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On Generalized b-Closed Sets

[1]AHMAD AL-OMARI AND [2]MOHD. SALMI MD. NOORANI
School of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
[1]omarimutah1@yahoo.com, [2]msn@pkrisc.cc.ukm.my

Abstract. In this paper, we study the class of generalized b-closed sets and use this notion to consider new weak and stronger forms of continuities associated with these sets. We apply these notions to give new characterization of extremally disconnected spaces and also T_{gs}-spaces.

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1. Introduction and preliminaries

Andrijević [2] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. This type of sets was discussed by Ekici and Caldas [11] under the name of γ-open sets. The class of b-open sets is contained in the class of semi-preopen sets and contains all semi-open sets and preopen sets. The class of b-open sets generates the same topology as the class of preopen sets. Since the advent of these notions, several research paper with interesting results in different respects came to existence (see [1, 2, 5, 11, 13, 17, 18, 27]). Levine [15] introduced the concept of generalized closed sets in topological space and a class of topological spaces called $T_{\frac{1}{2}}$-spaces. Extensive research on generalizing closedness was done in recent years as the notions of a generalized closed, generalized semi-closed, α-generalized closed, generalized semi-preopen closed sets were investigated in [3, 8, 15, 21, 22].

The aim of this paper is to continue the study of generalized closed sets. In particular, the notion of generalized b-closed sets and its various characterizations are given in this paper (see Section 2). In Section 3, we study various forms of continuity associated to generalized b-closed sets. Finally in Section 4, we characterize the T_{gs}-spaces in terms of these notions of continuity.

All through this paper, all spaces X and Y (or (X, τ) and (Y, σ)) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

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Definition 1.1. *A subset A of a space X is said to be:*

(1) *α-open* [19] *if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$;*
(2) *Semi-open* [16] *if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$;*
(3) *Preopen* [23] *or nearly open* [12] *if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$;*
(4) *β-open* [25] *or semi-preopen* [26] *if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$;*
(5) *b-open* [2] *or sp-open* [7], *γ-open* [11] *if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$.*

The complement of a b-open set is said to be b-closed [2]. The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by $\mathrm{bCl}(A)$. The union of all b-open sets of X contained in A is called b-interior of A and is denoted by $\mathrm{bInt}(A)$. The family of all b-open (resp. α-open, semi-open, preopen, β-open,, b-closed, preclosed) subsets of a space X is denoted by $bO(X)$ (resp. $\alpha O(X)$, $SO(X)$, $PO(X)$, $\beta O(X)$, $bC(X)$, $PC(X)$) and the collection of all b-open subsets of X containing a fixed point x is denoted by $bO(X,x)$. The sets $SO(X,x)$, $\alpha O(X,x)$, $PO(X,x)$, $\beta O(X,x)$ are defined analogously.

Definition 1.2. *A subset A of a space (X,τ) is called*

(1) *a generalized closed set (briefly g-closed)* [15] *if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open;*
(2) *an α-generalized closed set (briefly αg-closed)* [22] *if $\alpha\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open;*
(3) *a generalized preclosed set (briefly gp-closed)* [21] *if $\mathrm{pCl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open;*
(4) *a generalized semi-preclosed set (briefly gsp-closed)* [8] *if $\mathrm{spCl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open;*
(5) *a generalized semiclosed set (briefly gs-closed)* [3] *if $\mathrm{sCl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open;*
(6) *a generalized semiclosed set (briefly sg-closed)* [4] *if $\mathrm{sCl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open.*

Complement of g-closed (resp. gp-closed, etc.) sets are called g-open (resp. gp-open, etc.).

Definition 1.3. *A function $f : X \to Y$ is said to be b-continuous* [11] *if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in bO(X,x)$ such that $f(U) \subseteq V$.*

Definition 1.4. *A function $f : X \to Y$ is said to be contra continuous* [9] *(resp. contra b-continuous* [17]*) if $f^{-1}(V)$ closed (resp. b-closed) in X for each open set V of Y.*

Definition 1.5. [11] *A function $f : X \to Y$ is said to be b-irresolute if for each b-open set V in Y, $f^{-1}(V)$ is b-open in X.*

Definition 1.6. *A function $f : X \to Y$ is said to be b-closed (resp. b-open* [11]*) if for every b-closed (resp. b-open) subset A of X, $f(A)$ is b-closed (resp. b-open) in Y.*

Definition 1.7. *A map $f : X \to Y$ is said to be contra b-closed (resp. contra b-open) if $f(U)$ is b-open (resp. b-closed) in Y for each closed (resp. open) set U of X.*

We have the following implications for properties of subsets:

$$\begin{array}{ccccccc} \text{closed} & \Rightarrow & \alpha\text{-closed} & \Rightarrow & \text{pre-closed} & \Rightarrow & b\text{-closed} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{g-closed} & \Rightarrow & \alpha\text{g-closed} & \Rightarrow & \text{gp-closed} & \Rightarrow & \text{g}b\text{-closed} \end{array}$$

2. Generalized b-closed sets

In this section, we investigated the class of generalized b-closed sets and study some of its fundamental properties. Several characterizations of generalized b-closed sets are given.

Definition 2.1. [14] *Let X be a space. A subset A of X is called a generalized b-closed set (simply; gb-closed set) if* $\mathrm{bCl}(A) \subseteq U$*, whenever $A \subseteq U$ and U is open. The complement of a generalized b-closed set is called generalized b-open (simply; gb-open).*

Every b-closed set is gb-closed, but the converse is not true. And the collection of all gb-closed (resp. gb-open) subsets of X is denoted by g$bC(X)$ (resp. g$bO(X)$).

Example 2.1. Let $X = \{a, b, c\}$ and let $\tau = \{\phi, X, \{a\}\}$, then the family of all b-closed set of X is $bC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$ but the family of all gb-closed set of X is g$bC(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ then it is clear that $\{a, c\}$ is gb-closed but not b-closed in X.

Next, we give a necessary and sufficient condition for a gb-closed set to be b-closed.

Theorem 2.1. *Let A be a gb-closed subset of (X, τ). Then* $\mathrm{bCl}(A) - A$ *does not contain any non-empty closed sets.*

Proof. Let $F \in C(X)$ such that $F \subseteq \mathrm{bCl}(A) - A$. Since $X - F$ is open, $A \subseteq X - F$ and A is gb-closed, it follows that $\mathrm{bCl}(A) \subseteq X - F$ and thus $F \subseteq X - \mathrm{bCl}(A)$. This implies that $F \subseteq (X - \mathrm{bCl}(A)) \cap (\mathrm{bCl}(A) - A) = \phi$ and hence $F = \phi$. ∎

Corollary 2.1. *Let A a gb-closed set. Then A is b-closed if and only if* $\mathrm{bCl}(A) - A$ *is closed.*

Proof. Let A be gb-closed set. If A is b-closed, then we have $\mathrm{bCl}(A) - A = \phi$ which is closed set. Conversely, let $\mathrm{bCl}(A) - A$ be closed. Then, by Theorem 2.1, $\mathrm{bCl}(A) - A$ does not contain any non-empty closed subset and since $\mathrm{bCl}(A) - A$ is closed subset of itself, then $\mathrm{bCl}(A) - A = \phi$. This implies that $A = \mathrm{bCl}(A)$ and so A is b-closed set. ∎

Definition 2.2. *Let A be a subset of a space X. A point $x \in X$ is said to be a b-limit point of A if for each b-open set U containing x, we have $U \cap (A - \{x\}) \neq \phi$. The set of all b-limit points of A is called the b-derived set of A and is denoted by $D_b(A)$.*

Since every open set is b-open, we have $D_b(A) \subseteq D(A)$ for any subset $A \subseteq X$, where $D(A)$ is the derived set of A. Moreover, since every closed set is b-closed, we have $A \subseteq \mathrm{bCl}(A) \subseteq \mathrm{Cl}(A)$.
The proof of the following result is straightforward and thus omitted.

Lemma 2.1. *If $D(A) = D_b(A)$, then we have* $\mathrm{Cl}(A) = \mathrm{bCl}(A)$.

Corollary 2.2. *If $D(A) \subseteq D_b(A)$ for every subset A of X. Then for any subsets F and B of X, we have* $\mathrm{bCl}(F \cup B) = \mathrm{bCl}(F) \cup \mathrm{bCl}(B)$.

Corollary 2.3. *If A and B are gb-closed sets such that $D(A) \subseteq D_b(A)$ and $D(B) \subseteq D_b(B)$. Then $A \cup B$ is gb-closed.*

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B be gb-closed sets we have $\mathrm{bCl}(A) \subseteq U$ and $\mathrm{bCl}(B) \subseteq U$. Since $D(A) \subseteq D_b(A)$, thus $D(A) = D_b(A)$ and by Lemma 2.1, $\mathrm{Cl}(A) = \mathrm{bCl}(A)$. Similarly, $\mathrm{Cl}(B) = \mathrm{bCl}(B)$. Thus $\mathrm{bCl}(A \cup B) \subseteq \mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \mathrm{bCl}(A) \cup \mathrm{bCl}(B) \subseteq U$, which implies that $A \cup B$ is gb-closed. ∎

Let $B \subseteq A \subseteq X$. Then we say that B is gb-closed relative to A if $\mathrm{bCl}_A(B) \subseteq U$ where $B \subseteq U$ and U is open in A.

Theorem 2.2. *Let $B \subseteq A \subseteq X$ where A is a gb-closed and open set. Then B is gb-closed relative to A if and only if B is gb-closed in X.*

Proof. We first note that since $B \subseteq A$ and A is both a gb-closed and open set, then $\mathrm{bCl}(A) \subseteq A$ and thus $\mathrm{bCl}(B) \subseteq \mathrm{bCl}(A) \subseteq A$. Now from the fact that $A \cap \mathrm{bCl}(B) = \mathrm{bCl}_A(B)$, we have $\mathrm{bCl}(B) = \mathrm{bCl}_A(B) \subseteq A$. If B is gb-closed relative to A and U is open subset of X such that $B \subseteq U$, then $B = B \cap A \subseteq U \cap A$ where $U \cap A$ is open in A. Hence as B is gb-closed relative to A, $\mathrm{bCl}(B) = \mathrm{bCl}_A(B) \subseteq U \cap A \subseteq U$. Therefore B is gb-closed in X.

Conversely if B is gb-closed in X and U is an open subset of A such that $B \subseteq U$, then $U = V \cap A$ for some open subset V of X. As $B \subseteq V$ and B is gb-closed in X, $\mathrm{bCl}(B) \subseteq V$. Thus $\mathrm{bCl}_A(B) = \mathrm{bCl}(B) \cap A \subseteq V \cap A = U$. Therefore B is gb-closed relative to A. ∎

Corollary 2.4. *Let A be open gb-closed set. Then $A \cap F$ is gb-closed whenever $F \in bC(X)$.*

Proof. Since A is gb-closed and open, then $b\,\mathrm{Cl}(A) \subseteq A$ and thus A is b-closed. Hence $A \cap F$ is b-closed in X which implies that $A \cap F$ is gb-closed in X. ∎

Theorem 2.3. *If A is a gb-closed set and B is any set such that $A \subseteq B \subseteq \mathrm{bCl}(A)$, then B is a gb-closed set.*

Proof. Let $B \subseteq U$ where U is open set. Since A is gb-closed and $A \subseteq U$, then $\mathrm{bCl}(A) \subseteq U$ and also $\mathrm{bCl}(A) = \mathrm{bCl}(B)$. Therefore $\mathrm{bCl}(B) \subseteq U$ and hence B is a gb-closed set. ∎

Theorem 2.4. *A subset $A \subseteq X$ is gb-open if and only if $F \subseteq \mathrm{bInt}(A)$ whenever F is closed set and $F \subseteq A$.*

Proof. Let A be a gb-open set and suppose $F \subseteq A$ where F is closed. Then $X - A$ is a gb-closed set contained in the open set $X - F$. Hence $\mathrm{bCl}(X - A) \subseteq X - F$ and $X - \mathrm{bInt}(A) \subseteq X - F$. Thus $F \subseteq \mathrm{bInt}(A)$.

Conversely if F is a closed set with $F \subseteq \mathrm{bInt}(A)$ and $F \subseteq A$, then $X - \mathrm{bInt}(A) \subseteq X - F$. Thus $\mathrm{bCl}(X - A) \subseteq X - F$. Hence $X - A$ is a gb-closed set and A is a gb-open set. ∎

Proposition 2.1. [10] *Let A be a subset of a topological space (X, τ). If $A \in SO(X)$, then $p\,\mathrm{Cl}(A) = \mathrm{Cl}(A)$.*

Recall that a space (X, τ) is extremally disconnected if the closure of every open subset of X is open. The following result about the class of extremally disconnected space was prove in [6].

Theorem 2.5. [6] *For a space X, the following statements are equivalent:*

(1) *(X, τ) is extremally disconnected;*
(2) *$s\mathrm{Cl}(A \cup B) = s\mathrm{Cl}(A) \cup s\mathrm{Cl}(B)$ for all $A, B \subseteq X$;*
(3) *the union of two semi-closed subsets of X is semi-closed;*
(4) *the union of two sg-closed subsets of X is sg-closed;*
(5) *every semi-preclosed subset of X is preclosed;*
(6) *every sg-closed subset of X is perclosed;*
(7) *every semi-closed subset of X is preclosed;*
(8) *every semi-closed subset of X is α-closed;*
(9) *every semi-closed subset of X is $g\alpha$-closed.*

Next, we present one more characterization of extremally disconnectedness using generalized closed subsets.

Theorem 2.6. *A space X is extremally disconnected if and only if every gb-closed subset of X is gp-closed.*

Proof. Suppose that X is extremally disconnected. Let A be gb-closed and let U be an open set containing A. Then

$$\mathrm{bCl}(A) = A \cup [\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))] \subseteq U,$$

i.e. $[\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))] \subseteq U$. Since $\mathrm{Int}(\mathrm{Cl}(A))$ is closed, we have

$$\mathrm{Cl}(\mathrm{Int}(A)) \subseteq Cl[\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(A)] \subseteq [\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \cap \mathrm{Cl}(\mathrm{Int}(A))] \subseteq U.$$

It follows that $p\,\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(A)) \subseteq U$. Hence A is gp-closed.
To prove the converse, let every gb-closed subset of X be gp-closed. Let $A \subseteq X$ be regular open. Then

$$\mathrm{bCl}(A) = A \cup [\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))] = A \cup [A \cap \mathrm{Cl}(\mathrm{Int}(A))] \subseteq A.$$

Then A is gb-closed and so gp-closed. Since every regular open is semi open set by Proposition 2.1 and A is gp-closed we have $\mathrm{Cl}(A) = p\,\mathrm{Cl}(A) \subseteq A$. Therefore A is closed and X is extremally disconnected. ∎

3. Ap-b-continuous, ap-b-closed and contra-b-continuous maps

We introduce the following notion:

Definition 3.1. *A map $f : X \to Y$ is said to be approximately b-continuous (briefly ap-b-continuous) if $\mathrm{bCl}(F) \subseteq f^{-1}(U)$ whenever U is an open subset of Y and F is a gb-closed subset of X such that $F \subseteq f^{-1}(U)$.*

Definition 3.2. *A map $f : X \to Y$ is said to be approximately b-closed (briefly ap-b-closed) if $f(F) \subseteq \mathrm{bInt}(V)$ whenever V is a gb-open subset of Y, F is an closed subset of X and $f(F) \subseteq V$.*

Definition 3.3. *A map $f : X \to Y$ is said to be approximately b-open (briefly ap-b-open) if* $\mathrm{bCl}(F) \subseteq f(U)$ *whenever U is an open subset of X, F is a gb-closed subset of Y and $F \subseteq f(U)$.*

Theorem 3.1. *Let $f : X \to Y$ be a function. Then*

(1) *If f is contra b-continuous, then f is ap-b-continuous.*
(2) *If f is contra b-closed, then f is ap-b-closed.*
(3) *If f is contra b-open, then f ap-b-open.*

Proof.

(1) Let $F \subseteq f^{-1}(U)$, where U is open in Y and F is a gb-closed subset of X. Therefore $\mathrm{bCl}(F) \subseteq b\,\mathrm{Cl}(f^{-1}(U))$. Since f is contra b-continuous we have $\mathrm{bCl}(F) \subseteq \mathrm{bCl}(f^{-1}(U)) = f^{-1}(U)$. Thus f is ap-b-continuous.
(2) Let $f(F) \subseteq V)$, where F is a closed subset of X and V is a gb-open subset of Y. Therefore $f(F) = \mathrm{bInt}(f(F)) \subseteq \mathrm{bInt}(V)$. Thus f is ap-b-closed.
(3) Analogous to (1) and (2) making the obvious changes. ∎

The converse of Theorem 3.1 is not true. The following example shows that:

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, then $bC(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}\} = gbC(X)$. Let $f : (X, \tau) \to (X, \tau)$ be the identity map. Then f is ap-b-continuous (since every gb-closed is b-closed) but not contra b-continuous.

Example 3.2. Let $X = \{a, b, c\}$, with $\tau = \{\phi, X, \{a\}\}$, and $Y = \{a, b, c\}$, with $\sigma = \{\phi, Y, \{a\}, \{a, b\}\}$, then $bO(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$, g$bC(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. And $bO(Y) = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$, g$bC(Y) = \{\phi, Y, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is ap-b-closed (since the only gb-open subset of (Y, σ) containing the image of closed set V in X is Y) but not contra b-closed.

Also the function in Example 3.2 is ap-b-open, but not contra b-open.

Definition 3.4. [20] *A function $f : X \to Y$ is said to be perfectly continuous (resp. perfectly closed) if the inverse image (resp. image) of every open set in Y (resp. closed set in X) is clopen in X (resp. clopen in Y).*

A function $f : X \to Y$ is said to be pre-closed (resp. pre-open) if for every pre-closed (resp. pre-open) subset A of X, $f(A)$ is pre-closed (resp. pre-open) in Y. Clearly continuous maps are ap-b-continuous, pre-closed maps are ap-b-closed. Also pre-open maps are ap-b-open. The converse implications do not hold as it is shown in the following example.

Example 3.3. Let $X = \{a, b, c\}$, with $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, and $Y = \{a, b, c\}$, with $\sigma = \{\phi, Y, \{a, b\}\}$, then $bO(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, And $bO(Y) = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is ap-b-continuous (since f is contra b-continuous) but not continuous.

Example 3.4. Let $X = \{a, b, c\}$, with $\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, and $Y = \{a, b, c\}$, with $\sigma = \{\phi, Y, \{a\}\}$, then $bO(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. And $bO(Y) = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$, Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is ap-b-closed (since f is contra b-closed) but not pre-closed map.

Clearly, the following two diagrams holds and none of its implications is reversible:

$$\begin{array}{ccc} \text{perfectly continuous} & \Rightarrow & \text{contra-}b\text{-continuous} \\ \Downarrow & & \Downarrow \\ \text{continuous} & \Rightarrow & \text{ap-}b\text{-continuous} \end{array}$$

$$\begin{array}{ccc} \text{perfectly closed} & \Rightarrow & \text{contra-}b\text{-closed} \\ \Downarrow & & \Downarrow \\ \text{per-closed} & \Rightarrow & \text{ap-}b\text{-closed} \end{array}$$

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map.*

(1) *If the open and b-closed sets of (X, τ) coincide, then f is ap-b-continuous if and only if f is contra b-continuous.*
(2) *If the open and b-closed sets of (Y, σ) coincide , then f is ap-b-closed if and only if f is contra-b-closed.*
(3) *If the open and b-closed sets of (Y, σ) coincide, then f is ap-b-open if and only if f is contra-b-open.*

Proof.

(1) Assume f is ap-b-continuous. Let A be an arbitrary subset of (X, τ) such that $A \subseteq U$, where U is open in X. Then by hypothesis $\mathrm{bCl}(A) \subseteq \mathrm{bCl}(U) = U$. Therefore all subsets of (X, τ). are gb-closed (and hence all are gb-open). So for any open set V of (Y, σ), we have $f^{-1}(V)$ is gb-closed in (X, τ). Since f ap-b-continuous $\mathrm{bCl}(f^{-1}(V)) \subseteq f^{-1}(V)$. Therefore $\mathrm{bCl}(f^{-1}(V)) = f^{-1}(V)$. Thus $b\,\mathrm{Cl}(f^{-1}(V))$ is b-closed in (X, τ) and f is contra b-continuous. The converse is obvious by Theorem 3.1.
(2) Assume f is ap-b-closed. As in (1), we obtain that all subsets of (Y, σ) are gb-open. Therefore for any closed subset F of (X, τ), $f(F)$ is gb-open in Y. Since f is ap-b-closed, we have $f(F) \subseteq b\,\mathrm{Int}(f(F))$. Therefore $f(F) = \mathrm{bInt}(f(F))$, and f is contra b-closed. The converse is obvious by Theorem 3.1.
(3) Analogous to (1) and (2) making the obvious changes. ∎

Lemma 3.1. [2] *Let A be a subset of a space X. Then*

(1) $\mathrm{bCl}(A) = \mathrm{sCl}(A) \cap \mathrm{pCl}(A) = A \cup [\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Cl}(\mathrm{Int}(A))]$;
(2) $\mathrm{bInt}(A) = \mathrm{sInt}(A) \cup \mathrm{pInt}(A) = A \cap [\mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$;
(3) $\mathrm{bCl}(X - A) = X - \mathrm{bInt}(A)$;
(4) $\mathrm{bInt}(X - A) = X - \mathrm{bCl}(A)$.

Theorem 3.3. *If a map $f : X \to Y$ is surjective b-irresolute and ap-b-closed, then the inverse image of each gb-closed (resp. gb-open) set in Y is gb-closed (resp. gb-open) in X.*

Proof. Let A be a gb-closed subset of Y. Suppose that $f^{-1}(A) \subseteq U$ where U is an open subset of X. Taking complements we obtain $X - U \subseteq f^{-1}(Y - A)$ or $f(X - U) \subseteq Y - A$. Since f is ap-b-closed and by Lemma 3.1, then $f(X - U) \subseteq \mathrm{bInt}(Y - A) = Y - \mathrm{bCl}(A)$. It follows that $X - U \subseteq X - (f^{-1}(\mathrm{bCl}(A))$ and hence $f^{-1}(\mathrm{bCl}(A) \subseteq U$. Since f is b-irresolute $f^{-1}(\mathrm{bCl}(A))$ is b-closed. Thus we have

$\mathrm{bCl}(f^{-1}(A)) \subseteq \mathrm{bCl}(f^{-1}(\mathrm{bCl}(A))) = (f^{-1}(\mathrm{bCl}(A)) \subseteq U$. This implies that $f^{-1}(A)$ is gb-closed in X. A similar argument shows that inverse images of gb-open are gb-open. ∎

Theorem 3.4. *If a map $f : X \to Y$ is surjective b-irresolute and ap-b-open, then the inverse image of each gb-open set in Y is gb-open in X.*

Proof. Analogous to Theorem 3.3 making the obvious changes. ∎

Theorem 3.5. *If a map $f : X \to Y$ is ap-b-continuous and b-closed, then the image of each gb-closed set in X is gb-closed in Y.*

Proof. Let F be a gb-closed subset of X. Let $f(F) \subseteq V$ where V is an open set of Y. Then $F \subseteq f^{-1}(V)$ holds. Since f is ap-b-continuous we have $\mathrm{bCl}(F) \subseteq f^{-1}(V)$. Then $f(\mathrm{bCl}(F)) \subseteq V$. Therefore, we have $\mathrm{bCl}(f(F)) \subseteq \mathrm{bCl}(f(\mathrm{bCl}(F))) = f(\mathrm{bCl}(F)) \subseteq V$. Hence $f(F)$ is gb-closed in Y. ∎

Theorem 3.6. *If $f : X \to Y$ is a continuous and b-closed function, then $f(A)$ is gb-closed in Y for every gb-closed set A of X.*

Proof. Let A be gb-closed in X. Let $f(A) \subseteq V$, where V be any open set in Y. Since f is continuous, $f^{-1}(V)$ is open in X and $A \subseteq f^{-1}(V)$. Then we have $\mathrm{bCl}(A) \subseteq f^{-1}(V)$ and so $f(\mathrm{bCl}(A)) \subseteq V$. Since f is b-closed, $f(\mathrm{bCl}(A))$ is b-closed in Y and hence $\mathrm{bCl}(f(A) \subseteq \mathrm{bCl}(f(\mathrm{bCl}(A))) = f(\mathrm{bCl}(A)) \subseteq V$. This shows that $f(A)$ is gb-closed in Y. ∎

Definition 3.5. *A map $f : X \to Y$ is said to be contra-b-irresolute if $f^{-1}(U)$ is b-closed in X for each $U \in bO(Y)$.*

However the following theorem holds. The proof is easy and hence omitted.

Theorem 3.7. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps such that $(g \circ f) : X \to Z$,*

1. *If g is b-continuous and f is contra-b-irresolute, then $(g \circ f)$ is contra-b-continuous.*
2. *If g is b-irresolute and f is contra-b-irresolute, then $(g \circ f)$ is contra-b-irresolute.*

In an analogous way, we have the following.

Theorem 3.8. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps such that $(g \circ f) : X \to Z$.*

1. *If f is pre-closed and g is ap-b-closed, then $(g \circ f)$ is ap-b-closed.*
2. *If f is ap-b-closed and g is b-open and g^{-1} preserves gb-open sets, then $(g \circ f)$ is ap-b-closed.*
3. *If f is ap-b-continuous and g is continuous, then $(g \circ f)$ is ap-b-continuous.*

Proof.

1. Suppose B is an arbitrary closed subset in X and A is a gb-open subset of Z for which $(g \circ f)(B) \subseteq A$. Then $f(B)$ is closed in Y because f is pre-closed. Since g is ap-b-closed, $g(f(B)) \subseteq \mathrm{bInt}(A)$. This implies that $(g \circ f)$ is ap-b-closed.

(2) Suppose B is an arbitrary closed subset of X and A is a gb-open subset of Z for which $(g \circ f)(B) \subseteq A$. Hence $f(B) \subseteq g^{-1}(A)$. Then $f(B) \subseteq \mathrm{bInt}(g^{-1}(A))$ because $g^{-1}(A)$ is gb-open and f is ap-b-closed. Thus, $(g \circ f)(B) = g(f(B)) \subseteq g(\mathrm{bInt}(g^{-1}(A))) \subseteq \mathrm{bInt}(gg^{-1}(A)) \subseteq \mathrm{bInt}(A)$. This implies that $(g \circ f)$ is ap-b-closed.

(3) Suppose F is an arbitrary gb-closed subset of X and $U \in O(Z)$ for which $F \subseteq (g \circ f)^{-1}(U)$. Then $g^{-1}(U) \in O(Y)$ because g is continuous. Since f is ap-b-continuous, then we have $\mathrm{bCl}(F) \subseteq f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. This show that $(g \circ f)$ is ap-b-continuous. ∎

4. A characterization of T_{gs}-space

In the following theorems, we give a characterization of a class of topological space called T_{gs}-space by using the concepts of ap-b-continuous maps and ap-b-closed maps.

Definition 4.1. [14] *A topological space X is said to be T_{gs}-space if every gs-closed subset of (X, τ) is sg-closed.*

Proposition 4.1. [14] *A topological space X is said to be T_{gs}-space if and only if every gb-closed set is b-closed.*

Recall that a space X is $T_{\frac{1}{2}}$-space [15] if every g-closed set is closed or equivalently if every singleton is open or closed.

Theorem 4.1. *Every $T_{\frac{1}{2}}$-space is T_{gs}-space.*

Proof. Let (X, τ) be $T_{\frac{1}{2}}$ and suppose $A \subseteq X$ is not a b-closed set. Let $x \in \mathrm{bCl}(A) - A$, Then $\{x\} \subseteq \mathrm{bCl}(A) - A$. Since X is $T_{\frac{1}{2}}$, $\{x\}$ is a closed set and hence by Theorem 2.1, A is not a gb-closed set. This proves the theorem. ∎

Definition 4.2. [24] *A space (X, τ) is said to be a b-$T_{\frac{1}{2}}$ if every each singleton is either b-open or b-closed set.*

Lemma 4.1. [24] *Let (X, τ) be a topological space, then the space $bO(X, \tau)$ is b-$T_{\frac{1}{2}}$-space.*

Example 4.1. Let $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ then $bC(X) =$g$bC(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then X is a T_{gs}-space but not a $T_{\frac{1}{2}}$-space.

Example 4.2. Let $X = \{a, b, c, d\}$ with $\tau = \{\phi, X, \{a, b\}, \{b, c, d\}\}$ then $bC(X) = \{\phi, X, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. And $U = \{a, b, c\}$ is gb-closed since the only open set containing U is X. Therefore X is not a T_{gs}-space but b-$T_{\frac{1}{2}}$-space.

Clearly, the following diagram holds and none of its implications is reversible:

$$T_{\frac{1}{2}}\text{-space} \Rightarrow T_{gs}\text{-space} \Rightarrow b\text{-}T_{\frac{1}{2}}\text{-space}$$

Theorem 4.2. *For a topological space X, if every gb-closed subset of X is closed, then X is $T_{\frac{1}{2}}$-space.*

Proof. Let $x \in X$. If $\{x\}$ is not closed, then $A = X - \{x\}$ is not open and hence A is gb-closed, since the only open set containing A is X. Thus A is closed, and $\{x\}$ is open in X and so X is $T_{\frac{1}{2}}$-space. ∎

Definition 4.3. *A function $f : X \to Y$ is said to be gb-continuous (resp. gb-irresolute) if $f^{-1}(V)$ is gb-closed in X for every closed (resp. gb-closed) set V of Y.*

Clearly, $f : X \to Y$ is gb-continuous (resp. gb-irresolute) if $f^{-1}(V)$ is gb-open in (X, τ) for every open (resp. gb-open) set V of Y.

Theorem 4.3. *Let $f : X \to Y$ be a function.*

(1) *If f is gb-irresolute and X is T_{gs}-space, then f is b-irresolute.*
(2) *If f is gb-continuous and X is T_{gs}-space, then f is b-continuous.*

Proof.
(1) Let V be b-closed in Y. Then V is gb-closed in Y and since f is gb-irresolute, then $f^{-1}(V)$ is gb-closed in X. Since X is T_{gs}-space, $f^{-1}(V)$ is b-closed in X. Hence f is b-irresolute.
(2) Similar to (1). ∎

Theorem 4.4. *If the bijective $f : X \to Y$ is b-irresolute and open, then f is gb-irresolute.*

Proof. Let V be gb-closed and let $f^{-1}(V) \subseteq U$, where U is open in X. Clearly $V \subseteq f(U)$, since $f(U)$ is open (f is open map) and since V is gb-closed in Y, then $\mathrm{bCl}(V) \subseteq f(U)$ and thus $f^{-1}(\mathrm{bCl}(V)) \subseteq U$. Since f is b-irresolute and since $\mathrm{bCl}(V)$ is a b-closed set, then $f^{-1}(\mathrm{bCl}(V))$ is a b-closed in X. Thus $\mathrm{bCl}(f^{-1}(V)) \subseteq bCl(f^{-1}(\mathrm{bCl}(V))) = f^{-1}(\mathrm{bCl}(V)) \subseteq U$. So $f^{-1}(V)$ is gb-closed and f is gb-irresolute. ∎

The converse of above results need not be true may be seen by the following examples.

Example 4.3. Let $X = \{a, b, c\}$, with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, and $Y = \{a, b, c\}$, with $\sigma = \{\phi, Y, \{a\}\}$, then $bC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, and $bC(Y) = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$, Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is b-irresolute but not gb-irresolute since $V = \{a, b\} \in gb$-closed of (Y, σ) but not gb-closed of (X, τ).

Example 4.4. Let $X = \{a, b, c\}$, with $\tau = \{\phi, X, \{a\}\}$, then $bC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$, and g$bC(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$, Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = f(c) = b, f(b) = c$. Then f is gb-irresolute but not b-irresolute since $V = \{b\}$ is b-closed and $f^{-1}\{b\} = \{a, c\}$ is not b-closed of (X, τ).

Theorem 4.5. *If $f : X \to Y$ is an open, b-irresolute and b-closed surjection functions. If X is a T_{gs}-space, then Y is T_{gs}-space.*

Proof. Let F be a gb-closed set of Y. Let G be open subset of X such that $f^{-1}(F) \subseteq G$, then $F \subseteq f(G)$ and $f(G)$ is open. Since F is gb-closed, then $\mathrm{bCl}(F) \subseteq f(G)$ and $f^{-1}(\mathrm{bCl}(F)) \subseteq G$. But f is b-irresolute then $f^{-1}(\mathrm{bCl}(F))$ is b-closed and $\mathrm{bCl}(f^{-1}(\mathrm{bCl}(F))) = f^{-1}(\mathrm{bCl}(F)) \subseteq G$ also $\mathrm{bCl}(f^{-1}(F)) \subseteq \mathrm{bCl}(f^{-1}(\mathrm{bCl}(F))) \subseteq G$ thus $f^{-1}(F)$ is gb-closed in X. Since X is a T_{gs}-space, then $f^{-1}(F)$ is b-closed in

X. By the rest of the assumption it follows that F is b-closed in Y. Hence Y is T_{gs}-space. ∎

Theorem 4.6. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is T_{gs}-space.*
(2) *For every space Y and every map $f : X \to Y$, f is ap-b-continuous.*

Proof.
(1) ⇒ (2): Let F be a gb-closed subset of X and suppose that $F \subseteq f^{-1}(U)$, where U is open. Since (X, τ) is T_{gs}, then F is b-closed. Therefore $b\,\mathrm{Cl}(F) = F \subseteq f^{-1}(U)$. Then f is ap-b-continuous.
(2) ⇒ (1): Let B be a gb-closed subset of X and let Y be the set X with topology $\sigma = \{\phi, B, Y\}$. Let $f : X \to Y$ be the identity map. By assumption f is ap-b-continuous. Since B is gb-closed in X and open in Y and $B \subseteq f^{-1}(B)$, it follows that bCl$(B) \subseteq f^{-1}(B) = B$. Hence B is b-closed in X and hence (X, τ) is T_{gs}-space. ∎

Theorem 4.7. *Let Y be a topological space. Then the following statements are equivalent:*

(1) *Y is T_{gs}-space.*
(2) *For every space (X, τ) and every map $f : X \to Y$, f is ap-b-closed (or ap-b-open).*

Proof. Analogous to Theorem 4.6 making the obvious changes. ∎

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References

[1] D. Andrijević, Semipreopen sets, *Mat. Vesnik* **38** (1986), no. 1, 24–32.
[2] D. Andrijević, On b-open sets, *Mat. Vesnik* **48** (1996), no. 1–2, 59–64.
[3] S. P. Arya and T. M. Nour, Characterizations of s-normal spaces, *Indian J. Pure Appl. Math.* **21** (1990), no. 8, 717–719.
[4] P. Bhattacharyya and B. K. Lahiri, Semigeneralized closed sets in topology, *Indian J. Math.* **29** (1987), no. 3, 375–382 (1988).
[5] M. Caldas and S. Jafari, On some applications of b-open sets in topological spaces, *Kochi J. Math.* **2** (2007), 11–19.
[6] J. Cao, M. Ganster and I. Reilly, Submaximality, extremal disconnectedness and generalized closed sets, *Houston J. Math.* **24** (1998), no. 4, 681–688.
[7] J. Dontchev and M. Przemski, On the various decompositions of continuous and some weakly continuous functions, *Acta Math. Hungar.* **71** (1996), no. 1–2, 109–120.
[8] J. Dontchev, On generalizing semi-preopen sets, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.* **16** (1995), 35–48.
[9] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *Internat. J. Math. Math. Sci.* **19**(1996), 303–310.
[10] J. Dontchev, M. Ganster and T. Noiri, On p-closed spaces, *Int. J. Math. Math. Sci.* **24** (2000), no. 3, 203–212.
[11] E. Ekici and M. Caldas, Slightly γ-continuous functions, *Bol. Soc. Parana. Mat. (3)* **22** (2004), no. 2, 63–74.

[12] M. Ganster and I. L. Reilly, Locally closed sets and LC-continuous functions, *Internat. J. Math. Math. Sci.* **12** (1989), no. 3, 417–424.

[13] M. Ganster and M. Steiner, On some questions about b-open sets, *Questions Answers Gen. Topology* **25** (2007), no. 1, 45–52.

[14] M. Ganster and M. Steiner, On $b\tau$-closed sets, *Appl. Gen. Topol.* **8** (2007), no. 2, 243–247.

[15] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo (2)* **19** (1970), 89–96.

[16] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly* **70** (1963), 36–41.

[17] A. A. Nasef, Some properties of contra-γ-continuous functions, *Chaos Solitons Fractals* **24** (2005), no. 2, 471–477.

[18] A. A. Nasef, On b-locally closed sets and related topics, *Chaos Solitons Fractals* **12** (2001), no. 10, 1909–1915.

[19] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* **15** (1965), 961–970.

[20] T. Noiri, On δ-continuous functions, *J. Korean Math. Soc.* **16** (1979/80), no. 2, 161–166.

[21] H. Maki, J. Umehara and T. Noiri, Every topological space is pre-$T_{1/2}$, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.* **17** (1996), 33–42.

[22] H. Maki, R. Devi and K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.* **15** (1994), 51–63.

[23] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt No. 53* (1982), 47–53 (1983).

[24] M. E. Abd EL-Monsef, A. A. El-Atik and M. M. El-Sharkasy, Some topologies induced by b-open sets, *Kyungpook Math. J.* **45** (2005), no. 4, 539–547.

[25] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mapping, *Bull. Fac. Sci. Assiut Univ. A* **12** (1983), no. 1, 77–90.

[26] M. E. Abd El-Monsef and A. A. Nasef, On multifunctions, *Chaos Solitons Fractals* **12** (2001), no. 13, 2387–2394.

[27] J. H. Park, Strongly θ-b-continuous functions, *Acta Math. Hungar.* **110** (2006), no. 4, 347–359.