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On Finite Groups with Some Conditions on Subsets

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Abstract. Let n be a positive integer. We denote by $W^*(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exist a positive integer k, and a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$ and a function $f : \{0,1,2,\dots,k\} \longrightarrow X_0$, with $f(0) \neq f(1)$ and non-zero integers $t_0, t_1, \dots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_i := f(i)$, $i = 0, \dots, k$, and $x_j \in H$ whenever $x_j^{t_j} \in H$, for some subgroup $H \neq \left\langle x_j^{t_j} \right\rangle$ of G. If the integer k is fixed for every subset X we obtain the class $W_k^*(n)$. If one always has $t_i = 1$, $i = 0, 1, \dots, k$, and $x_0 = x, x_i = y$, $i = 1, 2, \dots, k$, one obtains the class $\mathcal{E}_k(n)$. In this paper, we prove that

(1) A finite semi-simple group has the property $W_k^*(5)$, for some k, if and only if $G \simeq A_5$ or S_5,
(3) A finite non-nilpotent group has the property $W_k^*(3)$, for some k, if and only if $G/Z^*(G) \simeq S_3$, where $Z^*(G)$ is the hypercenter of G,
(2) A finite semi-simple group has the property $\mathcal{E}_k(16)$, for some k, if and only if $G \simeq A_5$,

where A_n and S_n denote the alternating and symmetric groups of degree n respectively.

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1. Introduction and results

Our notation and terminology are standard and can be found in [17]. For a group G, and elements $x, y, x_1, \dots, x_k \in G$, we write

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 = x_1^{-1}x_1^{x_2}, \quad [x_1, \dots, x_k] = [[x_1, \dots, x_{k-1}], x_k]$$

$$[x, {}_0y] = x, \quad [x, {}_ky] = [[x, {}_{k-1}y], y].$$

A group is said to be k-Engel (respectively, Engel) if for all $x, y \in G$, $[x, {}_ky] = 1$ (respectively, there exists a positive integer t depending on x and y such that $[x, {}_ty] = 1$). The class of k-Engel (respectively, Engel) groups will be denoted by

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$\mathcal{E}_k$ (respectively, $\mathcal{E}$). For elements x, y, z of a group G, the following commutator identities will be frequently used without reference:

$$[xy, z] = [x, z]^y [y, z], \quad [x, yz] = [x, z][x, y]^z.$$

We denote by $W_k^*(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exist a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$ and a function $f : \{0, 1, 2, \ldots, k\} \longrightarrow X_0$, with $f(0) \neq f(1)$, $x_i := f(i)$, and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, and $x_i \in H$ whenever $x_i^{t_i} \in H$, for some subgroup $H \neq \langle x_i^{t_i} \rangle$ of G, $i = 0, 1, \ldots, k$. This class is subgroup and quotient closed, the facts which will be used in the proofs concerning $W_k^*(n)$. Note that the condition "$x_i \in H$ whenever $x_i^{t_i} \in H$, for some subgroup $H \neq \langle x_i^{t_i} \rangle$ of G" guarantees that $W_k^*(n)$ is quotient closed. To see this, let N be a normal subgroup of a $W_k^*(n)$-group G. Let Y be a subset of G/N of size $n+1$. Then there exists a subset X of G of size $n+1$ such that $Y = \{\bar{x} \mid x \in X\}$, where $\bar{x} = xN$. By hypothesis there exist a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$ and a function $f : \{0, 1, 2, \ldots, k\} \longrightarrow X_0$, with $f(0) \neq f(1)$, $x_i := f(i)$, and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, and $x_i \in H$ whenever $x_i^{t_i} \in H$, for some subgroup $H \neq \langle x_i^{t_i} \rangle$ of G, $i = 0, 1, \ldots, k$. Now we have $[\bar{x_0}^{t_0}, \bar{x_1}^{t_1}, \ldots, \bar{x_k}^{t_k}] = \bar{1}$. If $\bar{x_i}^{t_i} \in H/N$ with $H/N \neq \langle \bar{x_i}^{t_i} \rangle$, then $H \neq \langle x_i^{t_i} \rangle$ and $x_i^{t_i} \in H$. Thus $x_i \in H$ and so $x_i H \in H/N$.

If always $t_i = 1$, $i = 0, 1, \ldots, k$, we obtain the class $E_k(n)$. Also if one always has $|X_0| = 2$ and $t_i = 1$, $i = 0, 1, \ldots, k$, and $x_0 = x, x_i = y$, $i = 1, 2, \ldots, k$, one obtains the class $\mathcal{E}_k(n)$. Clearly

$$\mathcal{E}_k(n) \subseteq E_k(n) \subseteq W_k^*(n).$$

There are many results about these classes of groups. For example, a finite $\mathcal{E}_k(2)$-group is nilpotent and a finite $\mathcal{E}_k(15)$-group is soluble [2]. Every finite $W_k^*(2)$-group is nilpotent and every finitely generated soluble $W_k^*(n)$-group is finite-by-(nilpotent of k-bounded class) [22]. Every finitely generated soluble $\mathcal{E}_k(\infty)$-group is finite-by-nilpotent [9, 13]. A finitely generated soluble group has the property $W^*(\infty)$ if and only if it is finite-by-nilpotent [22].

The origin of these results is a following question of Paul Erdös. If in an infinite group all subsets consisting of mutually non-commuting elements are finite, is there a bound for the cardinalities of such sets? Neumann [16] obtained an affirmative answer by proving that a group is center-by-finite if and only if every infinite subset contains two different commuting elements. This result has initiated extensive study of researches toward the study of the structure of groups having some similar nature (for example, see [1–5], [9–11], [13–16], [20–25]).

Recall that a group G is semi-simple if G has no non-trivial normal Abelian subgroups. If G is a finite group, then the product of all minimal normal non-Abelian subgroup of G is said to be the centerless CR-radical of G, which is a direct product of non-Abelian simple groups (see [17, p. 88]). We prove that finite semi-simple groups with the property $W_k^*(5)$ or $\mathcal{E}_k(16)$ have restricted structures. In fact we have

Theorem 1.1. *Let G be a finite semi-simple group. Then $G \in W_k^*(5)$, for some k, if and only if $G \simeq A_5$ or S_5, where S_5 is the symmetric group on 5 letter.*

It is proved in [22] that every finite group with the property $W_k^*(2)$ is nilpotent, and $S_3 \in W_2^*(3)$. Here we prove that

Theorem 1.2. *Let G be a finite non-nilpotent group. Then $G \in W_k^*(3)$, for some k, if and only if $G/Z^*(G) \simeq S_3$, where $Z^*(G)$ is the hypercenter of G.*

Theorem 1.3. *Let G be a finite semi-simple group. Then $G \in \mathcal{E}_k(16)$, for some k, if and only if $G \simeq A_5$.*

Thus a finite semi-simple group G has the property $W_k^*(5)$, for some k, if and only if G has the property $\mathcal{E}_t(16)$, for some t.

2. Finite groups with the property $W_k^*(n)$

In this section, we prove Theorem 1.1. We begin with an easy lemma without proof.

Lemma 2.1. *Let G be a group, A be an Abelian normal subgroup of G, and g be an arbitrary element of G. Then for all distinct elements $a_1, \ldots, a_k$ of A, we have that*

$$[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = [g^{t_0}, a_r a_j^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_k}],$$

where $x_0 = g^{a_r} \neq x_1 = g^{a_j}$, and $x_i \in \{g^{a_1}, g^{a_2}, \ldots, g^{a_k}\}$.

Lemma 2.2. *Let G be a group in $W_k^*(n)$ and A be a normal Abelian subgroup of G. If there exists $g \in G$ such that $C_A(g^m) = 1$ for all integers m with $g^m \neq 1$, then $|A| \leq n$.*

Proof. Suppose that $|A| \geq n+1$. Then the set $g^A = \{g^a \mid a \in A\}$ has cardinality $\geq n+1$, as $C_A(g) = 1$. Now, since $G \in W_k^*(n)$, there exist distinct elements $a_i \in A$ and integers t_i such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, where $x_0 = g^{a_r} \neq x_1 = g^{a_j}$, and $x_i \in \{g^{a_1}, g^{a_2}, \ldots, g^{a_k}\}$, $i = 0, 1, 2, \ldots k$. Thus, by Lemma 2.1, $[g^{t_0}, a_r a_j^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_k}] = 1$. Now, since A is normal Abelian,

$$u = [g^{t_0}, a_r a_j^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_{k-1}}] \in C_A(g^{t_k}) = 1.$$

So $u = 1$. Continuing in this way we find that $[g^{t_0}, a_r a_j^{-1}] \in C_A(g^{t_1}) = 1$. So $a_r a_j^{-1} \in C_A(g^{t_0}) = 1$, which is a contradiction. ∎

Lemma 2.3. *Let p, q be distinct primes and $G \in W_k^*(n)$ be a finite minimal non-nilpotent group. If P is a normal Sylow p-subgroup of G, then $|P| \leq n$.*

Proof. By [12, Theorem 5.2, p. 281], we have $\Phi(P) \leq Z(G)$ and G has a cyclic Sylow q-subgroup Q such that $\Phi(Q) \leq Z(G)$, where $\Phi(G)$ denotes the Frattini subgroup of G. But since $Z(G) = 1$, P and Q are elementary Abelian. Thus $|Q| = q$ and $Q = \langle x \rangle$. Now $C_P(x) \leq Z(G) = 1$, and so by Lemma 2.2, $|P| \leq 3$. Thus $|P| = 2$, and therefore $P \leq Z(G) = 1$, a contradiction. ∎

Recall that a group G is p-nilpotent, if G has a normal complement to a Sylow p-subgroup, that is if $G = PK$, where P is a Sylow p-subgroup, K is a normal Hall subgroup and $P \cap K = 1$. We need the following useful result.

Lemma 2.4. *Let $G \in W_k^*(n)$ be a finite group. If p is a prime divisor of the order of G and $p > n$ then G is p-nilpotent.*

Proof. If the conclusion of the Lemma does not hold then choose a counter example G of smallest order. Then by a result of Itô (see [17, Theorem 10.3.3, p. 287]) G is minimal non-nilpotent. Thus G is a $\{p, q\}$-group, where p, q are distinct primes and G has a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup Q. Let $x \in Q$ be an element of order q. Then, since $Z(G) = 1$, we have $C_{Z(P)}(x) = 1$, and therefore, by Lemma 2.3, $|Z(P)| \le n$. But $n < p$, so $|Z(P)| < p$ and $Z(P) = 1$, a contradiction. Therefore G is p-nilpotent. ∎

Abdollahi [2] proved that if $G \in \mathcal{E}(15)$ is a finite group, then G is soluble, and $A_5 \in \mathcal{E}(16)$. It is proved in [22] that $W_k^*(4)$-groups are soluble. We give another simple proof of this result:

Proposition 2.1. *Every finite group $G \in W_k^*(4)$ is soluble.*

Proof. We use induction on G. If $p > 3$ is a prime divisor of $|G|$ then, by Lemma 2.4, G has a normal Hall p'-subgroup K. Since $G/K \simeq P$, where P is a Sylow p-subgroup of G, is nilpotent, and by induction hypothesis K is soluble, we conclude that G is soluble. Thus we may assume that $p \le 3$. So, by a famous Theorem of Burnside [17, Theorem 8.5.3, p. 240], G is soluble. ∎

Let us prove that $A_5 \in W_7^*(5)$, where A_5 is the alternating group on 5 letter. We must show that for any subset X of size 6, there exist subset $X_0 \subseteq X$, with $2 \le |X_0| \le 6$ and non-zero integers $t_0, t_1, \ldots, t_6$ such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_6^{t_6}] = 1$, where $x_0 \ne x_1$, and $x_i \in X_0$, $i = 0, \ldots, 6$. The alternating group of degree five has 5 Sylow 2-subgroups of order 4, 10 Sylow 3-subgroups of order 3, and 6 Sylow 5-subgroups of order 5. Suppose that X is a subset of A_5 of size 6. If two elements of X lie in a same Sylow subgroup, then these elements commute and so satisfy the condition. So we may assume that the elements of X lie in distinct Sylow subgroups. Now we may use the computer package GAP [19] obtain the desired result. We can see that there is a commutator, for a 3-cycle and a 5-cycle of weight at most 6, for a 3-cycle and an element of order two of A_5 of weight at most 5, for a 5-cycle and an element of order two of A_5 of weight at most 8, for two distinct 5-cycle of weight at most 6 is equal to identity. On the other hand every set of 3-cycles of size 5 contains a pair such that a commutator of weight at most 5 consisting of these elements is identity. Finally the set of 5-elements

$$X = \{(1, 2)(3, 4), (1, 2)(3, 5), (1, 2)(4, 5), (1, 3)(4, 5), (2, 3)(4, 5)\}$$

which contains one element from each Sylow 2-subgroup of A_5 is a maximal set that satisfies the condition $[x_0, \ldots, x_k] \ne 1$, $x_0 \ne x_1$, for all $x_0, \ldots, x_k \in X$ and for all positive integer k. (One can see this fact by a simple program in GAP and using the (second) function *allcomm* in Appendix). Thus $A_5 \in W_7^*(5)$, in fact $A_5 \in E_7(5)$.

Now we investigate the non-Abelian finite simple groups with the property $W_k^*(5)$.

Lemma 2.5. *Let G be a non-Abelian finite simple group. Then $G \in W_k^*(5)$, for some k, if and only if $G \simeq A_5$.*

Proof. Suppose, by contradiction, that there exists a non-Abelian finite simple group with the property $W_k^*(5)$ which is not isomorphic to A_5. Let G be such a group of least order. Therefore every proper non-Abelian simple section of G is isomorphic to A_5. Hence by [7, Proposition 3], it is isomorphic to one of the following:

(1) $PSL(2,2^p)$,$p=4$ or a prime;
(2) $PSL(2,3^p), PSL(2,5^p)$, p a prime;
(3) $PSL(2,p)$, p a prime ≥ 7;
(4) $PSL(3,3), PSL(3,5)$;
(5) $PSU(3,4)$, the projective special unitary group of degree 3 over a finite field of size 4^2; and
(6) $Sz(2^p)$, p an odd prime.

Let p be a prime devisor of $|G|$. Then, by Lemma 2.4, $p \leq 5$. Since

$$|PSL(n,q)| = (q^n-1)(q^n-q)\cdots(q^n-q^{n-2})q^{n-1}/\gcd\{n,q-1\},$$

among all projective special linear groups listed above, we only need to check the following: $PSL(2,2^2)$ which is isomorphic to A_5, and $PSL(2,3^2)$ which is isomorphic to A_6. But $A_6 \notin W_k^*(5)$, for all k, as the set

$$\{(1,2)(3,4),(1,2)(3,5),(1,3)(2,6),(2,6)(3,5),(2,5)(3,4),(2,4)(3,6)\}$$

shows. Now $|PSU(3,4)| = 2^6\cdot 3\cdot 5^2\cdot 13$ (see Theorem 10.12(d) of chapter II in [12]), and $|Sz(2^p)| = 2^{2p}(2^p-1)(2^{2p}+1)$, ($p$ an odd prime), so $PSU(3,4), Sz(2^p) \notin W_k^*(5)$, by Lemma 2.2. This completes the proof. ∎

The following lemma and its proof is similar to Lemma 2.2 of [6].

Lemma 2.6. *Let G_i be a finite group such that $G_i \notin W_k^*(n_i)$ for $i=1,\ldots,t$. Then $G := G_1\times\cdots\times G_t \notin W_k^*(n)$, where $n=(n_1+1)\cdots(n_t+1)-1$.*

Proof. By hypothesis, for every $i\in\{1,\ldots,t\}$, there exists a subset X_i in G_i of size n_i+1 such that for all $X_{i0}\subseteq X_i$, with $2\leq |X_{i0}|\leq n+1$ and for all $x_0,x_1,\ldots,x_k\in X_{i0}$ such that $x_0\neq x_1$, and for all non-zero integers $t_0,t_1,\ldots,t_k$, with $x_j\in H$ whenever $x_j^{t_j}\in H$, for some subgroup $H\neq \left\langle x_j^{t_j}\right\rangle$ of G, we have

$$[x_0^{t_0},x_1^{t_1},\ldots,x_k^{t_k}]\neq 1.$$

Now let $X = X_1\times\cdots\times X_t$. Suppose, for a contradiction, that there exists a subset X_0 of X and elements $a_0,a_1,\ldots,a_k\in X_0$, with $a_0\neq a_1$, and non-zero integers $t_0,t_1,\ldots,t_k$, with $a_j\in H$ whenever $a_j^{t_j}\in H$, for some subgroup $H\neq\left\langle a_j^{t_j}\right\rangle$ of G, such that $[a_0^{t_0},a_1^{t_1},\ldots,a_k^{t_k}]=1$. Then

$$1=[a_0^{t_0},a_1^{t_1},\ldots,a_k^{t_k}] = ([a_{0,1}^{t_0},a_{1,1}^{t_1},\ldots,a_{k,1}^{t_k}],\ldots,[a_{0,t}^{t_0},a_{1,t}^{t_1},\ldots,a_{k,t}^{t_k}]).$$

Thus $[a_{0,i}^{t_0},a_{1,i}^{t_1},\ldots,a_{k,i}^{t_k}]=1$, where $a_{r,s}$ is the rth component of a_s, for all $i=1,\ldots,t$. This is a contradiction. ∎

Lemma 2.7. *Let $G_1,\ldots,G_m$ be non-Abelian simple groups. Then*

$$G_1\times\cdots\times G_m\notin W_k^*(4^m-1).$$

Proof. By Proposition 2.1, $G_i\notin W_k^*(4)$. Thus $G_1\times\cdots\times G_m\notin W_k^*(4^m-1)$, by Lemma 2.6. ∎

Now we are in the position to prove the Theorem 1.1.

Proof of Theorem 1.1. Let R be the centerless CR-radical of G. Since R is a direct product of non-Abelian simple groups and has the property $W_k^*(5)$, it follows from lemmas 2.5 and 2.7 that R is isomorphic to A_5. Note that, since $C_G(R) = 1$, we have $G/R \hookrightarrow Out(R) \simeq C_2$. Therefore G is isomorphic to A_5 or S_5. Note that we have already seen that $A_5 \in W_7^*(5)$. We can verify (again by a simple program in GAP and using the (second) function *allcomm* in Appendix) that for any 5-cycle and any element of Sylow 2-subgroups of S_5 some commutator of weight at most 9 is identity. Therefore $S_5 \in W_8^*(5)$, if fact $S_5 \in E_8(5)$. ∎

3. Finite non-nilpotent groups with the property $W_k^*(3)$

In this section we prove Theorem 1.2. It is proved in [22] that every finite $W^*(2)$-group is nilpotent. First we give another proof of this result.

Proposition 3.1. *Every finite group $G \in W_k^*(2)$ is nilpotent.*

Proof. Suppose the assertion is false and choose a counter-example G of smallest order. If $Z(G) \neq 1$, then $G/Z(G)$ is nilpotent, by minimality of G. So G is nilpotent, a contradiction. Therefore $Z(G) = 1$. Now G is a finite minimal non-nilpotent group. By a result of Schmidt (see [17, Theorem 9.1.9]) G is a $\{p, q\}$-group, where p, q are distinct primes and G has a normal Sylow p-subgroup P and *a* cyclic Sylow q-subgroup Q. By Lemma 2.3, we have $|Z(P)| \leq 2$. Thus $Z(P) \leq Z(G) = 1$, a contradiction. Therefore G is nilpotent. ∎

Note that $S_3 \in W_2^*(3)$, so that the bound 2 in Proposition 3.1 cannot be improved.

Lemma 3.1. *Let G be a group in $W_k^*(3)$ such that 2 divides the order of G. Then G is 2-nilpotent.*

Proof. Suppose the assertion is false and choose a counter-example G of smallest order. Then by a result of Itô G is a finite minimal non-nilpotent group (see [17, Theorem 10.3.3, p. 287]) and G is a $\{2, q\}$-group, where $q > 2$ is a prime and G has a normal Sylow 2-subgroup P and a cyclic Sylow q-subgroup Q. By Lemma 2.3, $|P| \leq 3$. Thus $|P| = 2$, and therefore $P \leq Z(G) = 1$, a contradiction. ∎

The following two lemmas and their proofs are similar to those of Lemma 3.5 and 3.7 of [24].

Lemma 3.2. *Let G be a finite group in $W_k^*(3)$. Let $R = R_1 \times \cdots \times R_s$ be an elementary Abelian p-subgroup ($p > 2$ a prime) of G and let $x_1, x_2, \ldots, x_s$ be elements of order 2 of an Abelian 2-subgroup such that $[R_i, x_i] = R_i$, and $[R_i, x_j] = 1$, for $i \neq j$. Then $|R| \leq 3$.*

Proof. Let $x = x_1 x_2 \cdots x_s$ and suppose that $y \in C_R(x)$. Then we can write $y = y_1 y_2 \cdots y_s$, with $y_i \in R_i$. Now

$$1 = [y, x] = [y_1, x_1][y_2, x_2] \cdots [y_s, x_s].$$

Since $R = R_1 \times \cdots \times R_s$ and $[y_i, x] \in R_i$, we have $[y_i, x_i] = 1$ and so $y_i \in C_{R_i}(x) = 1$. Hence $C_R(x) = 1$ and $|R| \leq 3$, by Lemma 2.2. ∎

Lemma 3.3. *Let p be an odd prime. Let $G = RX$ be an extension of an elementary Abelian p-group R by an Abelian 2-subgroup X such that X acts faithfully on R and $R = [R, X]$. If $G \in W_k^*(3)$ then $|X| \leq 2$ and $|R| \leq 3$.*

Proof. Let $x_1 \in X$ be an element of order 2. Then $R = R_1 \times S_1$, where $R_1 = [R, x_1] = [R_1, x_1]$ and $S_1 = C_R(x_1)$. By Lemma 3.2, $|R_1| \leq 3$. Let $T_1 = C_X(R_1)$.

If $T_1 = 1$ then, since X acts faithfully on R_1, $|X| \leq 2$. We may assume that $X = \{1, x_1\}$, so $S_1 = C_R(x_1)$. If $S_1 \neq 1$ then $R = [R, X] = 1$, so we assume that $S_1 = 1$ and thus $|R| \leq 3$.

Suppose that $T_1 > 1$. Let $1 \neq x_2 \in T_1$. Now $S_1 = R_2 \times S_2$, where $R_2 = [S_1, x_2] = [R_2, x_2] = [R, x_2]$ and $S_2 = C_{S_1}(x_2)$. Note that $R = R_2 \times S_2 \times R_1$ and $S_2 \times R_1 = C_R(x_2)$. Now $R_1 \times R_2$ and x_1, x_2 are as in Lemma 3.2, so $|R_1 \times R_2| \leq 3$. As above we may assume that $T_2 = C_X(R_1 \times R_2) > 1$ and choose a non-trivial element $x_3 \in T_2$. Continuing in this way, we construct $R_1 \times \cdots \times R_s$ and $x_1, \ldots, x_s$ as in Lemma 3.2. If $T_i = C_X(R_1 \times \cdots \times R_s)$, then $X > T_1 > T_2 > \cdots > T_i$ so there is an s with $T_s = 1$ and, by Lemma 3.2, $|R_1 \times \cdots \times R_s| \leq 3$ so that $|X| \leq 2$ and $|R| \leq 3$. ∎

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By factoring out $Z^*(G)$, we may assume that G is a group with trivial center, satisfying the condition $W_k^*(3)$ and prove that $G \simeq S_3$. Note that G is soluble by Proposition 2.1. Let p be any prime divisor of $|G|$. If $p \neq 3$, then G is p-nilpotent, by Lemma 2.4, for any p, and G is nilpotent (see [18, exercise 563, p. 244]). Since G is not nilpotent, 3 divides $|G|$. Now if 2 does not divide $|G|$, then $|G| = 3^m p_1^{m_1} \cdots p_s^{m_s}$, where $p_i > 3$ is a prime and m_i is a positive integer, $i = 1, 2, \ldots, s$.

We claim that such a group G is 3-nilpotent and thus nilpotent (by [18, exercise 563, p. 244]), which is a contradiction. If the claim were not true, choose a counter-example G of smallest order. Then, as in Proposition 3.1, G is $\{3, q\}$-group, where q is prime, and G has a normal elementary Abelian Sylow 3-subgroup S and a cyclic Sylow q-subgroup $Q = \langle x \rangle$ of order q. Thus by Lemma 2.3 we have $|S| \leq 3$. Thus, by [18, exercise 564, p. 244], G is 3-nilpotent so that G is nilpotent, which a contradiction.

We thus conclude that 2 divides $|G|$, and $|G| = 2^n 3^m p_1^{m_1} \cdots p_s^{m_s}$. Let P be a Sylow 2-subgroup of G. Then since, by Lemma 3.1, G is 2-nilpotent there exists a normal subgroup K such that 2 does not divide $|K| = 3^m p_1^{m_1} \cdots p_s^{m_s}$, $P \cap K = 1$, and $G = PK$. Since $K \in W_k^*(3)$, K is nilpotent, by the above observation. But this means that G is 3-nilpotent, and so is nilpotent, unless $|K| = 3^m$. Thus K is in fact a Sylow 3-subgroup of G.

Now we follow the proof of [24, Theorem C]. We have $K = \mathrm{Fitt}(G) = O_3(G)$, the subgroup generated by all 3-elements of G. Let $\bar{G} = G/\Phi(K)$ and $\bar{K} = K/\Phi(K)$. By a result of Hall and Higman, $\bar{G}/\bar{K}$ acts faithfully on the $GF(3)$-vector space $\bar{K}$ (see for example [17, Theorem 9.3.2, p. 262]). We note that $\bar{K}$ is an elementary Abelian and normal 3-subgroup of $\bar{G}$, $\bar{K} = O_3(\bar{G})$, and $C_{\bar{G}}(\bar{K}) = \bar{K}$. Let $Q/\bar{K}$ be the socle of $\bar{G}/\bar{K}$ so that $Q/\bar{K}$ is an Abelian 2-subgroup. We may write $Q = \bar{K}X$, where X is an Abelian 2-subgroup of G. Let $R = [\bar{K}, Q]$, so that $\bar{K} = R \times C_{\bar{K}}(Q)$. If $C = C_{\bar{G}}(R)$, then $C \cap Q$ centralizes $R \times C_{\bar{K}}(Q)$ and so $C \cap Q = \bar{K}$. It follows that $C_{\bar{G}}(R) = \bar{K}$ and $\bar{G}/\bar{K}$ acts faithfully on R. Now R and X satisfy the condition of Lemma 3.3, so $|R| \leq 3$. Since $\bar{G}/\bar{K}$ acts faithfully on R, $|P| = |G/K| = |\bar{G}/\bar{K}| \leq 2$. Let $P = \{1, a\}$. Now we have $[C_K(P), {}_m G] = [C_K(P), {}_m K] = 1$, for some positive integer m. Thus $C_K(P) \leq Z^*(G) = 1$, and $C_K(a) = 1$. Hence $C_{Z(K)}(a) = 1$ and $|Z(K)| = 3$, by Lemma 2.2.

Let K be nilpotent of class c. Put $A = \Gamma_{[c/2]}(K)$, where $[c/2]$ equals $(c+1)/2$ if c is odd and $(c+2)/2$ if c is even ($\Gamma_{s+1}(K)$ is the sth term of the lower central series of K). Then $A \leq K$ is a normal Abelian subgroup of G, and so $|A| \leq 3$, by Lemma 2.2. Therefore $c \leq 2$. If $c = 1$ then K is Abelian and, by Lemma 2.2, we have $|K| \leq 3$, and so $G \simeq S_3$. So suppose that $c = 2$. Then $Z(K) = K'$.

Now suppose that G is not isomorphic to S_3 and choose such G of the least order. Therefore every centerless section of G is isomorphic to S_3. We claim that the center of $G/Z(K)$ is trivial. If not, let $H/Z(K) = Z(G/Z(K))$. Since $H/Z(K) = H/K'$ is Abelian, by a result of P. Hall (see [17, Theorem 5.2.10, p. 129]), H is a normal nilpotent subgroup of G. Since $|H| > 3 = |Z(K)|$ and $C_K(a) = 1$, it is clear that $H \not\leq K$, so that $H = PK_0$ where $K_0 \leq K$. In particular $Z(H) \neq 1$ and so $C_{K_0}(a) \neq 1$. This means that $C_K(a) \neq 1$, a contradiction. Therefore $Z(G/Z(K)) = 1$ and, by hypothesis, $G/Z(K) \simeq S_3$. Thus $|G| = 2 \times 3^2$, $|K| = 9$ and so K is a normal Abelian subgroup of G. Since $C_K(a) = 1$ we have that $|K| \leq 3$, by Lemma 2.2, which is a contradiction. Therefore $G \simeq S_3$. ∎

4. Finite semi-simple groups with the property $\mathcal{E}_k(16)$

In this short section, we prove Theorem 1.3. A similar result to that of Lemma 2.5, for non-Abelian simple groups in $\mathcal{E}_k(16)$ is hold by [3, Theorem 1.5]:

Lemma 4.1. *Let G be a non-Abelian finite simple group. Then $G \in \mathcal{E}_k(16)$, for some k, if and only if $G \simeq A_5$.*

Now as in the Lemma 2.7 we can prove that the direct product of m non-Abelian simple groups does not belong to $\mathcal{E}_k(13^m - 1)$. Therefore, using this fact and Lemma 4.1 and arguing as in the proof of Theorem 1.1, one can proof Theorem 1.3. We must also show that $S_5 \notin \mathcal{E}_k(16)$, for any k. But we even have that $S_5 \notin \mathcal{E}_k(24)$, for any k, as the set

$$\{(1,4,2,3),(1,4,3,2),(1,2,4,3),(1,2,5,3),(1,2,3,5),(1,5,2,3),\\ (2,3,5,4),(2,5,4,3),(2,5,3,4),(1,5,2,4),(1,5,4,2),(1,2,5,4),\\ (1,3,5,4),(1,3,4,5),(1,5,3,4),(1,2,3),(1,2,4),(1,2,5),(1,3,4),\\ (1,3,5),(1,4,5),(2,3,4),(2,3,5),(2,4,5),(3,4,5)\}$$

shows.

Appendix

In this appendix, we include a GAP program consisting of some functions which can be used to determine whether all two element subsets of a list of a group elements satisfy the conditions $W_k^*(n)$ or $\mathcal{E}_k(n)$. Note that if $K = \langle x, y \rangle$, $|K| = n$, and

$$X = \{[x_0^{t_0}, x_1^{t_1}], \ldots, [x_0^{t_0}, x_1^{t_1}, \ldots, x_n^{t_n}];\ x_i \in \{x, y\}, x_0 \neq x_1, 1 \leq t_i < O(x_i)\},$$

then $1 \in X$ if and only if $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, for some k.

```
repeatedcommutator:=function(L) # this function computes the
   local k,w;   w:=L[1];        # repeated commutator [x,y,z,...]
   for k in  [2..Length(L)] do
     w:= Comm(w,L[k]);
   od;
  return w;
end;;  #################################################################
power:=function(x)               # this function computes the
  local i,s; s:=[];              # powers of an element x
  for i in [1..Order(x)-1] do
      Add(s,x^i);
  od;
return s;
end;; #############################
bit:=function(num,bas,len)     # this function changes the number 'num'
  local L,tt; tt:=num;  L:=[]; # into the basis 'bas' with length 'len'
  while tt<>0 do               # this help to compute all possibilities
    Add(L, tt mod bas);        # [y_0,y_1,...,y_k] where y_i in {x,y}
    tt:=Int(tt/bas);
  od;
  for tt in [Length(L)+1..len] do
   L[tt]:=0;
  od;
 return Reversed(L);
end;;  #############################
allcomm:=function(x,y,k,C) # this function  computes
   local L,i; L:=[x];         # that whether [x,_k y]=1 Mod C
   for i in [1..k] do         #  or [y,_k x]=1 Mod C
     Add(L,y);
   od;
   if repeatedcommutator(L) in C then
                          return [L,"does equal to identity"];
   fi;
   L:=[y];
   for i in [1..k] do
     Add(L,x);
   od;
   if repeatedcommutator(L) in C  then
                         return [L, "does equal to identity"];
   fi;
  return ["any Engel commutator of elements does not equal to identity"];
end;;  ###################################################
allcomm:=function(x,y,k,C)        # this function  computes
 local s,px,py,t,i,j,temp,L,u,e;  # that whether [x_0^t_0,...,x_k^t_k]=1 Mod C
 px:=power(x); py:=power(y); temp:=power(x); Append(temp,py); t:=Length(temp);
 s:=Filtered(Tuples(temp,2),u-> ( (u[1] in px and u[2] in py) or
                                  (u[1] in py and u[2] in px)    ) );
 for i in [1..Length(s)] do
   L:=[]; j:=0; L[1]:=s[i][1];  L[2]:=s[i][2];
   while  j < t^(k-2) do
      e:=bit(j,t,k-2);
      for u in [3..k] do
        L[u]:=temp[e[u-2]+1];
      od;
      if repeatedcommutator(L) in C then
```

```
                              return [L, "does equal to identity"];
      fi;
      j:=j+1;
   od;
 od;
return ["any commutator of  powers of elements does not equal to identity"];
end;;  #############################
checkall:=function(L,C)                # this function  determines whether
   local t,i,x,y,s,n;                  # two element subsets of the  list L
   if Length(L) > Length(Set(L)) then  # satisfy the condition W^*_k or E_k Mod C
         return[" some elements of the list are equal"];
   fi;
   t:=Filtered( UnorderedTuples(L,2),u->Length( Set(u) )>1 );
   for i in [1..Length(t)] do
     x:=t[i][1];  y:=t[i][2];  n:=Size(Group(x,y)); s:=allcomm(x,y,n,C);
     if Length(s)=2 then
                 then return ["NOT Satisfy"];
     fi;
   od;
return ["YES Satisfy"];
end;;
```

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