**BULLETIN** *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

*Bull. Malays. Math. Sci. Soc. (2)* **32**(2) (2009), 101–117

# Records and Concomitants

M. Ahsanullah
Department of Management Sciences, Rider University,
2083 Lawrence Road, Lawrenceville, New Jersey 08648, USA
ahsan@rider.edu

**Abstract.** In this paper we consider the record values of independent and identically distributed (i.i.d.) random variables. Some basic properties of record values of univariate distributions are presented. Representation of records as sum of independent identical random variables are shown. Some characterizations of distributions using record values are given. The concomitants of record values of Morgenstern family copula are considered.

2000 Mathematics Subject Classification: 62E10, 62E15

Key words and phrases: Records, weak records, characterizations.

## 1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. absolutely continuous random variables with cumulative distribution function (cdf), $F(x)$, and probability density function (pdf), $f(x)$. Let

$$Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\}, \quad n \geq 1.$$

We say that $Y_j$ is an upper (lower) record value of $\{X_n,\ n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j \geq 1$. The indices at which the upper record values occur are given by record times $\{U(n),\ n > 0\}$, where

$$U(n) = \min\{j|\ j > U(n-1),\ X_j > X_{U(n-1)},\ n > 1\}$$

and $U(1) = 1$. We will denote $L(n),\ n \geq 1$, as the indices where lower record values occur. By our assumption $U(1) = L(1) = 1$. By definition $X_1$ an upper as well as a lower record value. For i.i.d. discrete random variables we have two types of record values: ordinary and weak records. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. discrete random variables taking values on $0, 1, \ldots$ with distribution function $F(x)$. The upper weak record times, $U_w(n)$, $n \geq 1$, are defined as follows: $U_w(1) = 1$ and

$$U_w(n+1) = \min\{j > U_w(n), X_j \geq \max(X_1, \ldots, X_{j-1})\}.$$

The corresponding weak upper record values are defined as $X_{U_w}(n+1)$, $n = 0, 1, 2, \cdots$. If in the above expression, we replace $\geq$ by $>$, then we obtain ordinary upper record times and upper record values. We consider an example to clarify the differences.

*Received:* June 28, 2008; *Revised:* October 30, 2008.

Suppose the discrete random variables $X_1, X_2, \ldots$ have geometric distribution with probability mass function (pmf) of the following form for $k = 0, 1, 2, \ldots,$

$$p(k) = P(X_i = k) = pq^k, \quad 0 < p < 1, \quad q = 1 - p.$$

It can be shown that the marginal pmf of $X_{U(n)}$ is

$$P_n(x) = P(X_{U(n)} = x) = \binom{x}{n-1} p^n q^{x-n+1}, \quad x = n-1, n, \ldots; \quad n \geq 1.$$

The marginal pmf of $X_{U_w n}(x)$ is as follows

$$P_{wn}(x) = P(X_{U_w n} = x) = \binom{x+n-1}{n-1} p^n q^x, \quad x = 0, 1, 2, \ldots; \quad n \geq 1.$$

In this paper we consider continuous random variables unless mentioned otherwise.

## 2. Main results

### 2.1. Distribution of records of continuous random variables

Many properties of the record value sequence can be expressed in terms of the function $R(x) = -\ln \bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$. If we define $F_n(x)$ as the distribution function of $X_{U(n)}$ for $n \geq 1$, then $F_1(x) = P[X_{U(1)} \leq x] = F(x)$ and

$$\begin{aligned} F_2(x) &= P[X_{U(2)} \leq x] \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=1}^{\infty} (F(u))^{i-1} \, dF(u) \, dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dF(u)}{1 - F(u)} \, dF(y) \\ &= \int_{-\infty}^{x} R(y) \, dF(y) \end{aligned}$$

If $F(x)$ has a density $f(x)$, then the pdf of $X_{U(2)}$ is $f_2(x) = R(x) f(x)$. The distribution function $F_3(x)$ of $X_{U(3)}$ is

$$\begin{aligned} F_3(x) &= P(X_{U(3)} \leq x) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \sum_{i=0}^{\infty} (F(u))^i \, R(u) \, dF(u) \, dF(y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{R(u)}{1 - F(u)} \, dF(u) \, dF(y) \\ &= \int_{-\infty}^{x} \frac{(R(u))^2}{2!} \, dF(u). \end{aligned}$$

The pdf $f_3(x)$ of $X_{U(3)}$ is $f_3(x) = (R(x))^2 f(x)/2!$, $-\infty < x < \infty$. It can similarly be shown that the cdf $F_n(x)$ of $X_{U(n)}$ is for $-\infty < x < \infty$

$$F_n(x) = \int_{-\infty}^{x} f(u_n) du_n \int_{-\infty}^{u_n} \frac{f(u_{n-1})}{1 - F(u_{n-1})} \, du_{n-1} \ldots \int_{-\infty}^{u_2} \frac{f(u_1)}{1 - F(u_1)} \, du_1$$

$$= \int_{-\infty}^{x} \frac{R^{n-1}(u)}{(n-1)!}\, dF(u).$$

This can be expressed as

$$F_n(x) = \int_{-\infty}^{R(x)} \frac{u^{n-1}}{(n-1)!}\, e^{-u}\, du$$
$$= 1 - e^{-R(x)} \sum_{j=0}^{n-1} \frac{(R(x))^j}{j!}$$

The pdf $f_n$(x) of $X_{U(n)}$ is

$$f_n(x) = \frac{(R(x))^{n-1}}{(n-1)!} f(x) \tag{2.1}$$

The joint pdf $f(x_1, x_2, \ldots, x_n)$ of the $n$ record values $X_{U(1)}$, $X_{U(2)}$, $\ldots$, $X_{U(n)}$ is given for $-\infty < x_1 < x_2 < \cdots < x_{n-1} < x_n < \infty$, by

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1)\, r(x_2) \ldots r(x_{n-1}) f(x_n) \tag{2.2}$$

where $r(x) = dR(x)/dx = f(x)/(1 - F(x))$, $0 < F(x) < 1$. The function $r(x)$ is known as hazard rate.

The joint pdf of $X_{U(i)}$ and $X_{U(j)}$ is

$$f_{i,j}(x_i, x_j) = \frac{(R(x_i))^{i-1}}{(i-1)!} r(x_i)\, \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} f(x_j) \tag{2.3}$$

for $-\infty < x_i < x_j < \infty$. The conditional distribution of $X_{U(j)}|X_{U(i)}$ is given by

$$f_{j|i}(x_j|x_i) = \frac{(R(x_j) - R(x_i))^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)}, \quad \text{for} \quad -\infty < x_i < x_j < \infty. \tag{2.4}$$

In particular, for $j = i + 1$ and $-\infty < x_i < x_{i+1} < \infty$, we have

$$f(x_{i+1}|X_{U(i)} = x_i) = \frac{f(x_{i+1})}{1 - F(x_i)}.$$

For $i > 0$, $1 \leq k < m$, the joint conditional pdf of $X_{U(i+k)}$ and $X_{U(i+m)}|X_{U(i)}$ is for $-\infty < z < x < y < \infty$

$$f_{i+k,,i+m}(x, y|\, X_{U(i)} = z)$$
$$= \frac{1}{\Gamma(m-k)} \frac{1}{\Gamma(k)} [R(y) - R(x)]^{m-k-1} [R(x) - R(z)]^{k-1} \frac{f(y)\, r(x)}{\bar{F}(z)}$$

The marginal pdf of the $n$th lower record value can be derived by using the same procedure as that of the $n$th upper record value. Let $H(u) = -\ln F(u)$, $0 < F(u) < 1$ and $h(u) = -dH(u)/du$, then

$$P(X_{L(n)} \leq x) = \int_{-\infty}^{x} \frac{\{H(u)\}^{n-1}}{(n-1)!}\, dF(u)$$

and the corresponding pdf $f_{(n)}$ can be written as

$$f_{(n)}(x) = \frac{(H(x))^{n-1}}{(n-1)!}\, f(x). \tag{2.5}$$

The joint pdf of $X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)}$ can be written as

$$f_{(1),(2),\ldots,(m)}(x_1, x_2, \ldots, x_m) = h(x_1)h(x_2)\ldots h(x_{m-1})f(x_m),$$

for $-\infty < x_m < \cdots < x_1 < \infty$.
The joint pdf of $X_{L(r)}$ and $X_{L(s)}$ is

$$f_{(r),(s)}(x,y) = \frac{(H(x))^{r-1}}{(r-1)!}\frac{[H(y)-H(x)]^{s-r-1}}{(s-r-1)!}h(x)\,f(y) \tag{2.6}$$

for $s > r$ and $-\infty < y < x < \infty$.

**Example 2.1.** Let us consider the exponential distribution with pdf $f(x) = e^{-x}$ and cdf $F(x) = 1 - e^{-x}$, $0 \le x < \infty$. Then $R(x) = x$ and

$$f_n(x) = \frac{x^{n-1}}{\Gamma(n)}e^{-x}, \qquad x \ge 0.$$

The joint pdf of $X_{U(m)}$ and $X_{U(n)}$, $n > m$ is

$$f_{m,n}(x,y) = \frac{x^{m-1}}{\Gamma(m)\Gamma(n-m)}(y-x)^{n-m-1}e^{-y}, \qquad 0 \le x < y < \infty.$$

The conditional pdf of $X_{U(n)}|X_{U(m)} = x)$ is

$$f(y|X_{U(m)} = x) = \frac{(y-x)^{n-m-1}}{\Gamma(n-m)}e^{-(y-x)}, \qquad 0 \le x < y < \infty.$$

Thus the conditional distribution of $X_{U(n)} - X_{U(m)}$ given $X_{U(m)}$ is the same as the unconditional distribution of $X_{U(n-m)}$ for $n > m$.

**Example 2.2.** Suppose that the random variable X has the Gumbel distribution with pdf $f(x) = \exp\{-x\}\exp\{-e^{-x}\}$, $-\infty < x < \infty$. Let $F_{(n)}$ and $f_{(n)}$ be the cdf and pdf of $X_{L(n).}$ It is easy to see that

$$F_{(n)}(x) = \int_{-\infty}^{x} \frac{e^{-nu}}{\Gamma(n)}e^{-e^{-u}}du \quad \text{and} \quad f_{(n)}(x) = \frac{e^{-nx}}{\Gamma(n)}e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Let $f_{(m,n)}$(x,y) be the joint pdf of $X_{L(m)}$ and $X_{L(n)}$, m< n. Using (2.6), we get for the Gumbel distribution

$$f_{(m,n)}\left(x,y\right) = \frac{\left(e^{-y}-e^{-x}\right)^{n-m-1}}{\Gamma\left(n-m\right)}\frac{e^{-mx}}{\Gamma\left(m\right)}e^{-y}e^{-e^{-y}}, \quad -\infty < y < x < \infty.$$

Thus the conditional pdf $f_{(n|m)}(y|x)$ of $X_{L(n)}|X_{L(m)} = x$ is given by

$$f_{(n|m)}(y|x) = \frac{(e^{-y}-e^{-x})^{n-m-1}}{\Gamma(n-m)}e^{-y}e^{-(e^{-y}-e^{-x})}, \quad -\infty < y < x < \infty.$$

## 2.2. Representation of records

Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be a sequence of independent and identically distributed random variable with cdf $F_0(x) = 1 - \exp(-x)$, $x > 0$. Further suppose that $X_1, X_2, \ldots$ be a sequence i.i.d. r.v.'s with continuous cdf $F$. Then one has

$$\ln(1 - F(x)) \stackrel{d}{=} Y_1.$$

The following theorem (Bairamov and Ahsanullah [8]) gives the representation of the $n$th record as sum of $n$ independent random variables.

**Theorem 2.1.**

$$X_{U(n)} \stackrel{d}{=} g_F^{-1}(g_F(X_1) + g_F(X_2) + \cdots + g_F(X_n)), \tag{2.7}$$

*where* $g_F(x) = -\ln(1 - F(x))$ *and* $g_F^{-1}(x) = F^{-1}(1 - e^{-x})$.

*Proof.* Using the result of Example 2.1, we obtain

$$\begin{aligned}
&P(X_{U(n)} \le x) \\
&= P(Y_1 + Y_2 + \ldots + Y_n) \\
&= P(-\ln(1 - F(x_1)) + (-\ln(1 - F(X_2))) + \ldots + (-\ln(1 - F(x_n)))) \\
&= P(g_F(X_1) + g_F(X_2) + \ldots + g_F(X_n)) \\
&= P(g_F^{-1}\{g_F(X_1) + g_F(X_2) + \ldots + g_F(X_n)\}).
\end{aligned}$$

Therefore

$$X_{U(n)} \stackrel{d}{=} g_F^{-1}(g_F(X_1) + g_F(X_2) + \cdots + g_F(X_n)).$$ ∎

We will give here the representation of the $n$th record value of several well known distribution (for more details see Ahsanullah [2]).

**Example 2.3.** For the two parameter exponential distribution, $E(\mu, \sigma)$, with cdf $F(x) = 1 - e^{-(x-\mu)/\sigma}$ , $x \ge 0$, we obtain

$$X_{U(n)} \stackrel{d}{=} X_1 + X_2 + \cdots + X_n \ - \ (n-1)\mu.$$

**Example 2.4.** Consider the Weibull distribution, $W(\alpha, \beta)$ with $F(x) = 1 - e^{-\beta x^{\alpha}}$ ($x \ge 0$, $\alpha > 0, \beta > 0$) Using (2.7), we obtain

$$X_{U(n)} \stackrel{d}{=} (X_1^{\alpha} + X_2^{\alpha} + \ldots + X_2^{\alpha})^{\frac{1}{\alpha}}.$$

**Example 2.5.** Consider the Power Function distribution, POW $(\theta, \alpha, \beta)$ with cdf

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \left(\frac{\beta - x}{\beta - \alpha}\right)^{\theta}, & \alpha \le x \le \beta, \ \ \theta > 0, \ \ -\infty \ < \ \alpha < \beta < \infty, \\ 1, & x \ge \beta. \end{cases}$$

For this distribution, the following representation is true

$$X_{U(n)} \stackrel{d}{=} \beta - (\beta - \alpha)^{-(n-1)} \, (\beta - X_1)(\beta - X_2) \ldots (\beta - X_n).$$

The following theorem can be proved using the same procedure as in the proof of Theorem 2.1. For details, see Ahsanullah and Nevzorov [4, page 242].

**Theorem 2.2.** *Suppose that* $\{X_1, X_2, \ldots\}$ *and* $\{Y_1, Y_2, \ldots\}$ *are two sequences of i.i.d. r.v.'s with continuous distribution functions* $F$ *and* $H$*, respectively. Then*

$$(X_{U(1)}, \ldots, X_{U(n)}) \stackrel{d}{=} G(H(Y_{U(1)})), \ldots, G(H(Y_{U(n)})),$$

*where* $G$ *is the inverse function of* $F$.

### 2.3. Records of discrete distributions

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence independent and identically distributed random variables taking values on $0, 1, 2, \ldots$ such that $F(n) < 1$ for all $n = 0, 1, 2, \ldots$ Let $p_k = P(X_1 = k)$, $P(k) = \sum_{j=0}^{k} p(j)$, $k \geq 0$ and $\bar{P}(k) = 1 - P(k)$ with $P(\infty) = 1$. The joint probability mass function (pmf) of $X_{U(1)}, \ldots X_{U(n)}$ is defined for $0 \leq x_1 < \ldots < x_n < \infty$ as

$$\begin{aligned} p_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) &= P(X_{U(1)} = x_1, X_{U(2)} = x_2, \ldots, X_{U(n)} = x_n) \\ &= \frac{p(x_1)}{\bar{P}(x_1)} \frac{p(x_2)}{\bar{P}(x_2)} \cdots \frac{p(x_{n-1})}{\bar{P}(x_{n-1})} p(x_n) \end{aligned} \tag{2.8}$$

The marginal pmf's of the upper record values are given as

$$p_1(x_1) = P(X_{U(1)} = x_1) = p(x_1), \quad x_1 = 0, 1, 2, \ldots,$$

$$p_2(x_2) = P(X_{U(2)} = x_2) = R_1(x_2)p(x_2),$$

where

$$R_1(k) = \sum_{0 \leq x_1 < x_2} B(x_1) \quad \text{and} \quad B(x) = \frac{p(x)}{\bar{P}(x)}, \quad x_2 = 1, 2, \ldots$$

Finally,

$$p_n(x_n) = P(X_{U(n)} = x_n) = R_{n-1}(x_n)p(x_n),$$

where

$$R_{n-1}(x_n) = \sum_{0 \leq x_1 < x_2 < \cdots < x_{n-1} < x_n} B(x_1)B(x_2)\ldots B(x_{n-1}), \qquad x_n = n-1, n, \ldots$$

The joint pmf of $X_{U(m)}$ and $X_{U(n)}$, $m < n$ is given by

$$\begin{aligned} p_{m,n}(x_m, x_n) &= P(X_{U(m)} = x_m, X_{U(n)} = x_n) \\ &= R_{m-1}(x_m)A(x_m)R_{m+1,n}(x_m, x_n)p(x_n), \ m \leq x_m < x_n - n + m < \infty, \end{aligned} \tag{2.9}$$

where

$$R_{m+1,n}(x, y) = \sum_{x_m < x_{m+1} < \cdots < x_n} B(x_{m+1})\ldots B(x_{n-1}), \quad m < n-1, \qquad R_{n,n}(x, y) = 1.$$

The conditional pmf of $X_{U(n)}$ given $X_{U(m)} = x_m$ is

$$p_{n|m}(X_{U(n)} = x_n | X_{U(m)} = x_m) = R_{m,n}(x_m, x_n)\frac{p(x_n)}{\bar{P}(x_m)},$$

for $x_m \leq x_n - n + m < \infty$ and

$$P_{n|n-1}(X_{U(n)} = x_n | X_{U(n-1)} = x_{n-1}) = \frac{p(x_n)}{\bar{P}(x_{n-1})}, \quad x_{n-1} < x_n. \tag{2.10}$$

It follows that the sequence of upper record values $X_{U(1)}, X_{U(2)} \ldots$ forms a Markov chain. Let $I_n = n$ if $n$ is a record value i.e. $X_{U(m)} = n$, for $m = 1, 2, \ldots$ and $I_n = 0$ if $X_{U(1)}, X_{U(2)}, \ldots$ does not contain the value $n$.

The following theorem is due to Aliev and Ahsanullah [7].

**Theorem 2.3.** *Let $X_1,\ X_2,\ldots$ be a sequence of independent and identically distributed random variables taking values on $0,1,2,\ldots$ with common distribution function $F$ such that $F(n)<1$ for all $n$ and $E(X_i^2)<\infty$. Suppose that $\{B_k,\ k=0,1,\ldots|$ be a sequence of numbers such that $2+2B_{n+1}-B_s-B_{s+2}\geq 0\}$. If there exits $F(x)$ such that $E\{(X_{U(2)}-X_{U(1)})^2|X_{U(1)}\}=B_s,\ s=0,1,2\ldots.$ Then $F(x)$ is unique.*

### 2.4. Geometric distribution

A discrete random variable $X$ is said to have geometric distribution if its probability mass function (pmf) is of the following form:

$$p(k)=P(X=k)=pq^{k-1},\qquad 0<p<1,\quad q=1-p,\quad k\in A_0$$

where $A_n$ is the set of integers $n+1,n+2,\ldots\ (n\geq 0)$. We say $X\in$ GE(p), if the pmf of $X$ is as given in. For $k>0$, we define $r(k)=P(X=k|X\geq k)$. We choose to distinguish between GE(p) and the larger class of distributions having geometric tail (GET). We write $X\in$ GET(s, p) if the pmf of $X$ is as follows:

$$p(k)=P(X=k)=cq^{k-1},\qquad q=1-p,\quad k\in A_s,$$

where $c$ is such that $\sum_{k=s+1}^{\infty}p(k)=1$. If $s=0$, then GET$(s,p)=$ GE$(p)$ with $c=p$.

The geometric distribution like the exponential distribution possesses the memory less property, i.e.

$$\bar{P}(r+s)=\bar{P}(r)\bar{P}(s),$$

where $r$ and $s$ are positive integers and $\bar{P}(j)=\sum_{k=j+1}^{\infty}p(k)$. Geometric distribution is said to be a discrete analogue of the exponential distribution.

If $X\in$ GE(p), then $\bar{P}(x)=q^x$ and $p(x)=pq^{x-1}$, for $x\in A_0$. Substituting the values of $\bar{P}(x_i)$ and $p(x_i)$ in (2.8), we get

$$p(x_1,x_2,\ldots,x_m)=p^nq^{x_m-m},\quad 1\leq x_1<x_2<\cdots<x_m<\infty.$$

The conditional pmf of $X_{U(n)}|X_{U(n-1)}=x_{n-1}$ is

$$P(X_{U(n)}=x_n|X_{U(n-1)}=x_{n-1})=pq^{x_n-x_{n-1}-1},\quad n-1\leq x_{n-1}<x_n<\infty.$$

Thus $X_{U(n)}-X_{U(n-1)}$ is independent of $X_{U(n-1)}$ and $X_{U(n)}-X_{U(n-1)}\in GE(p)$, $n=2,3,\ldots.$ Let $V_1=X_{U(1)}$, $V_2=X_{U(2)}-X_{U(1)}$ and $V_n=X_{U(n)}-X_{U(n-1)}$. Then $V_i$'s are independent and $V_i\in GE(p)$. We have

$$X_{U(n)}=V_1+V_2+\cdots+V_n. \tag{2.11}$$

It is known that if $X\in GE(p)$, then

$$E(s)^X=\sum_{x=1}^{\infty}s^xpq^{x-1}=\frac{ps}{1-qs}.$$

Using (2.11) we obtain

$$E(s^{X_{U(n)}})=E(s^{V_1+V_2+\cdots+V_n})=\left(\frac{ps}{1-qs}\right)^n.$$

The coefficient of s$^x$ in $(ps/(1-qs))^n$ is $\binom{x-1}{n-1} p^n q^{x-n}$ $(x \geq n)$. Thus the marginal pmf of $X_{U(m)}$ can be written as

$$p_m(x) = P(X_{U(m)} = x) = \binom{x-1}{m-1} p^{\,m} \ q^{\,x\,-\,m} \ , \quad x \in A_{m\,-\,1} \ , \quad m \geq 1.$$

We see that $X_{U(m)}$ has a negative binomial distribution with parameters $m$ and $p$. We can write for $n > m$,

$$X_{U(n)}|X_{U(m)} = x_m \quad \overset{d}{=} U_{m+1} + \cdots + U_n + x_m$$

and

$$E\left(s^{X_{U(n)}}|X_{U_{(m)}} = x_m\right) = s^{x_m}\left(\frac{ps}{1-qs}\right)^{n-m}.$$

Considering the coefficient of $s^y$ in $s^{x_n}$ $(ps/1-qs)^{n-m}$

$$p_{n|m}(y|x) = P(X_{U(n)} = y|X_{U(m)} = x) = \binom{y-x-1}{n-m-1} p^{n-m} q^{y-x-n+m}$$

we obtain the conditional pmf of $X_{U(n)}$ given $X_{U(m)}$ as

$$P(X_{U(n)} = y|X_{U(m)} = x) = \binom{y-x-1}{n-m-1} p^{n-m} q^{y-x-n+m}, \quad 0 < m \leq x \leq y{-}n{+}m < \infty.$$

But we know that the marginal pmf of $X_{U(m)}$ is

$$P(X_{U(m)} = x) = \binom{x-1}{m-1} p^m q^{x-m}, \quad x \in A_{m-1}, \quad m \geq 1.$$

Thus, the joint pmf of $X_{U(m)}$ and $X_{U(n)}$ is

$$P(X_{U(m)} = x, X_{U(n)} = y) = \binom{x-1}{m-1}\binom{y-x-1}{n-m-1} p^n q^{y-n}, \quad m \leq x < y{-}n{+}m < \infty.$$

Let $Z_{m,n} = X_{U(n)} - X_{U(m)}$, 0 < m < n < ∞, then

$$\begin{aligned} P(Z_{m,n} = z|X_{U(m)} = x) &= P(X_{U(n)} = z + x|X_{U(m)} = x) \\ &= \binom{z-1}{n-m-1} p^{n-m}(1-p)^{z-n+m}, \quad z \in A_{n-m-1}. \end{aligned}$$

Thus Z$_{n,m}$ and $X_{U(m)}$ are independent. Further Z$_{n,m}$ and $X_{U(n-m)}$ are identically distributed.

### 2.5. Characterizations

Here we present some characterizations of discrete distributions. For characterizations of continuous distributions, see Ahsanullah and Raqab [5]. There are several characterizations of the geometric distributions based on

(i) independence of $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$,
(ii) conditional distribution of $X_{U(n)}|X_{U(m)}$, and
(iii) moment properties of some functions of $X_{U(n)}$ for $n > m$.

The following Theorem is proved by Srivastava [12].

**Theorem 2.4.** *Suppose $F(x)$ is the distribution function of the sequence of i.i.d. random variables $\{X_n,\ n \geq 1\}$ with positive mass function only at $1, 2, \ldots$. Then $P(X_{U(2)} - X_{U(1)} = 1 | X_{U(1)} = i) = P(X_{U(2)} - X_{U(1)} = 1)$ for $i = 1, 2, \ldots,$ if and only if $X_n$ has the geometric distribution with $pmf$ given by*

$$p_j = P(X = j) = cp(1-p)^{j-2}, \quad j = 2, 3, \ldots$$

*and*

$$p_1 = 1 - \sum_{j=2}^{\infty} p_j = 1 - c, \quad 0 < p < 1, \quad 0 < c < 1.$$

**Theorem 2.5.** *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed discrete random variables with common distribution function $F$. Suppose $X_n$ is concentrated on the positive integers and $a = \sup\{x | F(x) < 1\} = \infty$. Then $X_n \in GET(n, p)$ for some fixed $n \geq 1$, if and only if $X_{U(n+1)} - X_{U(n)}$ and $X_{U(n)}$ are independent.*

*Proof.* We have already seen that for $u \in A_0$,

$$P(X_{U(n+1)} - X_{U(n)} = u | X_{U(n)} = y) = pq^{u-1} = P(X_k = u). \qquad ∎$$

Does the above condition characterize the geometric distribution? As an answer to that question we have the following theorem.

**Theorem 2.6.** *Let $\{X_n, n = 1, 2, ...\}$ be a sequence of independent and identically distributed discrete random variables with common distribution function $F$. Suppose $X_n$ is concentrated on the positive integers with $a = \sup\{x | F(x) < 1\} = \infty$. Further if $P(X_{U(n+1)} - X_{U(n)} = u | X_{U(n)} = y) = P(X_1 = u)$ for two fixed $y \in A_{n-1}$ and $y_1$ and $y_2$ are relatively prime and all $u \in A_0$, then $X_1 \in GET(n, p)$.*

*Proof.* Suppose that $P(X_{U(n+1)} - X_{U(n)} = u | X_{U(n)} = y) = P(X_1 = u)$, then from (2.10), we have

$$P(X_U(n+1) - X_U(n) = u | X_U(n) = y) = \frac{p(u+y)}{\bar{P}(y)} = p(u)$$

for two relatively prime $y_1, y_2 \in A_{n-1}$ and all $u \in A_0$. Summing with respect to $u$ from $u_0 + 1$ to $\infty$, we get $\bar{P}(u_0 + y)/\bar{P}(y) = \bar{P}(u_0)$ for two relatively prime $y_1,\ y_2 \in A_{n-1}$ and all $u \in A_0$. The general solution is

$$\bar{P}(x) = cp^x, \qquad x \in A_n,$$

and since $\bar{P}(\infty) = 0$, we must have

$$\bar{P}(x) = cp^x, \qquad 0 < q < 1, \quad x \in A_n. \qquad ∎$$

Srivastava [12] gave a characterization of the geometric distribution using the condition $E(X_{U(2)} | X_{U(1)} = y) = \alpha + y$. Ahsanullah and Holland [3] proved the following theorem which is a generalization of Srivastava's result.

**Theorem 2.7.** *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed discrete random variables with common distribution function $F$. Suppose $X_n$ is concentrated on the positive integers with $a = \sup\{x | F(x) < 1\} = \infty$. Further suppose $E(X_{U(n+1)})^2 < \infty$. If $E(X_{U(n+1)} | X_{U(n)} = y) = y + p^{-1}$ for all $y \in A_{n-1}$, then $X_1 \in GET(n, p)$.*

### 2.6. Weak records

Vervaat [14] introduced the concept of weak records of discrete distribution. For simplicity we will assume that the random variables $X_i$, $i = 1, 2, \ldots$ are concentrated on the non-negative integers. If in the definition of record times and record values we replace $>$ by $\geq$, then we obtain weak record times and weak record values. The joint pmf of $X_{U_w(1)}, X_{U_w(2)}, \cdots, X_{U_w(n)}$ is given by

$$(2.12)\quad p_{w,1,2,\ldots,n}(x_1, \ldots, x_n) = \left( \prod_{i=1}^{n-1} \frac{p(x_i)}{\bar{P}(x_i - 1)} \right) p(x_n), \quad 0 \leq x_1 \leq \ldots \leq x_n < \infty.$$

For any $m > 1$ and $n > m$, we can write

$$(2.13)\quad \begin{aligned} &P\left(X_{U_w(n)} = x_n, \ldots, X_{U_w(m+1)} = x_{m+1} | X_{U_w(m)} = x_m, \ldots, X_{U_w(1)} = x_1\right) \\ &= \left( \prod_{i=m+1}^{n-1} \frac{p(x_i)}{\bar{P}(x_i - 1)} \right) \frac{p(x_n)}{\bar{P}(x_m - 1)} \end{aligned}$$

It follows easily from (2.12) and (2.13) that the weak records, $X_{U_w(1)}, X_{U_w(2)}, \ldots$ form a Markov chain.

The marginal pmf's of the upper weak records are given by

$$P(X_{U_w(1)} = x_1) = p_{w,1}(x_1) = p(x_1), \quad x_1 = 0, 1, \ldots,$$

$$P(X_{U_w(2)} = x_2) = p_{w,2}(x_2) = R_{w,1}(x_2) p(x_2), \quad x_2 = 0, 1, \ldots$$

where

$$R_{w,1}(x_2) = \sum_{0 \leq x_1 \leq x_2} \frac{p(x_1)}{\bar{P}(x_1 - 1)}.$$

Finally,

$$P(X_{U_w(n)} = x_n) = p_{w,n}(x_n) = R_{w,n-1}(x_n) p(x_n),$$

where

$$R_{w,n-1}(x_n) = \sum_{0 \leq x_1 \leq x_2 \leq \cdots x_{n-1}} \prod_{i=1}^{n-1} \frac{p(x_i)}{\bar{P}(x_i - 1)}.$$

The joint pmf of $X_{U_w(m)}$ and $X_{U_w(n)}$ for $m < n$ is given by

$$(2.14)\quad P_{w,m,n}(x_m, x_n) = R_{w,m}(x_m) A_w(x_m) R_{w,m+1,n}(x_m, x_n) p(x_n),$$

for $m \leq x_m < x_n - n + m < \infty$, where

$$R_{w,m,n}(x, y) = \sum_{x_m \leq x_{m+1} \leq \cdots \leq n} A_w(x_{m+1}) \ldots A_w(x_{n-1}), \quad m < n - 1,$$

$R_{w,n-1,n}(x, y) = 1$ and $A_w(x) = p(x)/\bar{P}(x - 1)$. The conditional pmf of $X_{U_w(m)}$ given $X_{U_w(n)}$ is

$$(2.15)\quad P_{w,n|m}(X_{U_w(n)} = x_n | X_{U_w(m)} = x_m) = R_{w,m+1,n}(x_m, x_n) \frac{p(x_n)}{\bar{P}(x_m - 1)},$$

for $m \leq x_m \leq x_n < \infty$.

Thus, the pmf of $X_{U_w(n)}$ given $X_{U_w(n-1)}$ and $X_{U_w(n-2)}$ are respectively

$$P_{w,n|n-1}(X_{U_w(n)} = x_n | X_{U_w(n-1)} = x_{n-1}) = \frac{p(x_n)}{\bar{P}(x_{n-1} - 1)}$$

and

$$\begin{aligned} P_{w,n|n-2}&(X_{U_w(n)} = x_n | X_{U_w(n-2)} = x_{n-2}) \\ &= R_{w,n-2,n}(x_{n-2}, x_n) \frac{p(x_n)}{\bar{P}(x_{n-2} - 1)} \end{aligned} \tag{2.16}$$

**Theorem 2.8.** *Let $\{X_i,\ i = 1, 2, \ldots\}$ be sequence of independent and identically distributed random variables taking values on $0, 1, \ldots, n \le \infty$ with distribution $F$ such that $F(n) < \infty$ for $n < \infty$ and $E(X_1 \ln(1 + X_1) < \infty$. For some continuous function $\psi$, the condition $E(\psi(X_{U_w(n)} | X_{U_w(n-1)} = j) = g(j)$ determines the distribution.*

*Proof.* We have from

$$P(\psi(X_{U_w(n)} = y) | X_{U_w(n-1)} = x) = \frac{p_y}{q_x},$$

where $p_y = P(X = y)$ and $q_x = P(X \ge x)$. Now

$$E(\psi(X_{U_w(n)} | X_{U_w(n-1)} = j) = \frac{1}{q_j} \sum_{k=j}^{N} \psi(k) p_k.$$

Using the condition as given in the theorem, we can write the above expression as $g(j) q_j = \sum_{k=j}^{n} \psi(k) p_k$ . Taking first order difference, we obtain from

$$g(j) q_j - g(j+1) q_{j+1} = \psi(j)\, p_j.$$

Thus,

$$g(j) q_j - g(j+1)(q_j - p_j) = \psi(j) p_j$$

i.e.

$$p_j = \frac{g(j+1) - g(j)}{g(j+1) - \psi(j)} q_j\ .$$

Since $q_0 = 1$ and

$$q_j = \frac{q_j}{q_{j-1}} \frac{q_{j-1}}{q_{j-2}} \cdots \frac{q_1}{q_0},$$

we have

$$p_j = \frac{g(j+1) - g(j)}{g(j+1) - \psi(j)} \prod_{k-0}^{j-1} \frac{q_{k+1}}{q_k}\ .$$

We have

$$g(j) q_j - g(j+1) q_{j+1} = \psi(j)(q_j - q_{j+1})$$

i.e.

$$\frac{q_{j+1}}{q_j} = \frac{g(j) - \psi(j)}{g(j+1) - \psi(j)}.$$

Then we can write

$$p_j = \frac{g(j+1) - g(j)}{g(j+1) - \psi(j)} \prod_{k=0}^{j-1} \left( \frac{g(k) - \psi(k)}{g(k+1) - \psi(k)} \right).$$

∎

**Example 2.6.** (Geometric distribution) Let $X_1, X_2, \ldots$ be a sequence of independent and identically random variables with $p(k) = pq^k$ and $\bar{P}(k-1) = q^k$, $k = 0, 1, 2, \ldots$. Here $R_{w,1}(x_2) = \sum_{0 \le x_1 \le x_2} p(x_1)/\bar{P}(x_1 - 1) = x_2 p$. Thus,

$$p_{w,2}(k) = R_{w,1}(k)p(k) = (k+1)p^2 q^k, \; k = 0, 1, \ldots$$

Since

$$P_{w,n|n-1}(X_{U_w(n)} = x_n | X_{U_w(n-1)} = x_{n-1}) = \frac{p(x_n)}{\bar{P}(x_{n-1} - 1)} = pq^{x_n - x_{n-1}}, \qquad x_n \ge x_{n-1},$$

we obtain for $x_3 = 0, 1, \ldots$

$$\begin{aligned} p_{w,3}(x_3) &= \sum_{x_2=0}^{x_3} (x_2+1)p^2 q^{x_2} p q^{x_3 - x_2} \\ &= \sum_{x_2=0}^{x_3} (x_2+1)p^3 q^{x_3} \\ &= \frac{(x_3+2)(x_3+1)}{2} p^3 q^{x_3}. \end{aligned}$$

By induction it can be proved that when $n \ge 2$ and $x_n = 0, 1, 2, \ldots$

$$\begin{aligned} p_{w,n}(x_n) &= \frac{(x_n+1)(x_n+2)\ldots(x_n+n-1)}{(n-1)!} p^n q^{x_n} \\ &= \binom{x_n+n-1}{n-1} p^n q^{x_n} \end{aligned}$$

and

$$E(X_{U_w(n)} | X_{U_w(n-1)} = x_{n-1}) = \sum_{x_n = x_{n-1}}^{\infty} x_n p q^{x_n - x_{n-1}} = x_{n-1} + \frac{q}{p}.$$

The conditional pmf of $X_{U_w(n)}$ given $X_{U_w(n-2)}$ is for $0 \le x_{n-2} \le x_n < \infty$.

$$P_{w,n|n-2}(X_{U_w(n)} = x_n | X_{U(n-2)} = x_{n-2}) = R_{w,n-2,n}(x_n, x_{n-2}) \frac{p(x_n)}{\bar{P}(x_{n-2} - 1)},$$

where

$$R_{w,n-2,n}(x, y) = \sum_{x_{n-2} \le x_{n-1} \le x} A_w(x_{n-1}) = x - x_{n-2} + 1.$$

The conditional expectation of $X_{U_w(n)} | X_{U_w(n-2)} = x$ is

$$\begin{aligned} & E(X_{U_w(n)} | X_{U_w(n-2)} = x_{n-2}) \\ &= \sum_{x = x_{n-2}}^{\infty} x(x - x_{n-2} + 1)p^2 q^{x - x_{n-2}} \\ &= x_{n-2} + \sum_{x = x_{n-2}}^{\infty} (x - x_{n-2})(x - x_{n-2} + 1)p^2 q^{x - x_{n-2}} \\ &= x_{n-2} + \frac{2q}{p}. \end{aligned}$$

Stepanov [13] and Aliev [6] proved that if $E(X_{U_w(n)}|X_{U_w(n-1)} = x_{n-1}) = x_{n-1}+b$ then $X_i$'s $\in GE(1/(1+b))$. This result follows from the equation of $p_j$ if we take $\psi(x) = x$ and $g(x) = x + b$. A generalization is the following theorem due to Wesolowski and Ahsanullah [15].

**Theorem 2.9.** *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables taking values on $0, 1, \ldots, N$ with distribution function $F$ such that $F(n) < 1$ for any $n < N$ and $N$ need not be finite. If $E(X_{U_w(n)}|X_{U_w(n-2)} = x) = x + b$, $n > 2$, then $X_i$'s $\in GE(b/(2 + b))$. Wesolowski and Ahsanullah [15] proved that if $E(X_{U_w(n)}|X_{U(n-2)} = m) = am + b$, where $a < 1$, then the distribution of the $X$'s is Beta-Binomial.*

Hijab and Ahsanullah [10] proved the following characterization theorem.

**Theorem 2.10.** *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution function $F$. Suppose $X_n$ is concentrated on $0, 1, \ldots \infty$ and $a = \{x|F(x) < 1\} = \infty$. Then $X_j \in GE(p)$ iff $X_{U_w(n+1)} \stackrel{d}{=} W_n + W$, where $W_n$ has the same distribution as the sum of n i.i.d. geometric random variable and $W$ is independent of $W_n$ and has the same distribution as $X_i's$. $i \geq 1$.*

The following generalization of this Theorem is proved by Ahsanullah [1].

**Theorem 2.11.** *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution function F. Suppose $X_n$ is concentrated on $0, 1, \ldots \infty$ and $a = \{x|F(x) < 1\} = \infty$. Then $X_j \in GE(p)$ iff $X_{U_w(n)} \stackrel{d}{=} X_{U_w(m)} + W_{n-m}$, $n > m$, where $W_{n-m}$ has the same distribution as the sum of n-m i.i.d. geometric random variable and $W_{n-m}$ is independent of $X_{U_w(m)}$. It is an open problem whether $X_{U_w(n)} \stackrel{d}{=} X_{U_w(m)} + W_{U_w(n-m)}$ for $1 \leq m < n$ is a characterization property of the geometric distribution.*

### 2.7. Concomitants

Let $\{X_i, Y_i, i = 1, 2, ...\}$ be a sequence of i.i.d. bivariate random variables with cdf $F_{12}(x, y)$ and pdf $f_{12}(x, y)$. Let $X_{U(1)} = X_1, X_{U(2)}, ...$ be the upper record values of the $X$'s and $Y(1), Y(2), ...$ be the corresponding values in the sequence $\{X_i, Y_i, i = 1, 2, ...\}$. Then we will call $y(r), r = 1, 2, ...$ as the (upper) concomitant of $X_{U(r)}$.

The conditional distribution of $Y(k)|X_{U(k)}$ is the same as conditional pdf of $Y_j|X_j$, $j = 1, 2, \ldots$

The joint pdf $f_{12,k}$ of $X_{U(k)}$ and $Y(k)$ is given by

$$f_{12,k}(x, y) = f(x, y)\frac{-\ln(1 - F_1(x))^{k-1}}{\Gamma(k)},$$

where $F_1(x)$ is the marginal cdf of $X$'s.

The Morgenstern copula has the joint pdf as

$$f(x, y) = 1 + \alpha(2x - 1)(2y - 1),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ -1 < \alpha < 1$$

and the corresponding cdf is

$$F(x,y) = xy[1+\alpha(1-x)(1-y)]$$
$$F(0,0)=1,\ F(0,1)=0=F(1,0) \text{ and } F(1,1)=1.$$

This copula was first proposed by Morgenstern [11]. If $\alpha$ is zero, then the copula collapses to independence. It is attractive due to its simplicity.

The conditional pdf of $Y$ given $X$ is given by

$$f_{Y|X}(y|x) = 1+\alpha(2x-1)(2y-1).$$

The joint pdf $f_{12,k}$ of $X_{U(k)}$ and $Y(k)$ is given by

$$f_{12,k}(x,y) = [1+\alpha(2x-1)(2y-1)]\frac{(-\ln(1-x)^{k-1}}{\Gamma(k)} \tag{2.17}$$

The marginal pdf $f_{2,k}(y)$ of $Y(k)$ is given by

$$\begin{aligned} f_{2,k}(y) &= \int_0^1 [1+\alpha(2x-1)(2y-1)]\frac{-\ln(1-x)^{k-1}}{\Gamma(k)}dx \\ &= 1-\alpha\left(1-\frac{1}{2^{k-1}}\right)+2\alpha\left(1-\frac{1}{2^{k-1}}\right)y \\ &= 1+\alpha\left(1-\frac{1}{2^{k-1}}\right)(2y-1). \end{aligned}$$

Hence

$$\begin{aligned} f_{2,k}(y)-f_{2,1}(y) &= \alpha\left(1-\frac{1}{2^{k-1}}\right)(2y-1), \\ E(Y(k))^p - E(Y(1))^p &= \alpha\left(1-\frac{1}{2^{k-1}}\right)\left(\frac{2}{p+2}-\frac{1}{p+1}\right). \end{aligned}$$

Since

$$\begin{aligned} E(Y(1))^p &= \int_0^1 y^p dy = \frac{1}{p+1}, \\ E(Y(k))^p &= \left[1-\alpha\left(1-\frac{1}{2^{k-1}}\right)\right]\frac{1}{p+1}+2\alpha\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{p+2}. \end{aligned}$$

Thus

$$\begin{aligned} E(Y(k)) &= \frac{1}{2}-\frac{\alpha}{2}\left(1-\frac{1}{2^{k-1}}\right)+\frac{2\alpha}{3}\left(1-\frac{1}{2^{k-1}}\right) \\ &= \frac{1}{2}\left[1+\frac{\alpha}{3}\left(1-\frac{1}{2^{k-1}}\right)\right] \\ E(Y(k))^2 &= \frac{1}{3}-\frac{\alpha}{3}\left(1-\frac{1}{2^{k-1}}\right)+\frac{\alpha}{2}\left(1-\frac{1}{2^{k-1}}\right) \\ &= \frac{1}{3}+\frac{\alpha}{6}\left(1-\frac{1}{2^{k-1}}\right) \\ Var(Y(k)) &= \frac{1}{12}-\left(\frac{\alpha}{6}\left(1-\frac{1}{2^{k-1}}\right)\right)^2. \end{aligned}$$

For any fixed $\alpha$, $Var(Y(k))$ decreases as $k$ increases and $Y(k) \longrightarrow 1/12-(\alpha/6)^2$, as $k \to \infty$.

The joint pdf $f_{c,r,s}$ of $Y(r)\,Y(s)$, $r<s$, can be obtained from Beg and Ahsanullah [9, page 18] as a special case for uniform marginals. However we will present here $f_{r,s}$ for the Morgenstern copula for completeness.

$$f_{c,r,s}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{Y|X|}(y_1|x_1)f_{Y|X}(y_2|x_2)f_{r.s}(x_1,x_2)dx_1dx_2$$

where $f_{r.s}(x_1,x_2)$ is the joint pdf of $r$th and $s$th records. Using (2.16), we get

$$\begin{aligned}f_{c,r,s}(y_1,y_2)&=\int_0^1\int_{-\infty}^{x_2}[1+\alpha(2x_1-1)(2y_1-1)][1+\alpha(2x_2-1)(2y_2-1)]\\ &\times\left[\frac{-\ln(1-x_1)^{r-1}}{(i-1)!}\frac{1}{1-x_1}\frac{[-\ln(1-x_2)-(-\ln(1-x_1))]^{s-r-1}}{(j-i-1)!}\right]dx_1dx_2 \qquad (2.18)\end{aligned}$$

To simplify the $f_{2,r,s}(y_1,y_2)$, we need the following lemma.

**Lemma 2.1.** *Let $p$ and $q$ are real numbers, then*

$$\begin{aligned}I_{p,q}&=\int_0^1\int_0^{x_2}(1-x_1)^p(1-x_2)^q\ \frac{-\ln(1-x_1)^{r-1}}{(i-1)!}\frac{1}{1-x}\\ &\quad\times\frac{[-\ln(1-x_2)-(-\ln(1-x_1))]^{s-r-1}}{(j-i-1)!}dx_1dx_2\\ &=\frac{1}{(p+q+1)^r}\frac{1}{(1+q)^{s-r}}.\end{aligned}$$

*Proof.* Let $u=-\ln(1-x_1)$ and $v=-\ln(1-x_2)$, then

$$\begin{aligned}I_{p,q}&=\int_0^1\int_0^{x_2}(1-x_1)^p\ (1-x_2)^q\frac{-\ln(1-x_1)^{r-1}}{(r-1)!}\frac{1}{1-x_1}\\ &\quad\times\frac{[-\ln(1-x_2)+\ln(1-x_1)]^{s-r-1}}{(s-r-1)!}dx_1dx_2\\ &=\frac{1}{(r-1)!}\frac{1}{(s-r-1)!}\int_0^1(1-x_1)^{p-1}\ -\ln(1-x_1)^{r-1}\\ &\quad\times\left[\int_{x_1}^1(1-x_2)^q\ \frac{[-\ln(1-x_2)-(-\ln(1-x_1))]^{s-r-1}}{(s-r-1)!}dx_2\right]dx_1\end{aligned}$$

On substituting $1-x_2=(1-x_1)e^{-t}$, we obtain

$$\begin{aligned}I_{p,q}&=\frac{1}{(r-1)!}\frac{1}{(s-r-1)!}\int_0^1(1-x_1)^{p+q}-\ln(1-x_1)^{r-1}\left[\int_0^{\infty}e^{-(1+q)t}t^{s-r-1}dt\right]dx_1\\ &=\frac{1}{(1+q)^{s-r}}\frac{1}{(r-1)!}\int_0^1(1-x_1)^{p+q}-\ln(1-x_1)^{r-1}dx_1\\ &=\frac{1}{(p+q+1)^r}\frac{1}{(1+q)^{s-r}}.\end{aligned}$$

∎

We have, from Lemma 2.1, $I_{0,0}=1$, $I_{1,0}=1/2^r$, $I_{0,1}=1/2^s$, $I_{1,1}=1/3^r2^{s-r}$.

The joint pdf of of $Y(r)$ and $Y(s)$ is

$$f_{c,r,s}(y_1,y_2) = \int_0^1 \int_{-\infty}^{x_2} [1+\alpha(2x_1-1)(2y_1-1)][1+\alpha(2x_2-1)(2y_2-1)]$$
$$\times \left[\frac{-\ln(1-x_1)^{r-1}}{(i-1)!}\frac{1}{1-x_1}\frac{[-\ln(1-x_2)-(-\ln(1-x_1))]^{s-r-1}}{(j-i-1)!}\right] dx_1dx_2$$
$$= 1+\alpha(2y_1-1)\left(1-\frac{1}{2^{r-1}}\right)+\alpha(2y_2-1)\left(1-\frac{1}{2^{s-1}}\right)$$
$$+\alpha^2(2y_1-1)(2y_2-1)\left[\frac{1}{3^r2^{s--r}}-\frac{1}{2^{r-1}}-\frac{1}{2^{s-1}}+1\right]$$
$$\times\frac{1}{(p+q+1)^r}\frac{1}{(1+q)^{s-r}}.$$

For $1 \le r < s$,

$$E(Y(r)Y(s)) = \frac{1}{4}+\frac{\alpha}{12}\left(1-\frac{1}{2^{r-1}}\right)+\frac{\alpha}{12}\left(1-\frac{1}{2^{s-1}}\right)$$
$$+\frac{\alpha^2}{36}\left(\frac{1}{3^r2^{s-r-2}}-\frac{1}{2^{r-1}}-\frac{1}{2^{s-1}}+1\right),$$

$$c_{[r,s]} = Cov(Y(r)Y(s))$$
$$= \frac{1}{4}+\frac{\alpha}{12}\left(1-\frac{1}{2^{r-1}}\right)+\frac{\alpha}{12}\left(1-\frac{1}{2^{s-1}}\right)$$
$$+\frac{\alpha^2}{36}\left(\frac{1}{3^r2^{s-r-2}}-\frac{1}{2^{r-1}}-\frac{1}{2^{s-1}}+1\right)$$
$$-\frac{1}{4}\left[1+\frac{\alpha}{3}\left(1-\frac{1}{2^{r-1}}\right)\right]\left[1+\frac{\alpha}{3}\left(1-\frac{1}{2^{s-1}}\right)\right]$$
$$= \frac{\alpha^2}{36}\left[\frac{1}{3^r2^{s-r-2}}-\frac{1}{2^{r+s-2}}\right] = \frac{\alpha^2}{9}\frac{1}{2^s}\left[\left(\frac{2}{3}\right)^r-\frac{1}{2^r}\right].$$

## References


[1] M. Ahsanullah, Some characteristic properties of weak records of geometric distributions, *J. Stat. Theory Appl.* **7** (2008), no. 1, 81–92.

[2] M. Ahsanullah, *Record Values - Theory and Applications*, University Press of America, Lanham, MD, 2004.

[3] M. Ahsanullah and B. Holland, On the use of record values to estimate the location and Scale Parameters of the generalized extreme value distribution, *Sankhya A*, **56** (1984), 480–499.

[4] M. Ahsanullah and V. B. Nevzorov, *Ordered Random Variables*, Nova Sci. Publ., Huntington, NY, 2001.

[5] M. Ahsanullah and M. Z. Raqab, *Bounds and Characterizations of Record Statistics*, Nova Sci. Publ., Hauppauge, NY, 2006.

[6] F. A. Aliev, Characterization of distributions through weak records, *J. Appl. Statist. Sci.* **8** (1998), no. 1, 13–16.

[7] F. A. Aliev and M. Ahsanullah, On characterizations of discrete distribution through regressions of record values, *Pakistan J. Statist.* **18** (2002), no. 3, 415–421.

[8] I. G. Bairamov and M. Ahsanullah, Distributional relations between order statistics and the sample itself and characterization of exponential distribution, *J. Appl. Statist. Sci.* **10** (2000), no. 1, 1–16.

[9] M.I. Beg and M. Ahsanullah, Concomitants of generalized order statistics from from Farlie-Gumbel-Morgenstern distributions, *Statistical Methodology* **5** (2008), 1–20.

[10] O. Hijab and M. Ahsanullah, Weak records of geometric distribution and some characterizations, *Pakistan J. Statist.* **22** (2006), no. 2, 139–146.

[11] D. Morgenstern, Einfache Beispiele zweidimensionaler Verteilungen, *Mitteilungsbl. Math. Statist.* **8** (1956), 234–235.

[12] R. C. Srivastava, Two characterizations of the geometric distribution by record values, *Sankhyā Ser. B* **40** (1978/79), no. 3-4, 276–278.

[13] A. V. Stepanov, Teor. Veroyatnost. i Primenen. **38** (1993), no. 4, 903–905; translation in *Theory Probab. Appl.* **38** (1993), no. 4, 762–764.

[14] W. Vervaat, Limit theorems for records from discrete distributions, *Stochastic Processes Appl.* **1** (1973), 317–334.

[15] J. Wesołowski and M. Ahsanullah, Linearity of regression for non-adjacent weak records, *Statist. Sinica* **11** (2001), no. 1, 39–52.