BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **31**(2) (2008), 165–173

Essentially Slant Hankel Operators

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Abstract. The notion of an essentially slant Hankel operator is introduced and its algebraic properties are studied. The study is further carried to compressions of such operators. It is proved that a Rhaly operator is the compression of an essentially slant Toeplitz operator if and only if it is the compression of an essentially slant Hankel operator.

2000 Mathematics Subject Classification: 47B35

Key words and phrases: Essentially Toeplitz operator, essentially Hankel operator, essentially slant Hankel operator, Rhaly operator.

1. Introduction

The notion of Toeplitz operators was introduced by O. Toeplitz [13] in the year 1911. The formal companions of Toeplitz operators are the Hankel operators [11]. For basic facts about these classes of operators and the spaces L^2, H^2, L^∞ and ℓ^2, one can refer [4], [9] and [11]. These classes of operators can also be seen as solutions of some linear operator equations involving the unilateral forward shift and its adjoint. It is well-known that an operator A on the space H^2 is Hankel iff $S^*A = AS$ and an operator B on the space H^2 is Toeplitz iff $S^*BS = B$, where S denotes the unilateral forward shift on H^2.

In the year 1995, M. C. Ho [8] came up with a new class of operators having the property that the representation matrices of such operators with respect to the standard basis could be obtained from those of the matrices of Toeplitz operators just by eliminating every other row. These operators were termed as slant Toeplitz operators. Many operator-theorists like Goodman [6], Strang and Strela [7] connected the smoothness of wavelets with the spectral properties of slant Toeplitz operators. However, Ho made the first attempt to investigate the basic properties of slant Toeplitz operators and in particular, he gave the following characterization of slant Toeplitz operators [8]: *A bounded operator A on the space L^2 is a slant Toeplitz*

Received: August 29, 2007; *Revised:* February 27, 2008.

operator iff $M_zA = AM_{z^2}$*, where* M_z *denotes the multiplication operator induced by* z *on* L^2*.*

Motivated by Ho's work, S. C. Arora [1] along with his research associates introduced the class of slant Hankel operators in the year 2006 and obtained a characterization [1] of slant Hankel operators as follows: *A bounded operator* K *on the space* L^2 *is a slant Hankel operator iff* $M_{\bar{z}}K = KM_{z^2}$*.*

Later, the notion of slant Toeplitz operators was generalized to essentially slant Toeplitz operators [2] by Arora and Bhola.

We begin with the following definitions:

Definition 1.1. [3] *A bounded linear operator* A *on the space* H^2 *is said to be essentially Hankel if* $T_z^*A - AT_z$ *is compact, where* T_z *denotes the Toeplitz operator induced by* z*.*

Definition 1.2. [1] *A slant Hankel operator* K_ϕ *on the space* L^2 *is defined as*

$$K_\phi(z^j) = \sum_{i=-\infty}^{\infty} a_{-2i-j} z^i$$

for all $j \in \mathbb{Z}$*, where* $\phi = \sum_{i=-\infty}^{\infty} a_i z^i$ *is a bounded measurable function on the unit circle. It may be seen* [1] *that* $K_\phi = WJM_\phi$ *where* W *is defined on* L^2

$$\left.\begin{aligned} W(z^{2n}) &= z^n \\ W(z^{2n-1}) &= 0 \end{aligned}\right\} \text{ for all } n \in \mathbb{Z},$$

J *is the unitary flip operator on* L^2 *defined as* $J(f(z)) = f(\bar{z})$*,* M_ϕ *is the multiplication operator on* L^2 *induced by* ϕ *in* L^∞ *and* $\{e_n : e_n(z) = z^n, n \in \mathbb{Z}\}$ *is the standard orthonormal basis of* L^2*.*

Definition 1.3. [2] *A bounded linear operator* A *on the space* L^2 *is said to be an essentially slant Toeplitz operator if*

$$M_zA - AM_{z^2} = k,$$

for some compact operator k *on* L^2*.*

Definition 1.4. [2] *An operator* B *on the space* H^2 *is said to be the compression of an essentially slant Toeplitz operator to* H^2 *if*

$$T_zB - BT_{z^2} = k,$$

for some compact operator k *on* H^2*.*

The set of all essentially Hankel operators on H^2 is denoted by essHank [3]. The set of all essentially slant Toeplitz operators on L^2 is denoted by $\boldsymbol{ESTO}(L^2)$ [2]. The set of all compressions of essentially slant Toeplitz operators to H^2 is denoted by $\boldsymbol{ESTO}(H^2)$ [2].

In this paper, we introduce and study the class of **essentially slant Hankel operators**. The set of all essentially slant Hankel operators on the space L^2 is denoted by $\boldsymbol{ESHO}(L^2)$. It is shown here that, although this set is not an algebra of operators on L^2, it is nevertheless a norm-closed vector subspace of $\mathcal{B}(L^2)$,the space of all bounded linear operators on L^2. An example of an essentially slant Hankel operator which is not slant Hankel has also been presented. This paper also contains discussion of some conditions under which the product of two operators in $\boldsymbol{ESHO}(L^2)$ is again in $\boldsymbol{ESHO}(L^2)$. The second section deals with the counterpart of such operators on the space H^2. The final result of this section relates the class of Rhaly operators [10, 12], essentially slant Toeplitz operators and essentially slant Hankel operators on the space H^2. Precisely it is proved that the intersection of the class of Rhaly operators on H^2 with the class $\boldsymbol{ESTO}(H^2)$ and that with $\boldsymbol{ESHO}(H^2)$ is identical.

2. Essentially Slant Hankel Operators on L^2

We introduce the following:

Definition 2.1. *A bounded linear operator A on the space L^2 is said to be an essentially slant Hankel operator if*

$$M_{\bar{z}}A - AM_{z^2} = k,$$

for some compact operator k on L^2.

We denote the set of all essentially slant Hankel operators on L^2 by $\boldsymbol{ESHO}(L^2)$. It is clear that sum of a slant Hankel operator and a compact operator on L^2 is in $\boldsymbol{ESHO}(L^2)$. In particular every slant Hankel operator on L^2 is in $\boldsymbol{ESHO}(L^2)$. If $\mathcal{K}$ denotes the space of all compact operators on L^2, then

$$\boldsymbol{ESHO}(L^2) \cap \mathcal{K} = \mathcal{K}\,.$$

We now present an example of a non-compact essentially slant Hankel operator on L^2 which is not a slant Hankel operator.

Example 2.1. Let A on L^2 be defined as

$$Ae_n = \begin{cases} e_1 & \text{if } \ n = 0 \\ 0 & \text{if } \ n \neq 0, \ n \text{ is even} \\ e_m, \ \text{where } \ m = -\left(\dfrac{n+1}{2}\right), & \text{if } \ n \text{ is odd} \end{cases}$$

where $e_n(z) = z^n$ for all $n \in \mathbb{Z}$. The matrix representation of A with respect to $\{e_n\}_{n\in\mathbb{Z}}$ is given by

$$\left[\begin{array}{cccc|cccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\hline
\cdots & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]$$

If W is defined on L^2 as

$$W(z^n) = \begin{cases} z^{n/2} & \text{if } n \text{ is even;} \\ 0 & \text{otherwise;} \end{cases}$$

for all $n \in \mathbb{Z}$ and K is defined on L^2 as

$$Ke_n = \begin{cases} e_1 & \text{if } \ n = 0; \\ 0 & \text{otherwise}; \end{cases}$$

for all $n \in \mathbb{Z}$, then we have

$$A = WJM_z + K$$

where J denotes the unitary flip operator on L^2. Now

$$\begin{aligned} M_{\bar{z}}A - AM_{z^2} &= M_{\bar{z}}(WJM_z + K) - (WJM_z + K)M_{z^2} \\ &= M_{\bar{z}}K - KM_{z^2} \in \mathcal{K}\,. \end{aligned}$$

Therefore, $A \in \boldsymbol{ESHO}(L^2)$ but A is not a slant Hankel operator on L^2.

We now discuss some basic properties of the set $\boldsymbol{ESHO}(L^2)$.

Theorem 2.1. *$\boldsymbol{ESHO}(L^2)$ is a norm-closed vector subspace of $\mathcal{B}(L^2)$, the set of all bounded linear operators on the space L^2.*

Proof. For $H_1, H_2 \in \boldsymbol{ESHO}(L^2)$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} &M_{\bar{z}}(\alpha H_1 + \beta H_2) - (\alpha H_1 + \beta H_2)M_{z^2} \\ &\quad = \alpha(M_{\bar{z}}H_1 - H_1M_{z^2}) + \beta(M_{\bar{z}}H_2 - H_2M_{z^2}) \in \mathcal{K}\,. \end{aligned}$$

Also, if for each n, H_n is in $\boldsymbol{ESHO}(L^2)$ and $H_n \to H$ uniformly in $\mathcal{B}(L^2)$ then

$$M_{\bar{z}}H_n - H_nM_{z^2} \to M_{\bar{z}}H - HM_{z^2}$$

uniformly in $\mathcal{B}(L^2)$. Since $\mathcal{K}$ is uniformly closed, we have $H \in \boldsymbol{ESHO}(L^2)$. Thus, $\boldsymbol{ESHO}(L^2)$ is a norm-closed vector subspace of $\mathcal{B}(L^2)$. ∎

Theorem 2.2. *$\boldsymbol{ESHO}(L^2)$ is not an algebra of operators on L^2.*

The result follows since product of two essentially slant Hankel operators on L^2 is not necessarily an essentially slant Hankel operator as shown in the following:

Example 2.2. Let

$$A = B = WJM_z + K$$

where W, J, K are as defined in Example 2.1. Then $A, B \in \boldsymbol{ESHO}(L^2)$ but $C = AB \notin \boldsymbol{ESHO}(L^2)$ because

$$\begin{aligned} M_{\bar{z}}C - CM_{z^2} &= M_{\bar{z}}(WJM_z + K)(WJM_z + K) - (WJM_z + K)(WJM_z + K)M_{z^2} \\ &= M_{\bar{z}}(WJM_z)(WJM_z) - (WJM_z)(WJM_z)M_{z^2} \pmod{\mathcal{K}}\,. \end{aligned}$$

However

$$(M_{\bar{z}}(WJM_z)^2 - (WJM_z)^2M_{z^2})e_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ e_{-1} & \text{if } n = 1 \\ -e_1 & \text{if } n = 3 \\ e_0 & \text{if } n = 5 \\ -e_2 & \text{if } n = 7 \\ e_1 & \text{if } n = 9 \\ & \vdots \end{cases}$$

Therefore, $M_{\bar{z}}C - CM_{z^2} \notin \mathcal{K}$. Hence $C = AB \notin \boldsymbol{ESHO}(L^2)$.

Theorem 2.3. *$\boldsymbol{ESHO}(L^2)$ is not a self-adjoint set.*

The operator $A = WJM_z + K$, given above, belongs to $\boldsymbol{ESHO}(L^2)$ and $A^* \notin \boldsymbol{ESHO}(L^2)$.

Theorem 2.4. *If $H_1, H_2 \in \boldsymbol{ESHO}(L^2)$ then $H_1H_2 \in \boldsymbol{ESHO}(L^2)$ iff $H_1M_{\bar{z}}H_2 = H_1M_{z^2}H_2(\text{mod }\mathcal{K})$.*

Proof. Let $H_1, H_2 \in \boldsymbol{ESHO}(L^2)$. Then

$$\begin{aligned} M_{\bar{z}}H_1H_2 - H_1H_2M_{z^2} &= H_1M_{z^2}H_2 - H_1H_2M_{z^2} \ (\text{mod } \mathcal{K}) \\ &= H_1M_{z^2}H_2 - H_1M_{\bar{z}}H_2 \ (\text{mod } \mathcal{K})\,. \end{aligned}$$

Therefore, $H_1H_2 \in \boldsymbol{ESHO}(L^2)$ iff $H_1M_{\bar{z}}H_2 = H_1M_{z^2}H_2$ (mod $\mathcal{K}$). ∎

Theorem 2.5. *Let $A \in \boldsymbol{ESHO}(L^2)$ and $p \in \mathbb{N}$, $p > 1$. If $n(p)$ denotes the number of partitions of p as sum of two natural numbers, $p = m_i + n_i$; $i = 1, 2, \ldots, n(p)$, $m_i, n_i \in \mathbb{N}$; $A^{m_i}, A^{n_i} \in \boldsymbol{ESHO}(L^2)$, then the following are equivalent:*

(a) $A^p \in \boldsymbol{ESHO}(L^2)$
(b) $A^{m_i}M_{\bar{z}}A^{n_i} = A^{m_i}M_{z^2}A^{n_i}$ (mod $\mathcal{K}$), $i = 1, 2, \ldots, n(p)$
(c) $A^{n_i}M_{\bar{z}}A^{m_i} = A^{n_i}M_{z^2}A^{m_i}$ (mod $\mathcal{K}$), $i = 1, 2, \ldots, n(p)$

In addition to these properties, we have the following:

Theorem 2.6. (i) *If H_1 is in the essential commutant of $M_{\bar{z}}$ and $H_2 \in \boldsymbol{ESHO}(L^2)$, then $H_1H_2 \in \boldsymbol{ESHO}(L^2)$.*
(ii) *If $H_1 \in \boldsymbol{ESHO}(L^2)$ and H_2 is in the essential commutant of M_{z^2}, then $H_1H_2 \in \boldsymbol{ESHO}(L^2)$.*

Proof. (i) Let H_1 commute essentially with $M_{\bar{z}}$ and $H_2 \in \boldsymbol{ESHO}(L^2)$. Then

$$\begin{aligned} M_{\bar{z}}H_1H_2 - H_1H_2M_{z^2} &= H_1M_{\bar{z}}H_2 - H_1H_2M_{z^2} \ (\text{mod } \mathcal{K}) \\ &= H_1(M_{\bar{z}}H_2 - H_2M_{z^2}) \ (\text{mod } \mathcal{K}) \in \mathcal{K}\,. \end{aligned}$$

Therefore $H_1H_2 \in \boldsymbol{ESHO}(L^2)$.
(ii) Let $H_1 \in \boldsymbol{ESHO}(L^2)$ and H_2 commute essentially with M_{z^2}. Then

$$\begin{aligned} M_{\bar{z}}H_1H_2 - H_1H_2M_{z^2} &= M_{\bar{z}}H_1H_2 - H_1M_{z^2}H_2 \ (\text{mod } \mathcal{K}) \\ &= (M_{\bar{z}}H_1 - H_1M_{z^2})H_2 \ (\text{mod } \mathcal{K}) \in \mathcal{K}\,. \end{aligned}$$

Therefore $H_1H_2 \in \boldsymbol{ESHO}(L^2)$. ∎

Theorem 2.7. *If M_ϕ is a multiplication operator on L^2 induced by ϕ in L^∞ and $H \in \boldsymbol{ESHO}(L^2)$ then HM_ϕ and $M_\phi H$ both are in $\boldsymbol{ESHO}(L^2)$.*

Proof. Let $H \in \boldsymbol{ESHO}(L^2)$ and M_ϕ be the multiplication operator on L^2 induced by ϕ in L^∞. Then

$$M_{\bar{z}}(HM_\phi) - (HM_\phi)M_{z^2} = (M_{\bar{z}}H - HM_{z^2})M_\phi \in \mathcal{K}\,.$$

Also

$$M_{\bar{z}}(M_\phi H) - (M_\phi H)M_{z^2} = M_\phi(M_{\bar{z}}H - HM_{z^2}) \in \mathcal{K}\,.$$

Therefore HM_ϕ, $M_\phi H \in \boldsymbol{ESHO}(L^2)$. ∎

Theorem 2.8. *If $A, A^* \in \boldsymbol{ESHO}(L^2)$ then $A^*T = T^*A^*(\text{mod }\mathcal{K})$ where*

$$T = M_z(I + M_z).$$

Proof. Let $A, A^* \in \boldsymbol{ESHO}(L^2)$. Then

$$M_{\bar{z}}A - AM_{z^2} = k_1 \tag{2.1}$$

and

$$M_{\bar{z}}A^* - A^*M_{z^2} = k_2 \tag{2.2}$$

where $k_1, k_2 \in \mathcal{K}$. Taking adjoints on both sides of equation (2.1) and subtracting equation (2.2), we have

$$A^*(M_z + M_{z^2}) = (M_{\bar{z}} + M_{\bar{z}^2})A^* \pmod{\mathcal{K}}.$$

Therefore $A^*T = T^*A^* \pmod{\mathcal{K}}$, where $T = M_z(I + M_z)$. ∎

Corollary 2.1. *A necessary condition for an operator $A \in \boldsymbol{ESHO}(L^2)$ to be self-adjoint is that AT is essentially self-adjoint, where $T = M_z(I + M_z)$.*

3. Compressions of Essentially Slant Hankel Operators

In 2001, Arora and Zegeye [14] obtained a characterization of the compression of a slant Hankel operator to H^2 as follows: *An operator A on H^2 is the compression of a slant Hankel operator to H^2 iff $T_{\bar{z}}A = AT_{z^2}$, where T_z is the Toeplitz operator induced by z.*

Motivated by this, we define the compression of an essentially slant Hankel operator as follows:

Definition 3.1. *An operator A on the space H^2 is termed as the compression of an essentially slant Hankel operator if $T_{\bar{z}}A - AT_{z^2} = k$, for some compact operator k on H^2.*

We denote by $\boldsymbol{ESHO}(H^2)$, the set of all compressions of essentially slant Hankel operators to H^2. The set $\boldsymbol{ESHO}(H^2)$ has the following properties:

Theorem 3.1. *$\boldsymbol{ESHO}(H^2)$ is a norm-closed vector subspace of $\mathcal{B}(H^2)$ containing all the compressions of slant Hankel operators to H^2.*

Theorem 3.2. *$\boldsymbol{ESHO}(H^2)$ is not an algebra of operators on H^2.*

Theorem 3.3. *$\boldsymbol{ESHO}(H^2)$ is not a self-adjoint set.*

Theorem 3.4. *If $\mathcal{K}(H^2)$ denotes the space of all compact operators on H^2, then*

$$\boldsymbol{ESHO}(H^2) \cap \mathcal{K}(H^2) = \mathcal{K}(H^2).$$

Theorem 3.5. *If $A, B \in \boldsymbol{ESHO}(H^2)$, then*

$$AB \in \boldsymbol{ESHO}(H^2) \text{ iff } AT_{\bar{z}}B = AT_{z^2}B \pmod{\mathcal{K}(H^2)}.$$

Theorem 3.6. (i) *If A is in the essential commutant of $T_{\bar{z}}$ and $B \in \boldsymbol{ESHO}(H^2)$, then $AB \in \boldsymbol{ESHO}(H^2)$.*
(ii) *If $A \in \boldsymbol{ESHO}(H^2)$ and B is in the essential commutant of T_{z^2}, then $AB \in \boldsymbol{ESHO}(H^2)$.*

Theorem 3.7. *A necessary condition for an operator $A \in \boldsymbol{ESHO}(H^2)$ to be self-adjoint is that AT is essentially self-adjoint where $T = T_z(I + T_z)$.*

Note 3.1. Using the fact that any two multiplication operators commute, we have shown that if M_ϕ is any multiplication operator on L^2 and $H \in \boldsymbol{ESHO}(L^2)$ then HM_ϕ and $M_\phi H$ both are in $\boldsymbol{ESHO}(L^2)$. Since two Toeplitz operators do not commute in general [5, 9] therefore for any $A \in \boldsymbol{ESHO}(H^2)$ we immediately have the following:

(i) If ϕ is analytic then $AT_\phi \in \boldsymbol{ESHO}(H^2)$.
(ii) If ϕ is co-analytic then $T_\phi A \in \boldsymbol{ESHO}(H^2)$.

However using the fact that the commutator of any Toeplitz operator T_ϕ with $T_{\overline{z}}$ and that with T_{z^2} is compact, one can show that for $A \in \mathbf{ESHO}(H^2)$, AT_ϕ and $T_\phi A$ are in $\mathbf{ESHO}(H^2)$ for any Toeplitz operator T_ϕ.

Theorem 3.8. *There is no invertible operator in the set* $\boldsymbol{ESHO}(H^2)$*.*

Proof. If possible suppose $A \in \boldsymbol{ESHO}(H^2)$ be invertible. Then

$$T_{\overline{z}}A - AT_{z^2} = k_1 \quad \text{for some} \quad k_1 \in \mathcal{K}(H^2).$$

Post multiplying both sides by A^{-1} we have,

$$T_{\overline{z}} = k_2 + AT_{z^2}A^{-1} \quad \text{where} \quad k_2 \in \mathcal{K}(H^2).$$

Now $T_{\overline{z}}$ is a Fredholm operator of index 1 and $k_2 + AT_{z^2}A^{-1}$ is a Fredholm operator of index -2. Therefore, we have $1 = -2$ which is absurd.

Hence there is no invertible operator in the set $\boldsymbol{ESHO}(H^2)$. ∎

More generally, the set $\mathbf{ESHO}(H^2)$ does not contain any Fredholm operator.

Theorem 3.9. *If* $A \in$ *essHank and* $B \in \boldsymbol{ESHO}(H^2)$ *then* $AB \in \boldsymbol{ESTO}(H^2)$*.*

Proof. Let $A \in$ essHank. Then

$$T_z^* A - AT_z = k_1,$$

where $k_1 \in \mathcal{K}(H^2)$. As T_z is unitary in the Calkin algebra $\mathcal{B}(H^2)/\mathcal{K}(H^2)$, we have

$$AT_z^* - T_zA \in \mathcal{K}(H^2).$$

Also $B \in \boldsymbol{ESHO}(H^2)$. Therefore

$$T_z^* B - BT_{z^2} = k_2,$$

for some $k_2 \in \mathcal{K}(H^2)$. Consider

$$\begin{aligned} T_zAB - ABT_{z^2} &= T_zAB - AT_z^*B \ (\mathrm{mod}\ \mathcal{K}(H^2)) \\ &= (T_zA - AT_z^*)B \ (\mathrm{mod}\ \mathcal{K}(H^2)) \\ &\in \mathcal{K}(H^2)\,. \end{aligned}$$

Therefore $AB \in \boldsymbol{ESTO}(H^2)\,.$ ∎

Definition 3.2. *An infinite matrix of the form*

$$R = \begin{bmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

for some complex sequence $\{a_n\}_{n=0}^{\infty} \in \ell^2$*, is called a Rhaly matrix (or a terraced matrix) (see* [10, 12] *for more on this topic). It is known that a Rhaly matrix determines a bounded operator on* H^2 *if* $\{na_n\}$ *is bounded* [10]*, while a Rhaly matrix determines a compact operator if* $\{na_n\} \to 0$ [12]*. If the infinite matrix* R *determines a bounded operator on* H^2 *then the corresponding operator is called a Rhaly operator on* H^2*. We denote by* $\mathcal{R}$*, the set of all Rhaly operators on the space* H^2*.*

Our next result shows that a Rhaly operator on H^2 is the compression of an essentially slant Toeplitz operator iff it is the compression of an essentially slant Hankel operator on H^2. Precisely, we prove the following:

Theorem 3.10. $\mathcal{R} \cap \boldsymbol{ESTO}(H^2) = \mathcal{R} \cap \boldsymbol{ESHO}(H^2)$.

Proof. Let R be a Rhaly operator on H^2 corresponding to the complex sequence $\{a_n\}_{n=0}^{\infty} \in \ell^2$. Then

$$Re_n = \sum_{i=n}^{\infty} a_i e_i \quad \text{for all } n = 0, 1, 2, 3, \ldots$$

where $\{e_n\}_{n=0}^{\infty}$ is the standard basis of H^2.

Define the infinite matrix A as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ a_3 - a_2 & 0 & 0 & 0 & \cdots \\ a_4 - a_3 & a_4 - a_3 & 0 & 0 & \cdots \\ a_5 - a_4 & a_5 - a_4 & a_5 - a_4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Calculations show that the matrix of $T_z R - RT_{z^2}$ with respect to $\{e_n\}_{n=0}^{\infty}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ a_0 & 0 & 0 & 0 & 0 & \cdots \\ a_1 - a_2 & a_1 & 0 & 0 & 0 & \cdots \\ a_2 - a_3 & a_2 - a_3 & a_2 & 0 & 0 & \cdots \\ a_3 - a_4 & a_3 - a_4 & a_3 - a_4 & a_3 & 0 & \cdots \\ a_4 - a_5 & a_4 - a_5 & a_4 - a_5 & a_4 - a_5 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Therefore,

$$T_z R - RT_{z^2} = T_z \operatorname{diag}\{a_0, a_1, a_2, \cdots\} + T_{z^2} \operatorname{diag}\{a_1 - a_2, a_2 - a_3, \cdots\} - T_z A$$

Since $\operatorname{diag}\{a_0, a_1, a_2, \cdots\}$ and $\operatorname{diag}\{a_1 - a_2, a_2 - a_3, \cdots\}$ are compact operators on H^2, it follows that $R \in \boldsymbol{ESTO}(H^2)$ iff $T_z A$ is compact. Therefore, $R \in \boldsymbol{ESTO}(H^2)$ iff $A \in \mathcal{K}(H^2)$. Analogously, the matrix of $T_{\bar{z}} R - RT_{z^2}$ with respect to $\{e_n\}_{n=0}^{\infty}$ is given by

$$\begin{pmatrix} a_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & 0 & \cdots \\ a_3-a_2 & a_3 & a_3 & a_3 & 0 & \cdots \\ a_4-a_3 & a_4-a_3 & a_4 & a_4 & a_4 & \cdots \\ a_5-a_4 & a_5-a_4 & a_5-a_4 & a_5 & a_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Therefore

$$\begin{aligned} T_{\bar{z}}R - RT_{z^2} &= A + \operatorname{diag}\{a_1, a_2, a_3, \cdots\} + T_z \operatorname{diag}\{a_2, a_3, a_4, \cdots\} \\ &\quad + T_{\bar{z}} \operatorname{diag}\{a_1, a_2, a_3 \ \cdots\} \end{aligned}$$

Thus, $R \in \boldsymbol{ESHO}(H^2)$ iff $A \in \mathcal{K}(H^2)$. This concludes the proof. ∎

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