following corollary is in fact the same proof as the proof given in [1], Corollary 8. For the sake of completeness we repeat it here.

Corollary 3.1. *Let v be a typical weight. Suppose that φ, not an automorphism, has fixed point $a \in D$ and that $C_{\varphi,\psi} : H_v^0 \to H_v^0$ is bounded. Then*

$$\sigma_{H_v^0}(C_{\varphi,\psi}) = \{\lambda \in \mathbb{C};\ |\lambda| \leq r_{e,H_v^0}(C_{\varphi,\psi})\} \cup \{\psi(a)\varphi'(a)\}_{n=0}^\infty.$$

Proof. Since $C_{\varphi,\psi} : H_v^0 \to H_v^0$ is bounded, we get that $C_{\varphi,\psi}^{**} : H_v^\infty \to H_v^\infty$ is also bounded, and we can apply the previous theorem. Moreover $C_{\varphi,\psi}^n$ is bounded on H_v^∞ and H_v^0 and and $C_{\varphi,\psi}^n f(w) = \psi(w) \cdots \psi(\varphi_{n-1}(w)) C_{\varphi_n}(f(w))$. Hence $C_{\varphi,\psi}^n$ is a weighted bounded weighted composition operator on H_v^0 and H_v^∞ and we can apply Theorem 2 in [13] to get that the essential norm of $C_{\varphi,\psi}^n$ on H_v^0 coincides with the essential norm of $C_{\varphi,\psi}^n$ on H_v^∞.Thus, $r_{e,H_v^\infty}(C_{\varphi,\psi}) = r_{e,H_v^0}(C_{\varphi,\psi})$, and the claim follows. ∎

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