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Neighborhood Conditions and Fractional k-Factors

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Abstract. Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. In this paper, it is proved that a graph G has a fractional k-factor if $|N_G(x) \cup N_G(y)| \geq \max\{n/2, (n+k-2)/2\}$ for each pair of non-adjacent vertices $x, y \in V(G)$.

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1. Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. Set $d_G(S) = \sum_{x \in S} d_G(x)$, where S is a set of vertices of G. The minimum vertex degree of G is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$ the number of edges joining S and T. We write $I(G)$ for the set of isolated vertices in G and $i(G) = |I(G)|$. The number of connected components of G is denoted by $\omega(G)$.

Let k be an integer such that $k \geq 1$. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for all $x \in V(G)$. If $k = 1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x we have $d_G^h(x) = k$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional

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degree of x in G. If $k = 1$, then a fractional k-factor is a fractional 1-factor. The other terminologies and notations not given in this paper can be found in [1,2].

Many authors have investigated k-factors [3–7]. In [6], Iida and Nishimura gave a neighborhood condition for a graph to have a k-factor. Recently, Zhou obtained some sufficient conditions for graphs to have factors or fractional factors [8–10]. There is a necessary and sufficient condition for a graph to have a fractional k-factor which was given by Liu and Zhang [11,12].

Theorem 1.1. [11] *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| - k|T| + d_{G-S}(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$.

Theorem 1.2. [12] *Let G be a graph. Then G has a fractional k-factor if and only if for all subset S of $V(G)$,*

$$k|S| - \sum_{j=0}^{k-1}(k-j)p_j(G-S) \geq 0,$$

where $p_j(G-S)$ denotes the number of vertices in $G-S$ with degree j.

In Theorem 1.2, let $k = 1$, then we get the following result.

Theorem 1.3. [2] *A graph G has a fractional 1-factor if and only if $i(G-S) \leq |S|$ for all $S \subseteq V(G)$.*

In [6], Iida and Nishimura gave a neighborhood condition for a graph to have a k-factor.

Theorem 1.4. [6] *Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies*

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k-factor.

In this paper, we give a neighborhood condition for a graph to have fractional k-factors. The main result is an extension of Theorem 1.4. We extend Theorem 1.4 to fractional k-factors. The main results will be given in the following section.

2. Results and proofs

First we state the numerical result which is often applied in the proof of our Theorems.

Fact 2.1. [6] *Let k be an integer such that $k \geq 1$. Then*

$$9k - 1 - 4\sqrt{2(k-1)^2+2} \begin{cases} > 3k+5, & \text{for } k \geq 4 \\ > 3k+4, & \text{for } k = 3 \\ = 3k+3, & \text{for } k = 2 \\ > 2, & \text{for } k = 1 \end{cases}$$

Now, we give a lemma.

Lemma 2.1. *Let G be a connected graph of order n. If $|N_G(x) \cup N_G(y)| \geq n/2$ for each pair of nonadjacent vertices $x, y \in V(G)$, then $\omega(G-S) \leq |S|+1$ for all $S \subseteq V(G)$ with $|S| \geq 2$.*

Proof. Suppose that there exists a vertex subset $S \subseteq V(G)$ with $|S| \geq 2$ such that $\omega(G-S) \geq |S|+2$. We first claim that $2 \leq |S| \leq (n-2)/2$. For, if $|S| \geq (n-1)/2$, then

$$\omega(G-S) \leq n-|S| \leq n-\frac{n-1}{2}=\frac{n+1}{2}=\frac{n-1}{2}+1 \leq |S|+1,$$

a contradiction.

In the following, let $C_1, C_2, \cdots, C_\omega$ be the connected components of $G-S$. We have $\omega=\omega(G-S) \geq |S|+2 \geq 4$ since $|S| \geq 2$. Choose an arbitrary vertex $x_i \in V(C_i)$ $(1 \leq i \leq \omega(G-S))$. Then $x_i x_j \notin E(G)$ $(i \neq j)$. By the hypothesis of the lemma, then

$$\begin{aligned}
\frac{n}{2} &\leq |N_G(x_i) \cup N_G(x_j)| \\
&\leq d_{G-S}(x_i)+d_{G-S}(x_j)+|S| \\
&\leq |V(C_i)|-1+|V(C_j)|-1+|S|,
\end{aligned}$$

that is,

$$|V(C_i)|+|V(C_j)| \geq \frac{n+4}{2}-|S|$$

for $i \neq j$. Thus, we get that

$$\begin{aligned}
n &= |S|+|V(C_1)|+|V(C_2)|+\cdots+|V(C_\omega)| \\
&= |S|+\frac{1}{2}[2(|V(C_1)|+|V(C_2)|+\cdots+|V(C_\omega)|)] \\
&\geq |S|+\frac{\omega}{2}(\frac{n+4}{2}-|S|) \\
&\geq |S|+\frac{|S|+2}{2}(\frac{n+4}{2}-|S|) \\
&= -\frac{1}{2}|S|^2+\frac{n+4}{4}|S|+\frac{n+4}{2}. \qquad (2.1)
\end{aligned}$$

Let

$$f(|S|)=-\frac{1}{2}|S|^2+\frac{n+4}{4}|S|+\frac{n+4}{2}. \qquad (2.2)$$

In fact, the function $f(|S|)$ attains its minimum value at $|S|=2$ or $|S|=(n-2)/2$ which follows from $2 \leq |S| \leq (n-2)/2$, and the minimum value of the function $f(|S|)$ is $\min\{n+2,(5n+2)/4\}$. According to (2.1) and the minimum value of the function $f(|S|)$, we have $n>n$, which is a contradiction. This completes the proof of the lemma. ∎

We now give our main theorems and prove them.

Theorem 2.1. *Let G be a connected graph of order n such that $n \geq 3$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of non-adjacent vertices $x, y \in V(G)$, then G has a fractional 1-factor.

Proof. If G is a graph which satisfies the condition of the theorem and there is no fractional 1-factor in G, then by Theorem 1.3, there exists a vertex set $S \subseteq V(G)$ such that

$$i(G-S) > |S|. \tag{2.3}$$

Clearly, $S \neq \varnothing$ since G is connected. In the following, we assume $|S| \geq 1$. The proof splits into two cases.

Case 1. $|S| = 1$.

Thus, we get that $i(G-S) > |S| = 1$, that is, $i(G-S) \geq 2$. Let $x, y \in I(G-S)$, then $xy \notin E(G)$ and $d_{G-S}(x) = d_{G-S}(y) = 0$. According to the hypothesis of the theorem, we have

$$\frac{n}{2} \leq |N_G(x) \cup N_G(y)| \leq d_{G-S}(x) + d_{G-S}(y) + |S| = |S| = 1.$$

Therefore, we get $n \leq 2$. This contradicts $n \geq 3$.

Case 2. $|S| \geq 2$.

We have known that $\omega(G-S) \geq i(G-S)$ and it implies that $\omega(G-S) \geq |S|+1$ by (2.3). But $\omega(G-S) \leq |S|+1$ for all $S \subseteq V(G)$ with $|S| \geq 2$ by Lemma 2.1. Thus, we have

$$\omega(G-S) = |S|+1$$

for all $S \subseteq V(G)$ with $|S| \geq 2$. Moreover, we have

$$i(G-S) = |S|+1$$

and

$$|S| \leq \frac{n-1}{2}. \tag{2.4}$$

For any two vertices $x, y \in I(G-S)$, obviously, $xy \notin E(G)$ and $d_{G-S}(x) = d_{G-S}(y) = 0$. Then, we have

$$\frac{n}{2} \leq |N_G(x) \cup N_G(y)| \leq d_{G-S}(x) + d_{G-S}(y) + |S| = |S| \leq \frac{n-1}{2}$$

by the hypothesis of the theorem and (2.4). It is a contradiction. This completes the proof of the theorem. ∎

Theorem 2.2. *Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, and the minimum degree $\delta(G) \geq k$. If*

$$|N_G(x) \cup N_G(y)| \geq \max\left\{\frac{n}{2}, \frac{1}{2}(n+k-2)\right\}$$

for each pair of non-adjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

Proof. By Theorem 2.1, the result obviously holds for $k = 1$. If $k \geq 2$ and kn is even, then G has a k-factor by Theorem 1.4. We have known that a k-factor is a special fractional k-factor. Now we consider the case that k and n are both odd.

If G has no fractional k-factor, then by Theorem 1.1, there exists some $S \subseteq V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq -1, \quad (2.5)$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$.

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (2.5). As k is odd, therefore, $(k-1)$ is even and G has a fractional $(k-1)$-factor by Theorem 1.4.

Claim 1. $|T| \geq |S| + 1$.
Since G has a fractional $(k-1)$-factor, then we get

$$(k-1)|S| + d_{G-S}(T^{'}) - (k-1)|T^{'}| \geq 0$$

where $T^{'} = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k-1\}$. Moreover,

$$\begin{aligned} 0 &\leq (k-1)|S| + d_{G-S}(T^{'}) - (k-1)|T^{'}| \\ &= (k-1)|S| + d_{G-S}(T) - kp_k(G-S) - (k-1)(|T| - p_k(G-S)) \\ &= k|S| + d_{G-S}(T) - k|T| - |S| + |T| - p_k(G-S) \\ &= \delta_G(S,T) - |S| + |T| - p_k(G-S) \\ &\leq -1 - |S| + |T| - p_k(G-S) \end{aligned}$$

where $p_k(G-S)$ denotes the number of vertices in $G-S$ with degree k.
Thus we obtain

$$|T| \geq |S| + 1.$$

Claim 2. $|T| \geq k+1$.
If $|T| \leq k$, then by (2.5) and $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq k$, we obtain

$$\begin{aligned} -1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \geq 0, \end{aligned}$$

which is a contradiction.

Claim 3. $d_{G-S}(x) \leq k-1$ for all $x \in T$.
If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2.5). This contradicts the choice of S and T.

Claim 4. $|S| \leq (n-1)/2$.
By Claim 1 and $n \geq |S| + |T|$, we obtain

$$n \geq |S| + |T| \geq 2|S| + 1,$$

which implies $|S| \leq (n-1)/2$.

Define

$$h_1 = \min\{d_{G-S}(x)|x \in T\},$$
$$T^{"} = \{x \in T, d_{G-S}(x) = 0\}$$

and

$$p = |T^{"}|.$$

Then by Claim 3 and $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq k$, we have

$$h_1 \leq k-1, \qquad k \leq \delta(G) \leq h_1 + |S|. \tag{2.6}$$

Choose $x_1 \in T$ such that $d_{G-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \neq \varnothing$, let

$$h_2 = \min\{d_{G-S}(x)|x \in T \setminus N_T[x_1]\}.$$

Thus, we have $0 \leq h_1 \leq h_2 \leq k-1$ by Claim 3.

We shall consider various cases according to the value of p and derive contradictions.

Case 1. $p \geq 2$.

Then we have

$$i(G-S) = |I(G-S)| \geq 2.$$

Thus, there exist $x, y \in I(G-S)$ and $d_{G-S}(x) = d_{G-S}(y) = 0$ and $xy \notin E(G)$. By the hypothesis of the theorem and Claim 4 and $k \geq 3$, we get that

$$\frac{n+k-2}{2} \leq |N_G(x) \cup N_G(y)| \leq d_{G-S}(x) + d_{G-S}(y) + |S| = |S| \leq \frac{n-1}{2},$$

a contradiction.

Case 2. $p = 1$.

Obviously, we have $h_1 = 0$ and $|N_T[x_1]| = 1$. Thus, we get that $T \setminus N_T[x_1] \neq \varnothing$ and $1 \leq h_2 \leq k-1$ in view of Claim 2 and $p = 1$. Choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Clearly, $x_1x_2 \notin E(G)$. By the hypothesis of the theorem, we obtain

$$\frac{n+k-2}{2} \leq |N_G(x_1) \cup N_G(x_2)| \leq d_{G-S}(x_1) + d_{G-S}(x_2) + |S| = h_2 + |S|,$$

that is,

$$|S| \geq \frac{n+k-2}{2} - h_2. \tag{2.7}$$

According to Fact 2.1 and $k \geq 3$, we get $n > 3k+4$. By (2.5) and (2.7) and $n \geq |S| + |T|$, we have

$$\begin{aligned}
-1 \geq \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\
&\geq k|S| + h_2(|T|-1) - k|T| \\
&= k|S| - (k-h_2)|T| - h_2 \\
&\geq k|S| - (k-h_2)(n-|S|) - h_2 \\
&= (2k-h_2)|S| + (h_2-k)n - h_2 \\
&\geq (2k-h_2)(\frac{n+k-2}{2} - h_2) + (h_2-k)n - h_2
\end{aligned}$$

$$= h_2^2 + \frac{n-5k}{2}h_2 + k^2 - 2k$$
$$> h_2^2 + \frac{3k+4-5k}{2}h_2 + k^2 - 2k$$
$$= h_2^2 + 2h_2 + k(k - h_2 - 2).$$

If $1 \le h_2 \le k-2$, then

$$-1 \ge \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \ge h_2^2 + 2h_2 > 0$$

is a contradiction.

If $h_2 = k-1$, then

$$\begin{aligned} -1 \ge \delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\ge h_2^2 + 2h_2 + k(k - h_2 - 2) \\ &= k^2 - k - 1 > 0. \end{aligned}$$

It is a contradiction.

Case 3. $p = 0$.

Claim 5. $T \setminus N_T[x_1] \ne \varnothing$.

If $T = N_T[x_1]$, then we have $|T| = |N_T[x_1]| \le d_{G-S}(x_1) + 1 = h_1 + 1 \le k$, which contradicts Claim 2.

According to Claim 5, there exists $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Clearly, $x_1x_2 \notin E(G)$. In view of the hypothesis of the theorem, we have

$$\frac{n+k-2}{2} \le |N_G(x_1) \cup N_G(x_2)| \le d_{G-S}(x_1) + d_{G-S}(x_2) + |S| = h_1 + h_2 + |S|,$$

that is,

$$2|S| \ge n + k - 2(h_1 + h_2 + 1). \tag{2.8}$$

Since $p = 0$, then we have $1 \le h_1 \le h_2 \le k-1$. Moreover,

$$|N_T[x_1]| \le d_{G-S}(x_1) + 1 = h_1 + 1.$$

Obviously, $n - |S| - |T| \ge 0$ and $k - h_2 \ge 1$. Thus,

$$\begin{aligned} &(k - h_2)(n - |S| - |T|) - 1 \\ &\ge \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\ge k|S| - k|T| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|). \end{aligned}$$

Since $|N_T[x_1]| \le h_1 + 1$, this implies

$$(2k - h_2)|S| \le (k - h_2)n + (h_2 - h_1)(h_1 + 1) - 1. \tag{2.9}$$

By (2.8), (2.9), $1 \le h_1 \le h_2 \le k-1$ and $n \ge 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, we have

$$\begin{aligned} 0 &\le -h_2 n + 2(h_2 - h_1)(h_1 + 1) - 2 - k(2k - h_2) \\ &\quad + 2(2k - h_2)(h_1 + h_2 + 1) \\ &\le -4h_2^2 + (9k - n - 2)h_2 - (2k^2 - 4k + 2) \\ &\le -4h_2^2 + (4\sqrt{2(k-1)^2 + 2} - 1)h_2 - 2(k-1)^2 \end{aligned}$$

$$= -4\left(h_2 - \sqrt{\frac{(k-1)^2+1}{2}}\right)^2 - h_2 + 2,$$

that is,

$$0 \le -4\left(h_2 - \sqrt{\frac{(k-1)^2+1}{2}}\right)^2 - h_2 + 2. \tag{2.10}$$

Subcase 3.1. $3 \le h_2 \le k-1$.
In view of (2.10), we get

$$0 \le -4\left(h_2 - \sqrt{\frac{(k-1)^2+1}{2}}\right)^2 - h_2 + 2 \le -1.$$

This is a contradiction.
Subcase 3.2. $h_2 = 2$.
According to (2.10) and $k \ge 3$ (k is an odd integer), we obtain

$$0 \le -4\left(2 - \sqrt{\frac{(k-1)^2+1}{2}}\right)^2 < 0.$$

This is a contradiction.
Subcase 3.3. $h_2 = 1$.
By (2.10) and $k \ge 3$ (k is an odd integer), we have

$$0 \le -4\left(1 - \sqrt{\frac{(k-1)^2+1}{2}}\right)^2 + 1 < 0.$$

This is a contradiction.

From all the cases above, we deduced the contradiction. So the hypothesis can not be hold. Hence, G has a fractional k-factor. This completes the proof of the theorem. ∎

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