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Generalized Fuzzy Ideals of BCI-Algebras

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Abstract. The notions of $(\in, \in\vee \mathrm{q})$-fuzzy BCI-positive implicative (resp., BCI-implicative, BCI-commutative) ideals in BCI-algebras are introduced, and related properties are investigated. Some characterization theorems of these generalized fuzzy ideals are derived. The relationship among these generalized fuzzy ideals of BCI-algebras is considered. We also consider the concept of implication-based fuzzy implicative ideals in BCI-algebras. In particular, we discuss the implication operators in Lukasiewicz system of continuous-valued logic.

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1. Introduction and preliminaries

Logic appears in a 'sacred' form (resp., a 'profane') which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold; as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty (see [28] for generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life.

BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki (see [9, 10]) and have been extensively investigated by many

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researchers (cf. [6, 7], [11–24] and [29–34]). BCI-algebras are generalizations of BCK-algebras. Hence, most of the algebras related to the t-norm based logic, such as MTL [8], BL, hoop, MV [3] (i.e., lattice implication algebra) and Boolean algebras etc., are extensions of BCK-algebras (i.e., they are subclasses of BCK-algebras). This shows that BCK/BCI-algebras are considerably general structures.

After the introduction of fuzzy sets by Zadeh [27], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [25]. In fact, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in view, Jun [12, 13] introduced the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI-algebra and investigated related results. Recently, Davvaz [4] applied this theory to near-rings and obtained some useful results. Further, Davvaz and Corsini [5] redefined fuzzy H_v-submodule and many valued implications. For more details, the reader is referred to [4, 5, 12, 13].

As a continuation of [12, 13], we further discuss this topic in this paper. In Section 2, we first introduce the notion of $(\in, \in\vee\mathrm{q})$-fuzzy BCI-positive implicative ideals in BCI-algebras and investigate some of their related properties. Further, the notions of $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative and BCI-commutative ideals of BCI-algebras are introduced and the relationship among these generalized fuzzy ideals of BCI-algebras is considered. Finally, in Section 3 we consider the concept of implication-based fuzzy implicative ideals in BCI-algebras. In particular, we discuss the implication operators in Lukasiewicz system of continuous-valued logic.

By a BCI-algebra, we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the axioms:

(i) $((x*y)*(x*z))*(z*y)=0$;
(ii) $(x*(x*y))*y=0$;
(iii) $x*x=0$;
(iv) $x*y=0$ and $y*x=0$ imply $x=y$.

We can define a partial ordering "$\leq$" by $x\leq y$ if and only if $x*y=0$. If a BCI-algebra X satisfies $0*x=0$ for all $x\in X$, then we say that X is a BCK-algebra. In what follows, let X denote a BCI-algebra unless otherwise specified.

A non-empty subset I of X is called an *ideal* of X if it satisfies (I1) $0\in I$, (I2) $x*y\in I$ and $y\in I$ imply $x\in I$. A non-empty subset I of X is called a *BCI-positive implicative ideal* if it satisfies (I1) and (I3) $((x*z)*z)*(y*z)\in I$ and $y\in I$ imply $x*z\in I$. A non-empty subset I of X is called a *BCI-commutative ideal* if it satisfies (I1) and (I4) $(x*y)*z\in I$ and $z\in I$ imply $x*((y*(y*x))*(0*(0*(x*y))))\in I$. A non-empty subset I of X is called a *BCI-implicative ideal* if it satisfies (I1) and (I5) $(((x*y)*y)*(0*y))*z\in I$ and $z\in I$ imply $x*((y*(y*x))*(0*(0*(x*y))))\in I$ (see [21, 22]).

Definition 1.1. *A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies:*
(F1) $\mu(0)\geq\mu(x), \forall x\in X$,
(F2) $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}, \forall x,y\in X$.

Definition 1.2. [20] *A fuzzy set μ in X is called a fuzzy BCI-positive implicative ideal of X if it satisfies (F1) and*

(F3) $\mu(x * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y)\}$, *for all* $x, y, z \in X$.

Definition 1.3. [15] *A fuzzy set μ in X is called a fuzzy BCI-commutative ideal of X if it satisfies (F1) and*

(F4) $\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\mu((x * y) * z), \mu(z)\}$, *for all* $x, y, z \in X$.

Definition 1.4. [20] *A fuzzy set μ in X is called a fuzzy BCI-implicative ideal of X if it satisfies (F1) and*

(F5) $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu((((x*y)*y)*(0*y))*z), \mu(z)\}$, *for all* $x, y, z \in X$.

For any fuzzy set μ of X and $t \in (0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a *level subset* of μ.

By Fuzzy Transfer Principle, Liu et al. gave the following results:

Theorem 1.1. [20] *A fuzzy set μ in X is a fuzzy (resp., BCI-positive implicative, BCI-commutative, implicative) ideal of X if and only if each non-empty level subset μ_t is a (resp., BCI-positive implicative, BCI-commutative, implicative) ideal of X.*

A fuzzy set μ of a BCI-algebra X of the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t and is denoted by $U(x; t)$*. A fuzzy point $U(x; t)$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ, written as $U(x; t) \in \mu$ (resp. $U(x; t)q\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $U(x; t) \in \mu$ or (resp. and) $U(x; t)q\mu$, then we write $U(x; t) \in \vee\, \mathrm{q}$(resp. $\in \wedge\, \mathrm{q}$)μ. The symbol $\overline{\in \vee\, \mathrm{q}}$ means $\in \vee\, \mathrm{q}$ does not hold. Using the notion of " belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, the concept of (α, β)-fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in \vee\, \mathrm{q}, \in \wedge\, \mathrm{q}\}$ with $\alpha \neq \in \wedge\, \mathrm{q}$, was introduced in [2]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroup. For more information of $(\in, \in \vee\, \mathrm{q})$-fuzzy subgroups, the reader is referred to [1] if necessary.

In [12], Jun introduced the concept of (α, β)-fuzzy ideals of a BCK/BCI-algebra and investigated some related results.

Definition 1.5. [12] *A fuzzy set μ of X is called an $(\in, \in \vee\, \mathrm{q})$-fuzzy ideal of X if for all $t, r \in (0, 1]$ and $x, y \in X$,*

(F6) $U(x; t) \in \mu$ *implies* $U(0; t) \in \vee\, \mathrm{q}\mu$,

(F7) $U(x * y; t) \in \mu$ *and* $U(y; r) \in \mu$ *imply* $U(x; \min\{t, r\}) \in \vee\, \mathrm{q}\mu$.

Lemma 1.1. [12] *The conditions of (F6) and (F7) in Definition 1.5, are equivalent to the following conditions respectively:*

(F8) $\mu(0) \geq \min\{\mu(x), 0.5\}$, *for all* $x \in X$,

(F9) $\forall x, y \in X, \mu(x) \geq \min\{\mu(x * y), \mu(y), 0.5\}$.

Lemma 1.2. [12] *A fuzzy set μ in X is an $(\in, \in \vee\, \mathrm{q})$-fuzzy ideal of X if and only if the set μ_t is an ideal of X for all $0 < t \leq 0.5$.*

Lemma 1.3. [12] *Let μ be a fuzzy set in X. Then μ_t is an ideal of X for all $0.5 < t \leq 1$ if and only if it satisfies:*
(F10) $\forall x \in X, \max\{\mu(0), 0.5\} \geq \mu(x)$,
(F11) $\forall x, y \in X, \max\{\mu(x), 0.5\} \geq \min\{\mu(x * y), \mu(y)\}$.

2. Generalized fuzzy ideals

In this section, we introduce the concepts of some kinds of $(\in, \in \vee \mathrm{q})$-fuzzy ideals and discuss the related properties.

Definition 2.1. *An $(\in, \in \vee \mathrm{q})$-fuzzy ideal μ of X is called an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X if it satisfies:*
(F12) $\mu(x * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y), 0.5\}$, *for all* $x, y, z \in X$.

Example 2.1. Consider the BCI-algebra $X = \{0, 1, 2, 3\}$ with the following Caylay table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Define a fuzzy set μ in X by $\mu(0) = 0.8, \mu(1) = \mu(2) = \mu(3) = 0.4$. It is now routine to verify that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X.

Obviously, every fuzzy BCI-positive implicative ideal of X is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal. But the converse is not true in general from the following example:

Example 2.2. Consider the BCI-algebra $X = \{0, 1, 2\}$ with the following Caylay table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Define a fuzzy set μ in X by $\mu(0) = 0.7$, $\mu(1) = \mu(2) = 0.6$. By routine calculation we know that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X, but it is not a fuzzy BCI-positive implicative of X since

$$\mu(2 * 1) = 0.6 \not\geq \min\{\mu(((2 * 1) * 1) * (0 * 1)), \mu(0)\}.$$

The following Proposition is obvious and we omit the details.

Proposition 2.1. *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X. Then the following are equivalent:*

(i) *μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal,*
(ii) $\mu((x * y) * z) \geq \min\{\mu(((x * z) * z) * (y * z)), 0.5\}$, *for all* $x, y, z \in X$,
(iii) $\mu(x * y) \geq \min\{\mu(((x * y) * y) * (0 * y)), 0.5\}$, *for all* $x, y \in X$.

Now, we characterize the $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideals by their level BCI-positive implicative ideals.

Theorem 2.1. *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X. Then for all $0 < t \leq 0.5$, μ_t is an empty set or a BCI-positive implicative ideal of X. Conversely, if μ is a fuzzy set of X such that $\mu_t(\neq \emptyset)$ is a BCI-positive implicative ideal of X for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X.*

Proof. Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X and $0 < t \leq 0.5$. Then, by Lemma 1.2, we know that μ_t is an ideal of X. Let $((x*z)*z)*(y*z) \in \mu_t$ and $y \in \mu_t$, then $\mu(((x*z)*z)*(y*z)) \geq t$ and $\mu(y) \geq t$. It follows that $\mu(x*z) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t$, which implies, $x*z \in \mu_t$. Thus, μ_t is a BCI-positive implicative ideal of X.

Conversely, let μ be a fuzzy set of X such that $\mu_t(\neq \emptyset)$ is a BCI-positive implicative ideal of X for all $0 < t \leq 0.5$. Then, by Lemma 1.2, μ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X. We can write

$$\mu(((x*z)*z)*(y*z)) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y), 0.5\} = t_0,$$

$$\mu(y) \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y), 0.5\} = t_0.$$

Thus, $((x*z)*z)*(y*z), y \in \mu_{t_0}$, which implies $x*z \in \mu_{t_0}$, and so, $\mu(x*z) \geq t_0 = \min\{\mu(((x*z)*z)*(y*z)), \mu(y), 0.5\}$. Therefore, μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X. ∎

Naturally, we can also establish a corresponding result when μ_t is a BCI-positive implicative ideal of X, for all $0.5 < t \leq 1$.

Theorem 2.2. *Let μ be a fuzzy set of X. Then $\mu_t(\neq \emptyset)$ is a BCI-positive implicative ideal of X for all $0.5 < t \leq 1$ if and only if it satisfies (F10), (F11) and*
(F13) $\max\{\mu(x*z), 0.5\} \geq \min\{\mu(((x*z)*z)*(y*z)), \mu(y)\}$, *for all* $x, y, z \in X$.

Proof. Assume that $\mu_t(\neq \emptyset)$ is a BCI-positive implicative ideal of X. Then, it follows from Lemma 1.3 that (F10) and (F11) hold.

If there exist $x, y, z \in X$ such that $\max\{\mu(x*z), 0.5\} < \min\{\mu(((x*z)*z)*(y*z)), \mu(y)\} = t$, then $0.5 < t \leq 1$, $\mu(x*z) < t$, and $((x*z)*z)*(y*z), y \in \mu_t$. Since μ_t is a BCI-positive implicative ideal of X, we have $x*z \in \mu_t$. This leads a contradiction. Hence (F13) holds.

Conversely, suppose that the conditions (F10),(F11) and (F13) hold. By Lemma 1.3, we know that μ_t is an ideal of X. Assume that $0.5 < t \leq 1$, $((x*z)*z)*(y*z), y \in \mu_t$. Then $0.5 < t \leq \min\{\mu(((x*z)*z)*(y*z)), \mu(y)\} \leq \max\{\mu(x*z), 0.5\} < \mu(x*z)$. Therefore, μ_t is indeed a BCI-positive implicative ideal of X. ∎

Let μ be a fuzzy set of X and $J = \{t | t \in (0, 1]$ and μ_t is an empty set or a BCI-positive implicative ideal of $X\}$. In particular, if $J = (0, 1]$, then μ is an ordinary fuzzy BCI-positive implicative ideal of X (Theorem 1.1); if $J = (0, 0.5]$, μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X (Theorem 2.1).

In [26], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and also Bhkat and Das's fuzzy subgroup. Based on the results of [26], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy BCI-positive implicative ideal with thresholds in the following way:

Definition 2.2. *Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Then a fuzzy set μ of X is called a fuzzy BCI-positive implicative ideal with thresholds $(\alpha, \beta]$ of X if it satisfies,*

(F14) $\forall x \in X, \max\{\mu(0), \alpha\} \geq \min\{\mu(x), \beta\}$,

(F15) $\forall x, y, z \in X, \max\{\mu(x * z), \alpha\} \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y), \beta\}$.

Example 2.3. Consider the BCI-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	0	0

Define a fuzzy set μ in X by $\mu(0) = \mu(1) = 0.6, \mu(2) = 1, \mu(3) = 0$ and $\mu(4) = 0.2$. It is now routine to verify that μ is a fuzzy BCI-positive implicative ideal with thresholds (0.2, 0.6] of X. But μ is neither a fuzzy BCI-positive implicative ideal nor an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative ideal of X. By routine calculations, we also know that μ is not a fuzzy BCI-positive implicative ideal with thresholds (0.6, 1] of X.

We now characterize the fuzzy BCI-positive implicative ideals with threshold by their level BCI-positive implicative ideals.

Theorem 2.3. *A fuzzy set μ of X is a fuzzy BCI-positive implicative ideal with thresholds (α, β) of X if and only if $\mu_t(\neq \emptyset)$ is a BCI-positive implicative ideal of X for all $\alpha < t \leq \beta$.*

Proof. It is similar to the proof of Theorems 2.1 and 2.2. ∎

Next, we introduce the concepts of $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative (BCI- commutative) ideals in BCI-algebras and investigate some of their properties.

Definition 2.3. *An $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X is called an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X if it satisfies the following additional condition:*

(F16) $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu((((x*y)*y)*(0*y))*z), \mu(z), 0.5\}$, *for all $x, y, z \in X$.*

Example 2.4. Consider the BCI-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Define a fuzzy set μ in X by $\mu(0) = 0.8, \mu(1) = \mu(2) = \mu(3) = 0.3$. It is now routine to verify that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X.

The characterizations of $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideals are given by the following proposition. The proof is obvious and we omit the details.

Proposition 2.2. *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X. Then the following are equivalent:*

(i) *μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal,*

(ii) *$\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\mu(((x * y) * y) * (0 * y)), 0.5\}$, for all $x, y \in X$.*

Using the level BCI-implicative ideals of BCI-algebras, we can characterize the $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideals as follows:

Theorem 2.4. *Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X. Then for all $0 < t \leq 0.5$, μ_t is an empty set or a BCI-implicative ideal of X. Conversely, if μ is a fuzzy set of X such that $\mu_t (\neq \emptyset)$ is a BCI-implicative ideal of X for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X.*

Proof. It is similar to the proof of Theorem 2.1. ▮

Naturally, we can establish a corresponding result when μ_t is a BCI-implicative ideal of X, for all $0.5 < t \leq 1$.

Theorem 2.5. *Let μ be a fuzzy set of X. Then $\mu_t (\neq \emptyset)$ is a BCI-implicative ideal of A for all $0.5 < t \leq 1$ if and only if it satisfies (F10), (F11) and*

(F17) $\max\{\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))), 0.5\} \geq \min\{\mu((((x * y) * y) * (0 * y)) * z), \mu(z)\}$, *for all $x, y, z \in X$.*

Proof. It is similar to the proof of Theorem 2.2. ▮

Definition 2.4. *An $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X is called an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-commutative ideal of X if it satisfies the following additional condition:*

(F18) $\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\mu((x * y) * z), \mu(z), 0.5\}$, for all $x, y, z \in X$.

Example 2.5. Consider the BCI-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define a fuzzy set μ in X by $\mu(0) = 1, \mu(1) = \mu(2) = 0.3, \mu(3) = 0$. It is now routine to verify that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-commutative ideal of X.

Proposition 2.3. *An $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-commutative ideal of X if it satisfies:*

(F19) $\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\mu(x * y), 0.5\}$, for all $x, y \in X$.

Theorem 2.6. *Every $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X is an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-commutative ideal.*

Proof. Let μ be an $(\in, \in \vee \mathrm{q})$-fuzzy BCI-implicative ideal of X. Then, by Proposition 2.2, we have

$$\mu(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq \min\{\mu(((x * y) * y) * (0 * y)), 0.5\}.$$

Since $((x*y)*y)*(0*y) \leq x*y$, we have $\mu(((x*y)*(0*y)*(x*y)) \geq \min\{\mu(x*y), 0.5\}$. Thus, $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu(x*y), 0.5\}$.

By Proposition 2.3, we know that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-commutative ideal of X. ∎

The converse of Theorem 2.6 may not be true. In Example 2.5, we know that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-commutative ideal of X. But μ is not an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X, because

$$\mu(2*((1*(1*2))*(0*(0*(2*1))))) = \mu(1) = 0.3 \not\geq 0.5$$
$$= \min\{\mu(((((2*1)*1)*(0*1))*0), \mu(0), 0.5\}.$$

Theorem 2.7. *Every $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-positive implicative ideal.*

Proof. Assume that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X . Then, by Proposition 2.2, we have $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu(((x*y)*y)*(0*y)), 0.5\}$. Since $((y*(y*x))*(0*(0*(x*y))))*y = (0*(y*x))*(0*(0*(x*y))) \leq (0*(x*y))*(y*x) = 0$, which implies, $(y*(y*x))*(0*(*(x*y))) \leq y$, and so $x*((y*(y*x))*(0*(0*(x*y)))) \geq x*y$. Thus, $\mu(x*y) \geq \min\{\mu(x*((y*(y*x))*(0*(0*(x*y))))), 0.5\}$. Hence, we get $\mu(x*y) \geq \min\{\mu(((x*y)*y)*(0*y)), 0.5\}$. ∎

The converse of Theorem 2.7 may not be true in general. In Example 2.1, we know that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-positive implicative ideal of X. But it is not an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X, because:

$$\mu(1*((2*(2*1))*(0*(0*(1*2))))) = \mu(1) = 0.4 \not\geq 0.5$$
$$= \min\{\mu((((1*2)*2)*(0*2))*0), \mu(0), 0.5\}.$$

Finally, we give the relationship among these generalized fuzzy ideals in BCI-algebras.

Theorem 2.8. *A fuzzy set μ in X is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X if and only if it is both an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-positive implicative ideal and an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-commutative ideal.*

Proof. Necessity: By Theorems 2.6 and 2.7. Sufficiency: Let μ be both an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-positive implicative ideal and an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-commutative ideal of X. By Proposition 2.3, we have $\mu(x*((y*(y*x))*(0*(0*(x*y))))) \geq \min\{\mu(x*y), 0.5\}$. By Proposition 2.2, we have $\mu(x*y) \geq \min\{\mu(((x*y)*y)*(0*y)), 0.5\}$. Hence $\mu(x*((y*(y*x))*(0*(0*(x*y)))) \geq \min\{\mu(((x*y)*y)*(0*y)), 0.5\}$. It follows from Proposition 2.1 that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-implicative ideal of X. ∎

3. Implication-based fuzzy BCI-implicative ideals

Fuzzy logic is an extension of set theoretic variables in terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic can also be defined by using the truth tables. Also, the extension principle can be used to derive definitions of the operators.

In the fuzzy logic, we denote the truth value of fuzzy proposition P by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions:

$[x \in \mu] = \mu(x)$;
$[x \notin \mu] = 1 - \mu(x)$;
$[P \wedge Q] = \min\{[P], [Q]\}$;
$[P \vee Q] = \max\{[P], [Q]\}$;
$[P \to Q] = \min\{1, 1 - [P] + [Q]\}$;
$[\forall x P(x)] = \inf[P(x)]$;
$\models P$ if and only if $[P] = 1$ for all valuations.

Of course, various implication operators can be similarly defined. In this paper, we deal with six implication operators which are given in Table 1.

Table 1. Definition of Implication Operators

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha,\beta) = \max\{1-\alpha, \min\{\alpha,\beta\}\}$
Lukasiewicz	$I_a(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$
Standard Star(Godel)	$I_g(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \le \beta \\ \beta & \text{if} \quad \alpha > \beta \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \le \beta \\ 1-\alpha & \text{if} \quad \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha \le \beta \\ 0 & \text{if} \quad \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha,\beta) = \max\{1-\alpha, \beta\}$

where α is the degree of truth (or degree of membership) of the premise and β is the respective values for the consequence, and I is the resulting degree of truth for the implication. The "quality" of these implication operators could be evaluated either by empirically or by axiomatically methods. In the following definition, we consider the implication operators in the Lukasiewicz system of continuous-valued logic.

Definition 3.1. *A fuzzy set μ of X is called a fuzzifying BCI-implicative ideal of X if it satisfies the following conditions:*
(F20) for any $x \in X, \models [x \in \mu] \to [0 \in \mu]$,
(F21) for any $x, y, z \in X$, $\models [(((x * y) * y) * (0 * y)) * z \in \mu] \wedge [z \in \mu]$
$\to [x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \mu]$.

Clearly, Definition 3.1 is equivalent to Definition 1.4. Therefore a fuzzifying BCI-implicative ideal is an ordinary fuzzy BCI-implicative ideal.

Now, we introduce the concept of t-tautology, i.e.,

$$\models_t P \quad \text{if and only if} \quad [P] \ge t, \quad \text{for all valuations.}$$

Based on the results in [26], we can extend the concept of implication-based fuzzy implicative ideals in the following way:

Definition 3.2. *Let μ be a fuzzy set of X and $t \in (0, 1]$ is a fixed number. Then μ is called a t-implication-based fuzzy BCI-implicative ideal of X if the following conditions hold:*
(F22) for any $x \in X, \models_t [x \in \mu] \to [0 \in \mu]$,

(F23) for any $x, y, z \in X$, $\models_t [(((x * y) * y) * (0 * y)) * z \in \mu] \wedge [z \in \mu]$
$\rightarrow [x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \mu]$.

Now, if I is an implication operator then we have the following corollary:

Corollary 3.1. *A fuzzy set* μ *of* X *is a t-implication-based fuzzy implicative ideal of* X *if and only if it satisfies:*
(F24) for any $x \in X, I(\mu(x), \mu(0)) \geq t$,
(F25) for any $x, y, z \in X$, $I(\mu((((x * y) * y) * (0 * y)) * z)$
$\wedge \mu(z), \mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq t$.

Let μ be a fuzzy set of X. Then we have the following results:

Theorem 3.1.
(i) *Let* $I = I_{gr}$. *Then* μ *is an 0.5-implication-based fuzzy BCI-implicative ideal of* X *if and only if* μ *is a fuzzy BCI-implicative ideal with thresholds* $(r = 0, s = 1)$ *of* X;
(ii) *Let* $I = I_g$. *Then* μ *is an 0.5-implication-based fuzzy BCI-implicative ideal of* X *if and only if* μ *is a fuzzy BCI-implicative ideal with thresholds* $(r = 0, s = 0.5)$ *of* X;
(iii) *Let* $I = I_{cg}$. *Then* μ *is an 0.5-implication-based fuzzy BCI-implicative ideal of* X *if and only if* μ *is a fuzzy BCI-implicative ideal with thresholds* $(r = 0.5, s = 1)$ *of* X.

4. Conclusions

To investigate the structure of an algebraic system, it is clear that ideals with special properties play an important role. In this paper, we considered the notions of $(\in, \in \vee \mathrm{q})$-fuzzy BCI-positive implicative (resp., BCI-implicative, BCI-commutative) ideals in BCI-algebras and investigated the relationship among these generalized fuzzy ideals of BCI-algebras. Finally, we investigated the concept of implication-based fuzzy BCI-implicative ideals in BCI-algebras. It is our hope that this work would serve as a foundation for further study of the theory of BCK/BCI-algebras. In considering the notions of an (α, β)-fuzzy BCI-positive implicative (BCI-implicative and BCI-commutative) ideal in BCI-algebras, we can consider twelve different types of such structures resulting from three choices of α and four choices of β. In this article, we only discussed the $(\in, \in \vee \mathrm{q})$-type. Based on this paper, we will consider the (α, β)-fuzzy BCI-positive implicative (BCI-implicative and BCI-commutative) ideals in BCI-algebras, where α, β is any one of $\in, q, \in \vee \mathrm{q}$ or $\in \wedge \mathrm{q}$. We shall focus on other types and their relationships among them, and also we shall consider primeness and maximality in the (α, β)-fuzzy setting of BCI-algebras.

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