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Representations of Operators on Certain Function Spaces and Operator Matrices

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Abstract. Continuous linear operators from $\ell_1(A,X)$ and $c_0(A,X)$ into $\lambda(A,X)$, $\lambda = \ell_1$, ℓ_∞ or c_0, for a normed space X are investigated. It is shown that such an operator has an operator matrix form whenever A is the set of positive integers.

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1. Introduction

The space of operators between some classical Banach spaces such as c_0, ℓ_∞ and ℓ_p is characterized by unit vector bases of c_0 and ℓ_p. For example, a continuous linear operator T from c_0 into ℓ_∞ is equivalent, by the mapping $T \to \{T(e_n)\}$ where $\{e_n\}$ is the unit vector basis of c_0, to the sequence $\{y_n\} \subset \ell_\infty$ such that $\sum_n y_n$ is weakly unconditionally Cauchy [7, pp.165–169]. A similar study on the function spaces $\ell_1(A,X)$, $c_0(A,X)$ and $\ell_\infty(A,X)$ are presented in this work. These spaces are important generalizations of the classical Banach spaces c_0, ℓ_1 and ℓ_∞, and they have no Schauder bases in general. The problem is solved by a representation of elements of $\ell_1(A,X)$ and $c_0(A,X)$ given in [8].

Geometric and other structural properties of these vector-valued function spaces have been intensively studied in recent years. Some important references on structural properties of the space $\ell_\infty(A,X)$ of all bounded X-valued functions $x : A \to X$ such that $\sup\{\|x(a)\| = \|x_a\| : a \in A\} < \infty$ are [2, 3], where A is assumed to be an infinite set and X is a normed space. It was shown that many elementary properties

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of X such as completeness, separability, barreledness etc. directly effected the structure of the space. For example, there are many examples [2, 3] of normed spaces of $\ell_\infty(A,X)$ which are barreled whenever X is barreled.

Further, Ferrando and Lüdkovsky [4] conducted similar investigations for the function space $c_0(A,X)$, the linear space of all functions $x : A \to X$ such that for each $\varepsilon > 0$ the set $\{a \in A : \|x_a\| > \varepsilon\}$ is finite or empty. It is a normed space as well as $\ell_\infty(A,X)$ with the sup norm $\|x\|_\infty = \sup\{\|x_a\| : a \in A\}$. They also proved that $c_0(A,X)$ is either barreled, ultrabornological or unordered Baire-like if and only if X is, respectively barreled, ultrabornological or unordered Baire-like. An analogous result for the X-valued sequence space $c_0(X) = c_0(\mathbb{N},X)$ was obtained earlier by Mendoza [6]. In [8], for a locally convex space X, we dealt with the structural properties of the spaces $\ell_1(A,X)$ and $c_0(A,X)$ where $\ell_1(A,X)$ is defined in the next section. A basic result in [8] was a lemma introducing a representation for elements of these spaces, similar to those of spaces possesing a basis. Hence, we can easily study the separability of the spaces and linear functionals on them. The representations are given by a family $\{I_a : a \in A\}$ of continuous linear functions from X into $\lambda(A,X)$, $\lambda = \ell_1$or c_0. The family may be unordered and so the representations needs unordered summations. In another important section of the work we prove that this representation of bounded linear operator from $\lambda(A,X)$, $\lambda = \ell_1$ or c_0, into $\lambda(A,X)$, $\lambda = \ell_1, \ell_\infty$ or c_0, shows that the operators have to be in just a usual operator matrix form whenever $A = \mathbb{N}$.

2. Preliminaries

We use the notations $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{R}$ for the sets of all positive integers, complex numbers and real numbers. Further, for a fixed normed space X, we denote by B_X and S_X the closed unit ball and sphere of X. $\mathcal{L}(X,Y)$ denotes the space of all continuous operators from X into another normed space Y, and X' denotes the continuous dual of X.

Let A be a set, $\{x_a : a \in A\}$ be a family of vectors in a topological vector space (TVS, for short) X and let $\mathcal{F}$ denote the family of all finite subsets of A. Then $\mathcal{F}$ is directed by the inclusion relation $\subseteq$ and, for each $F \in \mathcal{F}$, we can form the finite sum $s(F) = \sum_{a\in F} x_a$. If the net $(s(F) : \mathcal{F})$ converges to some x in X, then we say that the family $\{x_a : a \in A\}$ is summable, or that the sum $\sum_{a\in A} x_a$ exists, and we write $\sum_{a\in A} x_a = x$. The definition of summable family does not involve any ordering of the index set A, and we may therefore say that the notion of a sum thus defined is commutative (unconditional). In case $A = \mathbb{N}$, to say that the family $\{x_n : n \in \mathbb{N}\}$ is summable to x is equivalent to saying that the series $\sum_{n\in\mathbb{N}} x_n$ is unconditionally convergent to x. Recall that, a series $\sum_{n\in\mathbb{N}} x_n$ in a TVS is said to be unconditionally convergent if and only if for each permutation σ of $\mathbb{N}$, $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ is convergent. The definition of a convergent series, essentially, involves the order structure of $\mathbb{N}$. If the series $\sum_{n\in\mathbb{N}} x_n$ is convergent, and if σ is a permutation of $\mathbb{N}$, then the series $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ may not be convergent, that is, $\{x_n : n \in \mathbb{N}\}$ may not be summable [1, p. 270].

Lemma 2.1. *Let X be a normed space, A be a set and $\{x_a : a \in A\}$ be a family of vectors in X. Suppose that the net $(s(F) : \mathcal{F})$ converges to some x in X where $s(F) = \sum_{a \in F} x_a$, $F \in \mathcal{F}$. Then $(s(F) : \mathcal{F})$ is bounded.*

Proof. By the hypothesis there exist some $F_0 \in \mathcal{F}$ such that $\|s(F) - x\| \leq 1$ for every $F_0 \subseteq F$. Let $n(F_0)$ be the cardinality of F_0. The number of subsets of F_0 is $2^{n(F_0)} < \infty$ so that

$$\sup_{F \subseteq F_0} \|s(F)\| = M < \infty.$$

Further, for every $F_0 \subseteq F$,

$$\|s(F)\| - \|x\| \leq \|s(F) - x\| \leq 1 \Leftrightarrow \sup_{F_0 \subseteq F} \|s(F)\| \leq 1 + \|x\|.$$

Consequently,

$$\sup_{F \in \mathcal{F}} \|s(F)\| \leq 1 + \|x\| + M.$$ ∎

Note that a convergent net may not be bounded in some cases; e.g., $x \to 0$ for positive real x with reversed usual order: $x \succeq y \Leftrightarrow x \leq y$.

Another important normed linear subspace of X^A other than $\ell_\infty(A, X)$ and $c_0(A, X)$ is $\ell_p(A, X)$, $1 \leq p < \infty$, the set of all functions $x : A \to X$ such that the family $\{\|x_a\|^p : a \in A\}$ is summable in $\mathbb{R}$, i.e.

$$\sum_{a \in A} \|x_a\|^p < \infty,$$

and it is a normed space with the norm

$$\|x\|_p = \left(\sum_{a \in A} \|x_a\|^p \right)^{1/p}.$$

$\ell_p(A, X)$, $\ell_\infty(A, X)$ and $c_0(A, X)$ are Banach spaces iff X is a Banach space. If A is a directed set, then $\lambda(A, X)$ ($\lambda = \ell_\infty$, c_0 or ℓ_p) is a linear space of all nets in X satisfying the corresponding property. They are usually denoted by $\lambda(X)$ in the case $A = \mathbb{N}$ and are called X-valued sequence spaces.

Let us state an important lemma from [8] giving a representation for the elements of the spaces $\lambda(A, X)$ ($\lambda = c_0$ or ℓ_p).

Lemma 2.2. *Let X be a normed space and A be a set. Then each $x \in \lambda(A, X)$ ($\lambda = c_0$ or ℓ_p) is represented by*

$$x = \sum_{a \in A} I_a(x_a)$$

where $I_a : X \to \lambda(A, X)$ defined by $I_a(t) = y$ such that $y(b) = 0$ if $b \neq a$ and $y(b) = t$ if $b = a$.

More precisely, the representation in the lemma means that the net $(\pi_F(x) : \mathcal{F})$ converges to x in the norm topology of X where $\pi_F(x) = \sum_{a \in F} I_a(x_a)$, for $F \in \mathcal{F}$.

It is easy to see that each linear mapping I_a in this Lemma is continuous, and $\|I_a\| = 1$.

For each $a \in A$, the ath projection $P_a : \lambda(A, X) \to X$ is defined to be $P_a(x) = x(a)$. They are continuous since the topology of $\lambda(A, X)$ is stronger than the natural product topology of X^A which is the weakest topology such that the projections $P_a : X^A \to X,\ a \in A$, are continuous. By the representation let us define projections $\pi_F(x)$ from $\lambda(A, X)$ into $\lambda(A, X)$ by

$$\pi_F(x) = \sum_{a \in F} I_a(x_a) = \sum_{a \in F} (I_a \circ P_a)(x)$$

for each $F \in \mathcal{F}$. These are natural projections corresponding the above representations and will perform very important roles in this work.

3. Representations of some operator spaces

Let us give the main result of this section.

Theorem 3.1. *Let X and Y be arbitrary Banach spaces and $\lambda = c_0$ or ℓ_1. Then the operator space $\mathcal{L}(\lambda(A, X), Y)$ is equivalent by the mapping $T \to \{T \circ I_a : a \in A\}$ to the Banach space*

$$E_\lambda = \left\{ \chi = \{\chi_a\} \in \mathcal{L}(X, Y)^A : \sup_{F \in \mathcal{F}} \sup_{x \in B_{\lambda(A,X)}} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\| < \infty \right\}$$

which is normed by

$$\|\chi\| = \sup_{F \in \mathcal{F}} \sup_{x \in B_{\lambda(A,X)}} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\|. \tag{3.1}$$

Proof. Let us first prove that (3.1) defines a norm. Say

$$\pi'_F(x) = \sum_{a \in F} (\chi_a \circ P_a)(x)$$

and let $\|\chi\| = 0$. Then

$$\sup_{x \in B_{\lambda(A,X)}} \|\pi'_F(x)\| = 0$$

for each $F \in \mathcal{F}$ and in particular, for $F = \{a\}\ (a \in A)$. This implies

$$\|(\chi_a \circ P_a)(x)\| = 0$$

for each a. Hence, $\chi_a = 0$ by the definition of P_a and so $\chi = 0$. Other properties of norm can easily be verified.

Now let $T \in \mathcal{L}(\lambda(A, X), Y)$ and say $\chi_a = T \circ I_a$. Then

$$\begin{aligned} Tx = T\left(\sum_{a \in A} (I_a \circ P_a)(x)\right) &= \sum_{a \in A} (T \circ I_a \circ P_a)(x) \\ &= \sum_{a \in A} (\chi_a \circ P_a)(x) = \lim_{F \in \mathcal{F}} \pi'_F(x) \end{aligned}$$

for $x \in \lambda(A, X)$, whence, the net $(\pi'_F(x) : \mathcal{F})$ is convergent so is bounded by Lemma 2.1. Hence

$$\sup_{F \in \mathcal{F}} \|\pi'_F\| = \sup_{F \in \mathcal{F}} \sup_{x \in B_{\lambda(A,X)}} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\| < \infty$$

by the uniform boundedness principle, that is, $\chi = \{\chi_a = T \circ I_a : a \in A\} \in E$. Further,

$$\begin{aligned} \|Tx\| &\leq \left\| \sum_{a \in A} (\chi_a \circ P_a)(x) \right\| \leq \sup_{F \in \mathcal{F}} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\| \\ &= \sup_{F \in \mathcal{F}} \|\pi'_F(x)\| \leq \sup_{F \in \mathcal{F}} \|\pi'_F\| \|x\| = \|\chi\| \|x\| \end{aligned}$$

so this implies $\|T\| \leq \|\chi\|$.

On the other hand, for each $F \in \mathcal{F}$,

$$\begin{aligned} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\| &= \left\| T \left(\sum_{a \in F} (I_a \circ P_a)(x) \right) \right\| \leq \|T\| \left\| \sum_{a \in F} (I_a \circ P_a)(x) \right\| \\ &= \|T\| \|\pi_F(x)\| \leq \|T\| \|\pi_F\| \|x\| = \|T\| \|x\| \end{aligned}$$

hence, $\|\chi\| \leq \|T\|$. This means we can set an isometry Λ from $\mathcal{L}(\lambda(A, X), Y)$ into E_λ. In fact, Λ is an equivalence since it is onto. Let us prove this. Pick some $\chi \in E_\lambda$ and define $Tx = \sum_{a \in A} (\chi_a \circ P_a)(x)$. We must show first that T is well-defined. This, obviously, is equivalent to the summability of the family $\{(\chi_a \circ P_a)(x) : a \in A\}$ to Tx, and so is equivalent to the convergence of the net $(\pi'_F(x) : F \in \mathcal{F})$ to Tx for each $x \in \lambda(A, X)$. For each $G \in \mathcal{F}$

$$\pi'_{G \cup F}(\pi_F(x)) = \pi'_F(\pi_F(x))$$

since $P_a \circ I_a = id_X$, (identity on X) if $a = b$ and $P_a \circ I_b = 0$, (zero operator on X) if $a \neq b$. This implies the net $(\pi'_G(\pi_F(x)) : G \in \mathcal{F})$ converges in Y. This means that for each element z of the dense subspace

$$\{\pi_F(x) : F \in \mathcal{F},\ x \in \lambda(A, X)\}$$

of $\lambda(A, X)$, the net $(\pi'_G(z) : G \in \mathcal{F})$ converges in Y. Since Y is complete, for each $x \in \lambda(A, X)$ we have $(\pi'_F(x) : \mathcal{F})$ is convergent (necessarily to Tx). Hence, T is well-defined and, furthermore, T is continuous because $\|T\| \leq \|\chi\|$. ∎

Theorem 3.1 can also be stated as follows:

Theorem 3.2. *Let X and Y be two Banach spaces and $\lambda = c_0$ or ℓ_1. Then the operator space $\mathcal{L}(\lambda(A, X), Y)$ is equivalent to the Banach space*

$$E_\lambda = \left\{ \chi = \{\chi_a\} \in \mathcal{L}(X, Y)^A : \{(\chi_a \circ P_a)(x) : a \in A\} \text{ is summable, } \forall x \in X \right\}$$

equipped with the norm

$$\|\chi\| = \sup_{F \in \mathcal{F}} \sup_{x \in B_{\lambda(A,X)}} \left\| \sum_{a \in F} (\chi_a \circ P_a)(x) \right\|.$$

Remark 3.1. It is important to realize that

$$\ell_1(A, \mathcal{L}(X, Y)) \subseteq E_{c_0} \tag{3.2}$$

in Theorem 3.1. Indeed, if $\chi \in \ell_1(A, \mathcal{L}(X,Y))$ then

$$\sup_{F\in\mathcal{F}} \sup_{x\in B_{c_0(A,X)}} \left\| \sum_{a\in F} (\chi_a \circ P_a)(x) \right\| \leq \sup_{F\in\mathcal{F}} \sup_{x\in B_{c_0(A,X)}} \sum_{a\in F} \|\chi_a\| \|x_a\|$$
$$= \sup_{F\in\mathcal{F}} \sum_{a\in F} \|\chi_a\| = \sum_{a\in A} \|\chi_a\| < \infty.$$

Example 3.1. The inclusion relation (3.2) may be strict. Indeed; let $X = Y = c_0$ and $A = \mathbb{N}$. Take $\chi = (\chi_n)$ as sequence of infinite matrices such that $\chi_n = \left(a_{ij}^n\right)$: $a_{nn}^n = 1$, $a_{ij}^n = 0$ otherwise. By the characterization of the matrix classes (c_0, c_0), obviously each $\chi_n \in (c_0, c_0) = \mathcal{L}(c_0, c_0)$. Further, for some $x = (x_n) \in c_0(c_0)$

$$\chi_n(x_n) = (0, ..., 0, x_n^n, 0, ...) = x_n^n e_n, \ n \in \mathbb{N},$$

where $x_n = \left(x_n^k\right)_{k=1}^{\infty} \in c_0$. Hence, for each finite $F = (n_1, n_2, \cdots, n_m) \subset \mathbb{N}$,

$$\sum_{n_i \in F} \chi_{n_i}(x_{n_i}) = \sum_{i=1}^{m} x_{n_i}^{n_i} e_{n_i}.$$

This means

$$\sup_{x\in B_{c_0(c_0)}} \left\| \sum_{n_i\in F} \chi_{n_i}(x_{n_i}) \right\| = \sup_{\sup \|x_n\|_\infty \leq 1} \left\| \sum_{i=1}^{m} x_{n_i}^{n_i} e_{n_i} \right\| = 1,$$

hence,

$$\sup_{F\in\mathcal{F}} \sup_{x\in B_{c_0(c_0)}} \left\| \sum_{n\in F} \chi_n(x_n) \right\| = 1.$$

This shows $\chi \in E_{c_0}$. However, $\sum \|\chi_n\| = \infty$, that is, $\chi \notin \ell_1(\mathcal{L}(c_0, c_0))$, because each $\|\chi_n\| = \sup_i \sum\limits_j \left|a_{ij}^n\right| = 1$.

Example 3.2. If $Y = \mathbb{C}$ in the above example then the inclusion relation (3.2) is an equality. Hence,

$$c_0(A, X)' = \ell_1(A, X').$$

This was also shown in [8] but we mention it again for greater understanding of the relation. Let $\chi \in E\left(= c_0(A, X)'\right)$, $F \in \mathcal{F}$ and $\varepsilon > 0$ be arbitrary. Consider a function $\varsigma \in \ell_1(A, \mathbb{C})$. For each $a \in F$, by the definition of $\|\chi_a\|$, we can find a function $\xi : A \to S_X$ such that

$$\|\chi_a\| < |\chi_a(\xi_a)| + \varepsilon |\varsigma_a|. \tag{3.3}$$

Write $\chi_a(\xi_a)$ in the polar form , i.e.,

$$\chi_a(\xi_a) = e^{i\theta_a} |\chi_a(\xi_a)|,$$

and define the function x from A to X by

$$x_a = \begin{cases} e^{-i\theta_a}\xi(a), & \text{if } a \in F \text{ and } \chi_a(\xi_a) \neq 0 \\ 0, & \text{if } a \notin F \text{ or } \chi_a(\xi_a) = 0 \end{cases}.$$

Then obviously $x \in c_0(A, X)$ and $\|x\|_\infty \leq 1$. Therefore, for this x, we have

$$\left|\sum_{a\in A} \chi_a(x_a)\right| = \left|\sum_{a\in F} e^{-i\theta_a}\chi_a(\xi_a)\right| = \sum_{a\in F} |\chi_a(\xi_a)| \leq \|\chi\|$$

This implies, by the inequality (3.3), that

$$\sum_{a\in A} \|\chi_a\| \leq \|\chi\| + \varepsilon \|\varsigma\|_1,$$

hence, $\sum_{a\in A} \|\chi_a\| \leq \|\chi\|$; that is, $\chi \in \ell_1(A, X')$.

We don't know whether this equality exist for an arbitrary finite-dimensional Banach space Y or not.

Corollary 3.1. $E_{\ell_1} = \ell_\infty(A, \mathcal{L}(X, Y))$

Proof. For each $v \in S_X$ and each $a \in A$ consider the functions $v^a = I_a(v)$. Hence for $v^a \in S_{\ell_1(A,X)}$ we have

$$\sum_{b\in F} (\chi_b \circ P_b)(v^a) = \begin{cases} \chi_a(v), & \text{if } a \in F \\ 0, & \text{if } a \notin F \end{cases},$$

and so

$$\sup_{F\in\mathcal{F}} \sup_{x\in B_{\ell_1(A,X)}} \left\|\sum_{a\in F} (\chi_a \circ P_a)(x)\right\| = \sup_{F\in\mathcal{F}} \sup_{v\in S_X} \left\|\sum_{b\in F} (\chi_b \circ P_b)(v^a)\right\|$$
$$= \sup_{a\in A} \sup_{v\in S_X} \|\chi_a(v)\| = \sup_{a\in A} \|\chi_a\|.$$

This implies the equality. ∎

Hence we have the equality $\ell_1(A, X)' = \ell_\infty(A, X')$ of [8] as a corollary of this result with $Y = \mathbb{C}$.

4. The operator matrix representation

Let $A = \mathbb{N}$ and X, Y be arbitrary Banach spaces. In the classical sequence space theory, each bounded linear operator from ℓ_1 or c_0 into another BK-space have to be given an infinite matrix of scalar terms. Is there such a useful representation of every operator from $\ell_1(X)$ or $c_0(X)$ into another Y-valued BK-spaces $\eta(Y)$? We mean by Y-valued BK-spaces a Banach space $\eta(Y)$ of Y-valued sequences $y = (y_n)$ such that all coordinate projections $Q_n : \eta(Y) \to Y$, $Q_n(y) = y_n$, $n = 1, 2, \ldots$ are continuous under the norm topology. The representations presented in Lemma 2.2 helps to give us an affirmative answer to this question. A deep investigation of operator matrices between Banach space-valued sequence spaces was given by Maddox [5]. Now let us record some fundamental notations on operator matrices from this paper.

Let X and Y be Banach space and $A = (A_{nk})_{n,k\in\mathbb{N}}$ an infinite matrix of linear, bounded operators A_{nk} on X into Y. Suppose $\zeta(X)$ and $\eta(Y)$ be nonempty sets of X and Y-valued sequences respectively. Then we define the matrix class $(\zeta(X), \eta(Y))$ by saying that $A \in (\zeta(X), \eta(Y))$ iff, for every $x = (x_k) \in \zeta(X)$

$$A_n(x) = \sum_{k=1}^{\infty} A_{nk} x_k$$

converges in the norm of Y, for each n, and the sequence

$$Ax = \left(\sum_{k=1}^{\infty} A_{nk} x_k \right)_{n=1}^{\infty}$$

belongs to $\eta(Y)$.

Now, we are ready to prove one of our main results.

Theorem 4.1. *Let X and Y be arbitrary Banach spaces and $\eta(Y)$ be a Y-valued BK-space. Then each continuous linear operator T from $\lambda(X)$ ($\lambda = c_0$ or ℓ_1) into $\eta(Y)$ can be given by an infinite operator matrix. More precisely; for each $T \in \mathcal{L}(\lambda(X), \eta(Y))$, there exist an infinite matrix $A = (A_{nk})_{n,k\in\mathbb{N}}$ of continuous linear operators A_{nk} on X into Y such that for each $n \in \mathbb{N}$ and each $x \in \lambda(X)$ the series $\sum_{k=1}^{\infty} A_{nk} x_k$ converges in Y and*

$$Ax = \left(\sum_{k=1}^{\infty} A_{nk} x_k \right)_{n=1}^{\infty} \in \eta(Y).$$

Conversely, if $A = (A_{nk})_{n,k\in\mathbb{N}}$ is any infinite matrix of continuous linear operators A_{nk} on X into Y such that

$$\left(\sum_{k=1}^{\infty} A_{nk} x_k \right)_{n=1}^{\infty} \in \eta(Y), \text{ for each } x \in \lambda(X)$$

then

$$Tx = \left(\sum_{k=1}^{\infty} A_{nk} x_k \right)_{n=1}^{\infty} \tag{4.1}$$

defines an operator $T \in \mathcal{L}(\lambda(X), \eta(Y))$.

Proof. Let $T \in \mathcal{L}(\lambda(X), \eta(Y))$, then since each $x \in \lambda(X)$ has the representation $x = \sum_{k=1}^{\infty} I_k(x_k)$, we can write

$$Tx = \sum_{k=1}^{\infty} (T \circ I_k)(x_k).$$

Say

$$A_{nk} = Q_n \circ T \circ I_k$$

where Q_n's are coordinate projections on $\eta(Y)$ and are continuous since $\eta(Y)$ is a Y-valued BK-space. This means each $A_{nk} \in \mathcal{L}(X, Y)$. Let us write $Tx = y = (y_n)$. We must show that, for each n, the series $\sum_{k=1}^{\infty} A_{nk} x_k$ converges to y_n in Y. For some $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} A_{nk} x_k = \sum_{k=1}^{m} Q_n (T(I_k(x_k))) = Q_n \left(\sum_{k=1}^{m} T(I_k(x_k)) \right).$$

Letting $m \to \infty$, we have $\sum_{k=1}^{m} T(I_k(x_k))$ converges to $Tx = y$ and hence by the continuity of Q_n

$$Q_n\left(\sum_{k=1}^{m} T(I_k(x_k))\right) \to Q_n(y) = y_n.$$

Conversely, suppose $A = (A_{nk})_{n,k\in\mathbb{N}}$ is any infinite operator matrix such that

$$\left(\sum_{k=1}^{\infty} A_{nk}x_k\right)_{n=1}^{\infty} \in \eta(Y)\,,\ \text{for each } x \in \lambda(X)\,.$$

Let us define the operator T by (4.1). It is clear that T is linear. For each n, the operator

$$T_n : \lambda(X) \to Y;\ T_n x = \sum_{k=1}^{\infty} A_{nk}x_k$$

is continuous. This is a consequence of Banach-Steinhaus theorem. Since

$$\pi_m(x) = \sum_{k=1}^{m} I_k(x_k)$$

converges to x in $\lambda(X)$, we have $(T_m \circ \pi_m)(x) \to T_m x$ as $m \to \infty$. Suppose $(T \circ \pi_m)(x) \to y$. We note that

$$Q_m \circ T \circ \pi_m = T_m \circ \pi_m,$$

and so

$$(Q_m \circ T \circ \pi_m)(x) = (T_m \circ \pi_m)(x) \to Q_m(y) = y_m.$$

That is $T_m x = y_m$ for each m, whence $Tx = y$. This means T is continuous by the closed graph theorem. ∎

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