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An Iterative Method for an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

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Abstract. In this paper, we introduce an iterative method for an infinite family of nonexpansive mappings in the framework of Hilbert spaces. Our results improve and extend the corresponding results announced by many others.

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1. Introduction and preliminaries

Throughout this paper, we always assume that H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and T a nonlinear mapping. We use $F(T)$ to denote the fixed point set of T. $D(T)$ and $R(T)$ denote the domain and range of the mapping T. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T).$$

Recall that a mapping f is contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in D(f).$$

An operator A is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in D(A). \tag{1.1}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see [4, 7, 16, 18, 19, 22] and the references therein).

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A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in D}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.2}$$

where D is the fixed point set of a nonexpansive mapping T and b is a given point in H.

In [18], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0\in H$ chosen arbitrarily,

$$x_{n+1}=(I-\alpha_nA)Tx_n+\alpha_nb,\quad \forall n\geq 0, \tag{1.3}$$

converges strongly to the unique solution of the convex minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [7] considered a general iterative scheme by the viscosity approximation method, which first introduced by Moudafi [9],

$$x_0\in H,\quad x_{n+1}=(I-\alpha_nA)Tx_n+\alpha_n\gamma f(x_n),\quad \forall n\geq 0. \tag{1.4}$$

They proved that the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*,x-x^*\rangle\geq 0,\quad \forall x\in D,$$

which is the optimality condition for the convex minimization problem

$$\min_{x\in D}\frac{1}{2}\langle Ax,x\rangle-h(x), \tag{1.5}$$

where D is the fixed point set of a nonexpansive mapping T, h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$.)

Recall that the normal Mann iterative process was introduced by Mann [8] in 1953. The normal Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1\in C,\quad x_{n+1}=(1-\alpha_n)x_n+\alpha_nTx_n,\quad \forall n\geq 1, \tag{1.6}$$

where the sequence $\{\alpha_n\}$ is in $(0,1)$.

If $T:C\to C$ is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ generated by the normal Mann iterative process (1.6) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [12]). In an infinite-dimensional Hilbert space, the normal Mann iteration algorithm has only weak convergence, in general, even for nonexpansive mappings. Therefore, many authors try to modify the normal Mann iteration process to have the strong convergence for nonlinear operators (see [6, 10, 23] and the references therein).

Kim and Xu [6] introduced the following iteration process:

$$\begin{cases} x_0=x\in C \ \text{ arbitrarily chosen},\\ y_n=\beta_nx_n+(1-\beta_n)Tx_n,\\ x_{n+1}=\alpha_nu+(1-\alpha_n)y_n,\quad \forall n\geq 0,\end{cases} \tag{1.7}$$

where T is a nonexpansive mapping of C into itself, $u \in C$ is a given point. They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a fixed point of T provided that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Concerning a family of nonexpansive mappings has been considered by many authors (see [1–4, 11, 15–18, 20, 22, 24] and the references therein). Recently, Shang et al. [15] improved the results of Kim and Xu [6] from a single mapping to a finite family of mappings in the framework of Hilbert spaces.

Now, we consider the mapping W_n defined, as in Shimoji and Takahashi [17], by

$$\begin{aligned} &U_{n,n+1} = I, \\ &U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I, \\ &U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I, \\ &\cdots \\ &U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I, \\ &u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I, \\ &\cdots \\ &U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I, \\ &W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I, \end{aligned} \tag{1.8}$$

where $\gamma_1, \gamma_2, \cdots$ are real numbers such that $0 \le \gamma_n \le 1$, $T_1, T_2, \cdots$ be an infinite family of mappings of H into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n, we have the following lemmas in a real Hilbert space which can be obtained from Shimoji and Takahashi [17].

Lemma 1.1. *Let H be a real Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings from H into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in H$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 1.1, one can define the mapping W from H into itself as follows

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in H.$$

Such a W is called the W-mapping generated by $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$.

Throughout this paper, we will assume that $0 < \gamma_n \le b < 1$ for all $n \ge 1$.

Lemma 1.2. *Let H be a real Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings of H into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

In this paper, motivated by Halpern [5], Kim and Xu [6], Marino and Xu [7], Moudafi [9], Reich [13], Shang *et al.* [16] and Yao *et al.* [23], we introduce the composite iteration scheme as follows:

$$(1.9)\qquad \begin{cases} x_1 = x \in H \quad \text{arbitrarily chosen,} \\ z_n = \lambda_n x_n + (1-\lambda_n)W_n x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A)z_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad \forall n \geq 1, \end{cases}$$

where W_n is defined by (1.8), f is a contraction on H, $\gamma > 0$ is a constant and A is a strongly positive linear bounded self-adjoint operator with the the coefficient $\bar{\gamma} > 0$.

We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$, that $\{x_n\}$ defined by (1.9) converges strongly to a common fixed point of the infinite family nonexpansive mappings, which solve some variation inequality and is also the optimality condition for the convex minimization problem (1.5). Our results improve and extend the corresponding ones announced by many others.

In order to prove our main results, we need the following lemmas.

Lemma 1.3. *In a Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 1.4. [14] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5. [19] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.6. [7] *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with the coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 1.7. [7] *Let H be a Hilbert space. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $T : H \to H$ be a nonexpansive mapping with a fixed point $x_t \in H$ of the contraction $x \mapsto t\gamma f(x) + (1 - tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality:*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - z\rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\bar{x} = \bar{x}$.

2. Main results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let H be a real Hilbert space H and f be a contraction on H with the coefficient $(0 < \alpha < 1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$ and $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from H into itself. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be the composite process generated by (1.9), where $\{W_n\}$ is a sequence defined by (1.8), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ in $[0,1]$. If the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;
(ii) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iv) *there exists a constant $\lambda \in [0,1)$ such that $\lambda_n \leq \lambda$ for all $n \geq 1$,*

then $\{x_n\}$ converges strongly to $q \in F$, which also uniquely solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F. \tag{2.1}$$

Proof. We divide the proof into three parts.
Step 1: First, we observe that $\{x_n\}$ is bounded.
Indeed, take a point $p \in F$ and notice that

$$\|z_n - p\| \leq \lambda_n \|x_n - p\| + (1 - \lambda_n)\|W_n x_n - p\| \leq \|x_n - p\|.$$

From the condition (i), we may assume, with no loss of generality, that $\beta_n < \|A\|^{-1}$ for all $n \geq 1$. From Lemma 1.6, we know that, if $0 < \beta_n \leq \|A\|^{-1}$, then $\|I - \beta_n A\| \leq 1 - \beta_n \bar{\gamma}$. Therefore, we obtain that

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n(\gamma f(z_n) - Ap) + (I - \beta_n A)(z_n - p)\| \\
&\leq \beta_n \|\gamma f(z_n) - Ap\| + \|I - \beta_n A\| \|z_n - p\| \\
&\leq \beta_n \gamma \|f(z_n) - f(p)\| + \beta_n \|\gamma f(p) - Ap\| + (1 - \beta_n \bar{\gamma})\|x_n - p\| \\
&\leq [1 - \beta_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \beta_n \|\gamma f(p) - Ap\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\
&\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)[(1 - \beta_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| + \beta_n \|\gamma f(p) - Ap\|] \\
&= [1 - \beta_n(\bar{\gamma} - \gamma\alpha)(1 - \alpha_n)]\|x_n - p\| + \beta_n(\bar{\gamma} - \gamma\alpha)(1 - \alpha_n)\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}.
\end{aligned}$$

By simple induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|Ap - \gamma f(p)\|}{\bar{\gamma} - \gamma\alpha}\right\},$$

which gives that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

Step 2: In this part, we claim that $\lim_{n\to\infty} \|W x_n - x_n\| = 0$.

In fact, it follows from (1.9) that

$$\begin{aligned} z_{n+1} - z_n = \lambda_{n+1}(x_{n+1} - x_n) + (x_n - W_n x_n)(\lambda_{n+1} - \lambda_n) \\ + (1 - \lambda_{n+1})(W_{n+1}x_{n+1} - W_n x_n). \end{aligned}$$

This implies that

$$\begin{aligned} \|z_{n+1} - z_n\| \leq \lambda_{n+1}\|x_{n+1} - x_n\| + \|x_n - W_n x_n\||\lambda_{n+1} - \lambda_n| \\ + (1 - \lambda_{n+1})\|W_{n+1}x_{n+1} - W_n x_n\|. \end{aligned} \tag{2.2}$$

Since T_i and $U_{n,i}$ are nonexpansive, from (1.8), we have

$$\begin{aligned} \|W_{n+1}x_n - W_n x_n\| &= \|\gamma_1 T_1 U_{n+1,2}x_n - \gamma_1 T_1 U_{n,2}x_n\| \\ &\leq \gamma_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\ &= \gamma_1\|\gamma_2 T_2 U_{u+1,3}x_n - \gamma_2 T_2 U_{n,3}x_n\| \\ &\leq \gamma_1\gamma_2\|U_{u+1,3}x_n - U_{n,3}x_n\| \\ &\leq \cdots \\ &\leq \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\ &\leq M_1\prod_{i=1}^{n}\gamma_i, \end{aligned} \tag{2.3}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 1$. Substituting (2.3) into (2.2), we arrive at

$$\begin{aligned} \|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - W_n x_n\||\lambda_{n+1} - \lambda_n| \\ + (1 - \lambda_{n+1})M_1\prod_{i=1}^{n}\gamma_i. \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} \|y_n - y_{n+1}\| = \|(I - \beta_{n+1}A)(z_{n+1} - z_n) - (\beta_{n+1} - \beta_n)Az_n \\ + \gamma[\beta_{n+1}(f(z_{n+1}) - f(z_n)) + f(z_n)(\beta_{n+1} - \beta_n)]\| \\ \leq \|z_{n+1} - z_n\| + |\beta_{n+1} - \beta_n|M_2, \end{aligned} \tag{2.5}$$

where M_2 is an appropriate constant such that $M_2 \geq \sup_{n\geq 1}\{\|Az_n\| + \gamma\|f(z_n)\|\}$. Substitute (2.4) into (2.5) yields that

$$\begin{aligned} \|y_n - y_{n+1}\| \leq \|x_{n+1} - x_n\| + \|x_n - W_n x_n\||\lambda_{n+1} - \lambda_n| \\ + (1 - \lambda_{n+1})M_1\prod_{i=1}^{n}\gamma_i + |\beta_{n+1} - \beta_n|M_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|y_n - y_{n+1}\| - \|x_{n+1} - x_n\| \leq \|x_n - W_n x_n\||\lambda_{n+1} - \lambda_n| + (1 - \lambda_{n+1})M_1\prod_{i=1}^{n}\gamma_i \\ + |\beta_{n+1} - \beta_n|M_2. \end{aligned}$$

Using the conditions (i), (ii) and noting that $0 < \gamma_i \leq b < 1$ for all $i \geq 1$, we have

$$\limsup_{n\to\infty}\{\|y_n - y_{n+1}\| - \|x_{n+1} - x_n\|\} \leq 0.$$

By virtue of Lemma 1.4, we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.6}$$

Notice that

$$\begin{aligned}\|W_n x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - W_n x_n\| \\ &\le \|x_n - y_n\| + \beta_n \|\gamma f(z_n) - A z_n\| + \lambda_n \|x_n - W_n x_n\|,\end{aligned}$$

which in turn implies that

$$(1-\lambda_n)\|W_n x_n - x_n\| \le \|x_n - y_n\| + \beta_n \|\gamma f(z_n) - A z_n\|. \tag{2.7}$$

It follows from (2.6) and the conditions (i), (iv) that

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{2.8}$$

On the other hand, we have

$$\|W x_n - x_n\| \le \|W x_n - W_n x_n\| + \|W_n x_n - x_n\|,$$

From Remark 3.1 of Yao et al. [24] (see also Remark 2.2 of Ceng and Yao [3]), it follows that, for any $\epsilon > 0$, there exists N such that $\|Wx - W_n x\| \le \epsilon$ for all $x \in \{x_n\}$ and for all $n \ge N$. Therefore, we have $\|W x_n - W_n x_n\| \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \|W x_n - x_n\| = 0. \tag{2.9}$$

Step 3: Finally, we show that $x_n \to q$ as $n \to \infty$.

To this end, first we claim that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \le 0, \tag{2.10}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto t\gamma f(x) + (I - tA) W x.$$

Thus we have

$$\|x_t - x_n\| = \|(I - tA)(W x_t - x_n) + t(\gamma f(x_t) - A x_n)\|.$$

For any $t \le \min\{\|A\|^{-1}, 1\}$, it follows from Lemma 1.3 that

$$\begin{aligned}\|x_t - x_n\|^2 &= \|(I - tA)(W x_t - x_n) + t(\gamma f(x_t) - A x_n)\|^2 \\ &\le (1 - \bar{\gamma} t)^2 \|W x_t - x_n\|^2 + 2t\langle \gamma f(x_t) - A x_n, x_t - x_n\rangle \\ &\le (1 - 2\bar{\gamma} t + (\bar{\gamma} t)^2)\|x_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle \gamma f(x_t) - A x_t, x_t - x_n\rangle + 2t\langle A x_t - A x_n, x_t - x_n\rangle,\end{aligned} \tag{2.11}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - W x_n\|)\|x_n - W x_n\| \to 0 \quad (n \to 0). \tag{2.12}$$

Noticing that A is strongly positive linear mapping and using (1.1), we have

$$\langle A x_t - A x_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \ge \bar{\gamma}\|x_t - x_n\|^2. \tag{2.13}$$

Combining (2.11) with (2.13), we have

$$\begin{aligned}2t\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle &\le (\bar\gamma^2t^2-2\bar\gamma t)\|x_t-x_n\|^2+f_n(t)+2t\langle Ax_t-Ax_n,x_t-x_n\rangle\\ &\le (\bar\gamma t^2-2t)\langle A(x_t-x_n),x_t-x_n\rangle+f_n(t)\\ &\quad+2t\langle Ax_t-Ax_n,x_t-x_n\rangle\\ &\le \bar\gamma t^2\langle A(x_t-x_n),x_t-x_n\rangle+f_n(t).\end{aligned}$$

It follows that

$$\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle \le \frac{\bar\gamma t}{2}\langle Ax_t-Ax_n,x_t-x_n\rangle+\frac{1}{2t}f_n(t). \tag{2.14}$$

Letting $n\to\infty$ in (2.14) and noting that (2.12) yields

$$\limsup_{n\to\infty}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle \le \frac{t}{2}M_3, \tag{2.15}$$

where $M_3>0$ is a constant such that $M_3\ge\bar\gamma\langle Ax_t-Ax_n,x_t-x_n\rangle$ for all $t\in(0,\min\{\|A\|^{-1},1\})$ and $n\ge 1$. Taking $t\to 0$ from (2.15), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t-\gamma f(x_t),x_t-x_n\rangle\le 0. \tag{2.16}$$

On the other hand, we have

$$\begin{aligned}\langle\gamma f(q)-Aq,x_n-q\rangle &= \langle\gamma f(q)-Aq,x_n-q\rangle-\langle\gamma f(q)-Aq,x_n-x_t\rangle\\ &\quad+\langle\gamma f(q)-Aq,x_n-x_t\rangle-\langle\gamma f(q)-Ax_t,x_n-x_t\rangle\\ &\quad+\langle\gamma f(q)-Ax_t,x_n-x_t\rangle-\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\\ &\quad+\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle &\le \|\gamma f(q)-Aq\|\|x_t-q\|+\|A\|\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &\quad+\gamma\alpha\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &\quad+\limsup_{n\to\infty}\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle.\end{aligned}$$

Therefore, from (2.16), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle\\ &\le \limsup_{t\to 0}\|\gamma f(q)-Aq\|\|x_t-q\|\\ &\quad+\limsup_{t\to 0}\|A\|\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &\quad+\limsup_{t\to 0}\gamma\alpha\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &\quad+\limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\\ &\le 0.\end{aligned}$$

Hence (2.10) holds. On the other hand, we have

$$\begin{aligned}\langle\gamma f(q)-Aq,y_n-q\rangle &= \langle\gamma f(q)-Aq,y_n-x_n\rangle+\langle\gamma f(q)-Aq,x_n-q\rangle\\ &\le \alpha_n\|\gamma f(q)-Aq\|\|y_n-x_n\|+\langle\gamma f(q)-Aq,x_n-q\rangle\end{aligned}$$

and so, noticing (2.6),

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, y_n - q\rangle \le 0. \tag{2.17}$$

Now, from Lemma 1.3, we have

$$\begin{aligned}\|y_n - q\|^2 &= \|(I-\beta_n A)(z_n - q) + \beta_n(\gamma f(z_n) - Aq)\|^2\\ &\le \|(I-\beta_n A)(z_n - q)\|^2 + 2\beta_n\langle \gamma f(z_n) - Aq, y_n - q\rangle\\ &\le \|(I-\beta_n A)(z_n - q)\|^2 + 2\beta_n\gamma\alpha\|x_n - q\|\|y_n - q\|\\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle\\ &\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|x_n - q\|^2 + \|y_n - q\|^2)\\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle,\end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned}\|y_n - q\|^2 &\le \frac{(1-\beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle \gamma f(q) - Aq, y_n - q\rangle\\ &\le [1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n - q\|^2 + \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle\\ &\quad + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_4],\end{aligned} \tag{2.19}$$

where M_4 is an appropriate constant such that $M_4 \ge \sup_{n\ge1}\{\|x_n - q\|\}$.

On the other hand, we have

$$\|x_{n+1} - q\|^2 \le \alpha_n\|x_n - q\|^2 + (1-\alpha_n)\|y_n - q\|^2. \tag{2.20}$$

Substituting (2.19) into (2.20) yields that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le [1 - (1-\alpha_n)\frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n - q\|^2\\ &\quad + (1-\alpha_n)\frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_4],\end{aligned} \tag{2.21}$$

Put $j_n = (1-\alpha_n)\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\alpha\gamma}$ and

$$t_n = \frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_4.$$

Then, from (2.21), it follows that

$$\|x_{n+1} - q\|^2 \le (1 - j_n)\|x_n - q\| + j_n t_n. \tag{2.22}$$

It follows from the condition (i) and (2.22) that $\lim_{n\to\infty} j_n = 0, \sum_{n=1}^{\infty} j_n = \infty$ and $\limsup_{n\to\infty} t_n \le 0$. Applying Lemma 1.5 to (2.22), we can obtain $x_n \to q$ as $n \to \infty$. This completes the proof. ∎

If $S_i = I$ (the identity mapping) for each $i \ge 1$, then $W_n = I$ and so the following results can be obtained immediately from Theorem 2.1.

Corollary 2.1. *Let H be a real Hilbert space H and f be a contraction on H with coefficient $(0 < \alpha < 1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$ and T be a nonexpansive mapping from H into itself such that $F(T) \neq \emptyset$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be the composite process generated by the following manner:*

$$\begin{cases} x_1 = x \in H \quad \textit{arbitrarily chosen}, \\ z_n = \lambda_n x_n + (1-\lambda_n) T x_n, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) z_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$. If the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \beta_n = \infty,\ \lim_{n\to\infty} \beta_n = 0$;
(ii) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(iv) *there exists a constant $\lambda \in [0,1)$ such that $\lambda_n \leq \lambda$ for all $n \geq 1$,*

then $\{x_n\}$ converges strongly to $q \in F(T)$, which also uniquely solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F(T).$$

If $\lambda_n = 0$ for each $n \geq 1$, $\gamma = 1$ and $A = I$ (: the identity mapping), then we have the following result immediately from Corollary 2.1.

Corollary 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $(0 < \alpha < 1)$. Let $\{x_n\}$ be the composite process generated by the following manner:*

$$\begin{cases} x_1 = x \in C \quad \textit{arbitrarily chosen}, \\ y_n = \beta_n f(z_n) + (1-\beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$. If the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \beta_n = \infty,\ \lim_{n\to\infty} \beta_n = 0$;
(ii) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

then $\{x_n\}$ converges strongly to $q = P_{F(T)} q$.

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