BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **32***(2) (2009), 187–210*

On Perfect-Like Binary and Non-Binary Linear Codes – A Brief Survey

[1]BAL KISHAN DASS AND [2]PANKAJ KUMAR DAS
[1]Department of Mathematics, University of Delhi,
Delhi–110 007, India
[2]Department of Mathematics, Shivaji College
(University of Delhi), Delhi–110 027, India
[1]dassbk@rediffmail.com, [2]pankaj4thapril@yahoo.co.in

Abstract. This work presents a brief survey on codes (with respect to Hamming metric) which are perfect-like i.e. codes which are capable to correct a given set of error vectors and no others. Several kinds of such codes which exist in the literature have been discussed and possibilities of the existence of such codes in some cases have been explored.

2000 Mathematics Subject Classification: 94B25, 94B60, 94-02

Key words and phrases: Perfect codes, linear codes, existence of perfect-like codes, binary and non-binary linear codes.

1. Introduction

One of the problems in coding theory is the study of perfect codes with respect to Hamming metric. It is known that perfect codes emerge from the well known Rao-Hamming Bound with equality viz.

$$q^{n-k} = \sum_{i=0}^{t} \binom{n}{i} (q-1)^i \tag{1.1}$$

where the parameters have the usual meanings e.g. the code is an (n, k) linear code over $\mathrm{GF}(q)$ and t denotes the number of errors which the code is capable to correct.

It is already known that there are perfect codes for $t = 1$, $t = 2$ and $t = 3$. For $t = 1$, there is an infinite class of single error correcting perfect codes popularly known as Hamming codes [12] whereas for $t = 2$ and $t = 3$, there is one example each of $(11, 6)$ ternary double error correcting and $(23, 12)$ triple error correcting code, both due to Golay [11]. It has also been established that there is no other Hamming metric perfect code except the repetition codes which are trivial (refer Tietäväinen and Perko [17], Tietäväinen [18], and van Lint [19]).

Received: March 25, 2008; *Revised:* August 27, 2008.

Many researchers have studied codes which are near to perfect codes. They have been termed as quasi-perfect codes, optimal codes, close-packed codes etc. There have also been some efforts to look for codes that meet the equality sign in the corresponding bound according to the requirements. Such known codes have been discussed in this paper, e.g. in second section, we deal with Adjacent Error Correcting Binary Perfect Codes; in the third section, we discuss another kind of such codes viz. Anti Perfect Codes. The fourth section discusses $(1,2)$-optimal binary codes. The last section viz. Section 5 deals with several $(1,2)$ non-binary optimal codes.

2. Adjacent error correcting binary perfect codes

In this section, we discuss a class of linear codes in the binary case which are called "adjacent error correcting perfect codes" (see Sharma and Dass [15]). These codes are called perfect in the sense that for a fixed positive integer b, these correct all b adjacent errors and no others. To be precise, we put the definition as follows:

Definition 2.1. *A b-adjacent error is a word with nonzero entries in some b consecutive positions and zeros elsewhere. A linear code is b-adjacent error correcting perfect if it corrects all b-adjacent errors and no others i.e. if every coset of the standard array (except itself) contains exactly one b-adjacent error.*

Now considering the positive integers n (code length) and k (number of information digits) and for a given positive integer b, the total number of n-tuples which have only b consecutive nonzero positions is clearly $n-b+1$. Thus the number of correctable patterns including the one with all zeros becomes $1+(n-b+1)$. Since all these should belong to different cosets and the total number of cosets is 2^{n-k}, a code will be b-adjacent error correcting perfect if it corrects b-adjacent errors and no others i.e. every coset of it (except itself) contains exactly one b-adjacent error i.e. the number of cosets will be equal to the number of b-adjacent error patterns i.e.

$$\begin{aligned} & 2^{n-k} = 1+(n-b+1) \\ (2.1) \qquad \Longrightarrow \quad & 2^{n-k} = n-b+2\,. \end{aligned}$$

We shall consider examples for $b = 1, 2$ and 3 only.

Example 2.1. When $b = 1$, i.e., when we wish to consider the correction of all single errors, (2.1) gives

$$2^{n-k} = n+1 \tag{2.2}$$

which leads to an infinite class of single error correcting perfect codes due to Hamming (1950).

Example 2.2. When $b = 2$, (2.1) reduces to

$$2^{n-k} = n \tag{2.3}$$

which shows that whenever n is a power of 2, we may get 2-adjacent error correcting perfect code. To put more light on it, we discuss two examples.

Consider the parity-check matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix gives rise to a $(4,2)$ linear code, the code being the null space of this matrix. The syndromes of all 2-adjacent error patterns can be seen to be all different (see Table 1).

Table 1. Error Vectors – Syndromes

Error patterns	Syndromes
0 0 0 0	0 0
1 1 0 0	1 1
0 1 1 0	0 1
0 0 1 1	1 0

Therefore the code may be used for double adjacent error correction. Further, it is perfect by Definition 2.1, since each nonzero 2-tuple is a syndrome of some double adjacent error pattern.

From (2.3), the next higher admissible value of n is 8 $(= 2^3)$, for which $k = 5$. Consider the parity-check matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The syndromes of double adjacent error patterns can be seen to be all distinct and exhaustive as in Table 1, we conclude that the $(8,5)$ code under discussion is a 2-adjacent error correcting perfect code.

Example 2.3. We now come to examples when $b = 3$. In this case (2.1) reduces to

$$2^{n-k} = n - 1\,. \tag{2.4}$$

This shows that a 3-adjacent error correcting perfect code may be obtained if n is one more than a power of 2.

A parity-check matrix for the first admissible set of values of n and k satisfying (2.4) gives rise to a $(5,3)$ code so as to be 3-adjacent error correcting perfect may be given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here also all the syndromes of 3-adjacent error patterns are distinct (see Table 2).

Table 2. Error Vectors – Syndromes

Error patterns	Syndromes
0 0 0 0 0	0 0
1 1 1 0 0	1 1
0 1 1 1 0	0 1
0 0 1 1 1	1 0

Another example for the next higher value of n is that of a $(9, 6)$ code whose parity-check matrix may be taken as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

It can be verified that all the syndromes of all 3-adjacent error patterns constitute an exhaustive set of all nonzero 3-tuples. Therefore the code under discussion is 3-adjacent error correcting perfect code.

Remark 2.1. An interesting point to be noted in this example is that if we drop the last column from the parity-check matrix of the $(9, 6)$ code then the resulting matrix coincides with the parity-check matrix of the $(8, 5)$ code which has been considered as the 2-adjacent error correcting perfect code. Also if we delete the last two columns, we obtain the parity-check matrix of a perfect 1-error correcting code.

It may be pointed out that the parity-check matrices considered in the examples have been constructed by the following synthesis procedure according to which nonzero columns are added in the matrices in such a way that the linear combinations i.e. sums (since the case is binary) of any b consecutive columns are always different and nonzero.

In fact, for a fixed positive integer b, after having selected $n-1$ columns suitably of the parity-check matrix where the first $b-1$ columns may be chosen arbitrarily, n^{th} column may be chosen in such a way that it is not a sum of the immediately preceding $b-1$ columns together with any b consecutive columns among all the $n-1$ columns. This gives rise to a total of $n-b+1$ possible sums. In the worst case when these sums are distinct and nonzero, the n^{th} column can be added if these $(n-b+1)$ possible sums do not exhaust all the $2^{n-k}-1$ nonzero $(n-k)$-tuples, i.e. if

$$\begin{aligned} & 2^{n-k} - 1 > n - b + 1 \\ \Longrightarrow \qquad & 2^{n-k} > n - b + 2\,. \end{aligned}$$

However, the necessary bound for such a code (2.1) suggests that

$$2^{n-k} = n - b + 2\,.$$

Therefore, if we could construct a matrix in which all possible sums are not distinct or if any two possible sums coincide or even if any one sum becomes zero vector, we shall be able to form a desired code. This can be done by a suitable choice of columns and is not difficult.

Thus, for a fixed positive integer b, such codes which correct all b-adjacent errors and no other satisfying (2.1) can always be constructed. Therefore, for a fixed b, these codes form an infinite class. If we vary b, these form a doubly infinite class.

Observation 2.1. It may be noted that the parity-check matrices given in the examples discussed are not unique in general. For example, another parity check matrix for 2-adjacent error correcting $(8,5)$ perfect code may be given as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Open Question 2.1. We suspect that it should be possible to evolve a procedure for constructing these codes systematically. Further, it is not known if b-adjacent error correcting perfect codes exist for values of b other than 1 in the non-binary case.

3. Anti-perfect codes

The study of burst error correcting codes has been made with a view to correct errors amongst a specific number of consecutive positions. There are many situations where errors occurs, of course, in the form of a burst but not all the digits inside the burst get disturbed. Such a situation can be studied by imposing a suitable weight constraint over the detectable/correctable bursts. An attempt in this direction to find codes capable of detecting and correcting low-density bursts would find references in papers by Sharma and Dass [14], and Dass [4, 5].

In this section, we deal with a related problem, i.e., the problem of correcting bursts of length b or less that are of weight w or more ($w \leq b$). Such errors may be called *high-density bursts*. In the following, a sufficient condition for the existence of linear codes which are capable of correcting such errors has been given.

As a special case, by taking $b = n$, we have discussed linear codes which for some t correct all t and more errors and no others. Such codes have been termed as 'Anti perfect codes', as the character of such codes is directly opposite to that of perfect codes. Further, we have considered a special type of binary linear codes which correct all $n-1$ and n errors and no others. A relation between these codes and the well known Hamming codes has also been discussed (see Sharma, Dass and Gupta [16]).

3.1. A sufficient condition

We start by stating a lemma which is easy to prove:

Lemma 3.1. *If $\mathcal{B}(n,b,w)$ denotes the number of bursts of length b or less with weight w or more ($w \leq b$) over the space of all n-tuples over* GF(q)*, then*

$$\mathcal{B}(n,b,w) = \mu + J(n,b,w) \tag{3.1}$$

where

$$J(n,b,w) = \sum_{i=c}^{b} \left[(n-i+1) \sum_{j=w-2}^{b-2} \binom{i-2}{j} (q-1)^{j+2} \right],$$

$$c = \max\{2, w\}$$

and

$$\mu = \begin{cases} 1 + n(q-1)\,, & \textit{if } \; w = 0 \\ n(q-1)\,, & \textit{if } \; w = 1 \\ 0\,, & \textit{if } \; w \geq 2\,. \end{cases}$$

The following theorem is any easy application of the above lemma.

Theorem 3.1. *For an (n, k) linear code that corrects all bursts of length b or less with weight w or more $(w \leq b)$, we must have*

$$q^{n-k} \geq p + \mathcal{B}(n, b, w) \tag{3.2}$$

where

$$p = \begin{cases} 0\,, & \textit{if } \; w = 0 \\ 1\,, & \textit{if } \; w > 0 \end{cases}$$

and $\mathcal{B}(n, b, w)$ is given by Lemma 3.1.

In the next theorem, we give a sufficient condition for the existence of a high-density-burst correcting linear code.

Theorem 3.2. *Given non negative integers w and b $(w \leq b)$; a sufficient condition for the existence of an (n, k) linear code capable of correcting all single bursts of length b or less that are of weight w or more, is*

$$q^{n-k} > [M(q) + N(q) + B(n-b, b^*, \bar{w})\mathcal{B}(b-1, b-1, w-1)] \tag{3.3}$$

where $\bar{w} = \max\{1, w\}$, $b^* = \max\{n-b, b\}$, $\mathcal{B}(n, b, w)$ is given by Lemma 3.1 *and $M(q)$ and $N(q)$ are given as follows:*

$$N(q) = \sum_{i=0}^{p} \binom{b-1}{i} + \sum_{i=\alpha}^{b-1} \binom{b-1}{i}, \qquad \textit{if } \; q = 2$$

where p is the largest odd integer satisfying

$$\left[\frac{p}{2}\right] \leq b - w - 1\,, \qquad \alpha = \max\{p+1, w-1\}$$

and $[x]$ denoting the integral part of x. Also

$$N(q) = q^{b-1} \quad \textit{if} \quad q \geq 3\,.$$

Also if $q \geq 3$, then

$$M(q) = \sum_{k=0}^{b-2} \left[\sum_{r_1=w-1}^{b-k-2} \binom{b-k-2}{r_1} (q-1)^{r_1+1} \right] \times \left[\sum_{\substack{r_2, r_3 \\ r_2 + r_3 \leq w-2}} \binom{b-k-2}{r_3} \binom{k+1}{r_2} (q-1)^{r_2+r_3} \right]$$

$$+\sum_{k=0}^{b-2}\left[\sum_{r_1=w-1-(k+1)}^{w-2}\binom{b-k-2}{r_1}(q-1)^{r_1+1}\right]$$

$$\times\left[\sum_{r_2=0}^{k+1}\binom{k+1}{r_2}(q-1)^{r_2}\right]$$

$$\times\left[\sum_{r_3=w-1-(k+1)}^{b-k-2}\binom{b-k-2}{r_3}(q-1)^{r_3}\right]$$

and if $q=2$, then the r_i satisfy the additional constraints:

if $w-1-(k+1)\le r_1\le w-2$, then only those values of r_2 and r_3 are permissible for which there exists a positive integer p $(\le r_2)$ such that

$$r_1+(k+1-r_2)+p\ge w-1$$

and

$$r_3+(k+1-r_2)+(r_2-p)\ge w-1\,.$$

Remark 3.1. This result turns out to be a generalization of the result obtained by Camponiano given in Theorem 4.10 of Peterson and Weldon [13].

3.2. Anti-perfect codes

Whenever one obtains a bound, it is desirable to check the suitable values of the parameter for which the bound is tight. In the binary case, for $t=n-1$, we show that the bound obtained in Theorem 3.2 may be used to derive a class of anti-perfect codes. As proposed earlier, by an anti-perfect code we mean a code which for some t corrects all t and more errors and no others.

An expression determining n, the code length, and k, the number of information digits, may be obtained for such codes over $\mathrm{GF}(q)$ by putting $w=n-1$ and $b=n$ in (3.2) in the case of equality.

Therefore, an anti perfect code for $t=n-1$ will exist if it corrects all $n-1$ and n errors and satisfies the equation

$$2^{n-k}=n+2 \tag{3.4}$$

i.e. whenever n is 2 less than a power of 2, we may get $(n-1)$ and n error correcting anti-perfect codes.

To illustrate the idea fully, we discuss two examples. Consider a $(6,3)$ code whose parity-check matrix is given by

$$H=\begin{bmatrix}0&0&0&1&1&1\\0&1&1&0&0&1\\1&0&1&0&1&0\end{bmatrix}.$$

The table depicting Error patterns and their corresponding Syndromes is as follows:

Table 3. Error Vectors – Syndromes

Error patterns	Syndromes
0 1 1 1 1 1	1 1 0
1 0 1 1 1 1	1 0 1
1 1 0 1 1 1	1 0 0
1 1 1 0 1 1	0 1 1
1 1 1 1 0 1	0 1 0
1 1 1 1 1 0	0 0 1
1 1 1 1 1 1	1 1 1

It is seen that the syndromes of all the error patterns of weight 5 and 6 are distinct. Therefore, this code may be used for correcting these errors. Further, all the 3-tuples have been accounted for as syndromes, so it gives an anti-perfect code for $t = 5$.

The next higher admissible value of (n, k) satisfying (3.4) is $(14, 10)$. Consider the parity-check matrix H given by

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This gives rise to a $(14, 10)$ linear code. Also the syndromes of all patterns of weight 13 and 14 being different and exhaustive can be verified, so we conclude that this code is also anti-perfect for $t = 13$.

In fact for every set of values of n and k satisfying (3.4) such a code exists. This can be established with the help of the following relation between these codes and the Hamming's single error-correcting binary perfect codes.

3.3. Hamming codes and anti-perfect codes

It can be proved with a little effort that if we drop any column from the parity-check matrix of a Hamming code, the resultant matrix gives rise to an anti-perfect code with same number of information digits but with length one less than that of the corresponding Hamming code.

The existence of Hamming codes for every set of values of n and k satisfying

$$2^{n-k} = n + 1 \tag{3.5}$$

thus ensures the existence of anti-perfect codes for all values of n and k satisfying

$$2^{n-k} = n + 2\,.$$

3.4. A decoding process

It is interesting to note that if we choose the parity check matrix H of an anti-perfect code in such a way that its columns are the binary representations of their column numbers, then the syndrome of an error pattern of weight $(n - 1)$ is same as that of an error pattern of weight one obtained from it by interchanging 0 and 1 and reversing its order i.e. changing $(a_1, a_2, \ldots, a_n)$ to

$$(1-a_n, 1-a_{n-1}, \ldots, 1-a_2, 1-a_1)\,.$$

By choosing the columns of the parity-check matrix H in the above-mentioned way, the decoding process of anti-perfect codes becomes very simple. If the syndrome of a received pattern coincides with the ith column of H, this means that all the digits in the received pattern except the $(n-i)$th digit, are in error. If the syndrome does not coincide with any of the columns of H, this means that all the digits are in error.

3.5. An application

In a binary channel, suitable for tolerating single errors, if all the arrangements get reversed and their restructuring becomes either cumbersome or expensive, then the codes discussed above may be employed.

3.6. Open problem

It may be pointed out that the existence of such anti-perfect codes for other values of t and q is not yet known.

4. Binary $(1,2)$-optimal codes

Dass and Tyagi [9] obtained a lower bound over the necessary number of parity-check digits for an $(n = n_1+n_2, k)$ linear code that corrects bursts of length b_1(fixed) in the first block of length n_1 and bursts of length b_2(fixed) in the second block of length n_2. A *burst of length b(fixed)* has been considered as an n-tuple whose only nonzero components are confined to b consecutive positions, the first of which is nonzero and the number of its starting positions is the first $(n-b+1)$ components. This definition is a modification due to Dass [6] over the definition of a burst considered by Chien and Tang [3] and has been found useful in error analysis experiments on telephone lines (Alexander *et al.* [1]) and in some space channel models in which an amplitude modulated carrier is generated aboard a satellite and transmitted to an earth antenna. The result proved by Dass and Tyagi [9] is as follows:

Theorem 4.1. *The number of parity-check digits in an (n,k) linear code correcting all bursts of length $b_1(fixed)$ in the first block of length n_1 and all bursts of length $b_2(fixed)$ in the second block of length n_2 $(n = n_1+n_2)$ is at least*

$$\log_q\left[1+\{(n_1-b_1+1)q^{b_1-1}+(n_2-b_2+1)q^{b_2-1}\}(q-1)\right]\,. \tag{4.1}$$

The purpose of this section is to show that if we fix up $b_1 = 1$ and $b_2 = 2$, we can obtain optimal codes. The codes are optimal in the sense that these can be used to correct all single errors in the first block of length n_1 and all bursts of length 2(fixed) in the second block of length n_2 and no more. It is shown that for $n_1 + n_2 \leq 50$, such codes exist for all possible values of the parameters (refer Dass and Tyagi [10]). Such codes have been termed as $(1,2)$-optimal codes.

4.1. Optimal codes

Consider the inequality in (4.1) for the binary case with equality for $b_1 = 1$ and $b_2 = 2$. We get

$$2^{n_1+n_2-k} = 2n_2 + n_1 - 1\,. \tag{4.2}$$

We now examine the possibilities of the existence of codes for the values of n_1, n_2 and k satisfying (4.2) such that $n_1 + n_2 \le 50$. For this, we first note that the values of n_1 satisfying (4.2) should always be odd. We shall find out all possible values of n_2 and k by assigning values to n_1 as 1, 3, 5, successively.

Let $n_1 = 1$. The values of n_2 and k satisfying (4.2) such that $n_1 + n_2 \le 50$ are $(n_2, k) = (2,1), (4,2), (8,5), (16,12)$ and $(32,27)$ which shows the possibility of the existence of $(1+2,1)$, $(1+4,2)$, $(1+8,5)$, $(1+16,12)$, $(1+32,27)$ codes which may be $(1,2)$-optimal. Consider the matrices in the following examples :

Example 4.1. $\begin{bmatrix} 1 & 0\,1 \\ 0 & 1\,0 \end{bmatrix}$

Example 4.2. $\begin{bmatrix} 0 & 0\,0\,1\,0 \\ 1 & 0\,1\,0\,1 \\ 1 & 1\,0\,0\,1 \end{bmatrix}$

Example 4.3. $\begin{bmatrix} 1 & 0\,0\,0\,1\,0\,1\,0\,1 \\ 1 & 0\,0\,1\,0\,1\,1\,1\,1 \\ 0 & 0\,1\,0\,0\,1\,1\,0\,1 \\ 1 & 1\,0\,0\,0\,1\,0\,1\,1 \end{bmatrix}$

Example 4.4. $\begin{bmatrix} 0 & 1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,1\,0\,1 \\ 0 & 0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,0\,1\,0\,0 \\ 0 & 0\,0\,1\,0\,0\,1\,1\,1\,1\,0\,1\,0\,1\,0\,0\,1 \\ 1 & 0\,1\,0\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,1\,0\,0 \\ 1 & 1\,0\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,1\,0 \end{bmatrix}$

Example 4.5. $\begin{bmatrix} 1 & 1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,1\,0\,1 \\ 1 & 0\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,1\,0\,0\,1\,0\,1\,0\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,0\,0\,0 \\ 1 & 1\,0\,0\,0\,0\,1\,0\,0\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1\,0\,1\,0\,0\,0\,0\,1\,0\,1\,0\,0\,1 \\ 1 & 0\,0\,0\,0\,1\,0\,0\,0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,1\,1\,1\,1\,0 \\ 0 & 0\,0\,1\,1\,0\,0\,0\,0\,1\,1\,0\,0\,0\,1\,0\,1\,1\,0\,0\,1\,0\,0\,1\,0\,1\,1\,1\,0\,1\,0\,1\,0 \\ 0 & 0\,1\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1 \end{bmatrix}$

These matrices [Examples (4.1) to (4.5)] considered as parity-check matrices for a code give rise to $(1+2,1)$, $(1+4,2)$, $(1+8,5)$, $(1+16,12)$ and $(1+32,27)$ codes. Moreover, these codes can correct all bursts of length 1 in the first block of length 1 and all bursts of length 2(fixed) in the second block of lengths 2, 4, 8, 16 and 32 respectively and no other error pattern. For the sake of illustration, we list in Table 4 the error patterns and their syndromes for the $(1+8,5)$ code which is the null space of the matrix given in example (4.3). The syndromes being altogether different mean that the code under discussion is an $(1,2)$-optimal code.

Table 4. Error Vectors – Syndromes

Error patterns	Syndromes
100000000	1101
011000000	0011
010000000	0001
001100000	0110
001000000	0010
000110000	1100
000100000	0100
000011000	1111
000010000	1000
000001100	1001
000001000	0111
000000110	1011
000000100	1110
000000011	1010
000000010	0101

Let $n_1 = 3$. We see that the values of n_2 and k such that $n_1 + n_2 \leq 50$ and satisfying (4.2) may give rise to $(3+3, 3)$, $(3+7, 6)$, $(3+15, 13)$ and $(3+31, 28)$ codes. We give below the parity-check matrix for the $(3+31, 28)$ code as Example 4.6 and the matrices for the remaining three cases can be written down by a procedure to be given later.

Example 4.6.
$$\begin{bmatrix} 110 & 1001000100010000010010101010110 \\ 100 & 0001001000110010100111010101100 \\ 110 & 1000010001110101011101000010100 \\ 111 & 0000100011101010111010001001111 \\ 011 & 0011000011000101100100101110101 \\ 010 & 0110000010001001001001010101010 \end{bmatrix}$$

As before, one can easily verify that the codes which are null spaces of these matrices correct all bursts of length 1 in the first block of length 3 and all bursts of length 2(fixed) in the second block of length 3, 7, 15 and 31 respectively and no other error pattern.

Again for $\boldsymbol{n_1 = 5}$, the corresponding values of n_2 and k are given in Table 5.

Table 5

n_2	k
2	4
6	7
14	14
30	29

We give below the parity-check matrix for the $(5+30, 29)$ code. The matrices for the remaining three cases can be constructed by the procedure to be given later.

Example 4.7. For $n_1 = 5$, $n_2 = 30$ and $k = 29$, the requisite code is $(5+30, 29)$ whose parity-check matrix may be considered as

$$\begin{bmatrix} 11011 & 100100010001000001001010101011 \\ 10000 & 000100100011001010011101010110 \\ 11000 & 100001000111010101110100001010 \\ 11101 & 000010001110101011101000100111 \\ 01110 & 001100001100010110010010111010 \\ 01011 & 011000001000100100100101010101 \end{bmatrix}$$

and as pointed out earlier, one can easily verify that the codes which are null spaces of these matrices correct all bursts of length 1 in the first block and all bursts of length 2(fixed) in the second block.

We now give in Table 6 all possible suitable values of n_2 and k for fixed value of n_1.

The parity-check matrices of $(7+29, 30)$, $(9+28, 31)$, $(11+27, 32)$, $(13+26, 33)$, $(15+25, 34)$, $(17+24, 35)$, $(19+23, 36)$, $(21+22, 37)$ and $(23+21, 38)$ codes are given below as these do not follow the procedure to be given later.

For $(7+29, 30)$ code, the parity-check matrix is

$$\begin{bmatrix} 1001011 & 00100010001000100100101010101 \\ 0010101 & 00100100011001001001110101011 \\ 1011101 & 00001000111010110111010000101 \\ 0111011 & 00010001110101001110100010011 \\ 0111101 & 01100001100010011001001011101 \\ 1011110 & 11000001000100010010010101010 \end{bmatrix}.$$

For $(9+28, 31)$ code, the parity-check matrix is

$$\begin{bmatrix} 100101110 & 0010001000100010010010101010 \\ 001010101 & 0010010001100100100111010101 \\ 101110110 & 0000100011101011011101000010 \\ 011101101 & 0001000111010100111010001001 \\ 011110110 & 0110000110001001100100101110 \\ 101111011 & 1100000100010001001001010101 \end{bmatrix}.$$

For $(11+27, 32)$ code, the parity-check matrix is

$$\begin{bmatrix} 10010111011 & 001000100010001001001010101 \\ 00101010101 & 001001000110010010011101010 \\ 10111011011 & 000010001110101101110100001 \\ 01110110101 & 000100011101010011101000100 \\ 01111011011 & 011000011000100110010010111 \\ 10111101101 & 110000010001000100100101010 \end{bmatrix}.$$

For $(13+26,33)$ code, the parity-check matrix is

$$\begin{bmatrix} 0111111110011 & 00000100010001001001010101 \\ 1000111101110 & 00001000110010010011101010 \\ 1011000111011 & 00010001110101101110100001 \\ 0101001101110 & 00100011101010011101000100 \\ 1001011111011 & 01000011000100110010010111 \\ 1111100011110 & 10000010001000100100101010 \end{bmatrix}.$$

Table 6

n_1	n_2	k
7	5	8
	13	15
	29	30
9	4	9
	12	16
	28	31
11	3	10
	11	17
	27	32
13	2	11
	10	18
	26	33
15	9	19
	25	34
17	8	20
	24	35
19	7	21
	23	36
21	6	22
	22	37
23	5	23
	21	38
25	4	24
	20	39
27	3	25
	19	40
29	2	26
	18	41
31	17	42
33	16	43
35	15	44

For $(15+25, 34)$ code, the parity-check matrix is

$$\begin{bmatrix} 011111011101101 & 0000010001000100100101010 \\ 100011111011011 & 0000100011001001001110101 \\ 101100001111101 & 0001000111010110111010000 \\ 010100111011000 & 0010001110101001000100010 \\ 100101011111110 & 0100001100010011001001011 \\ 111110100111011 & 1000001000100010010010101 \end{bmatrix}.$$

For $(17+24, 35)$ code, the parity-check matrix is

$$\begin{bmatrix} 01111111101100111 & 000001000100010010010101 \\ 10001111011011110 & 000010001100100100111010 \\ 10110001111100100 & 000100011101011011101000 \\ 01010011011010011 & 001000111010100111010001 \\ 10010111111101001 & 010000110001001100100101 \\ 11111000111011110 & 100000100010001001001010 \end{bmatrix}.$$

For $(19+23, 36)$ code, the parity-check matrix is

$$\begin{bmatrix} 0111111110011101101 & 00000100010001001001010 \\ 1000111101110111011 1 & 00001000110010010011101 \\ 1011000111011100000 1 & 00010001110101101110100 \\ 0101001101110001110 1 & 00100011101010011101000 \\ 1001011111011010110 1 & 01000011000100110010010 \\ 1111100011110111011 1 & 10000010001000100100101 \end{bmatrix}.$$

For $(21+22, 37)$ code, the parity-check matrix is

$$\begin{bmatrix} 011111111001110111011 & 0000010001000100100101 \\ 100011110111011101101 & 0000100011001001001110 \\ 101100011101110000000 & 0001000111010110111010 \\ 010100110111000111000 & 0010001110101001110100 \\ 100101111101101011011 & 0100001100010011001001 \\ 111110001111011101111 & 1000001000100010010010 \end{bmatrix}.$$

For $(23+21, 38)$ code, the parity-check matrix is

$$\begin{bmatrix} 01111111100111011101111 & 000001000100010010010 \\ 10011111011101110110111 & 000010001100100100111 \\ 10110001110111000000011 & 000100011101011011101 \\ 01010011011100011100000 & 001000111010100111010 \\ 10010111110110101101110 & 010000110001001100100 \\ 11111000111101110110111 & 100000100010001001001 \end{bmatrix}.$$

4.2. Construction of $(1, 2)$-optimal codes

In the previous section, we have considered codes for all possible values of n_1, n_2 and k for which $n_1 + n_2 \leq 50$ and we have seen that these codes correct all bursts of length 1 in the first block of length n_1 and all bursts of length 2(fixed) in the second block of length n_2 and no more and therefore, are called $(1,2)$-optimal (n_1+n_2,k) linear codes.

For the construction of the parity-check matrices of these codes, it is sufficient to construct the 2nd block because of the following reason:

Suppose the second block of length n_2 that corrects all bursts of length 2(fixed) is constructed. This means that the syndromes of the first n_2-1 single error patterns and all double adjacent error pattern are different. (The last and the first component is not taken as an adjacent error pattern). If the number of all such syndromes is deleted from the total nonzero $(n-k)$-tuples, we remain with exactly n_1 $(n-k)$-tuples. Since we are correcting only single errors in the first block, the remaining $(n-k)$-tuples can be taken as the columns of the first block irrespective of their order. The construction of the second block is as follows:

Select any nonzero $(n-k)$-tuple as the first column of the parity-check matrix H. Subsequent columns are added to H such that after having selected n_2-1 columns $h_1, h_2, \ldots, h_{n_2}-1$, a column h_{n_2} is added provided that

$$h_{n_2} \neq u_{n_2-1}h_{n_2-1} + v_i h_i + v_{i+1}h_{i+1}$$

where either both v_i and v_{i+1} are zero or if v_s $(s=i$ or $i+1)$ is the last nonzero coefficient, then $2 \leq s \leq n_2-2$, and $u_{n_2-1} \in \mathrm{GF}(2)$.

We now propose a technique of constructing parity-check matrices for such codes in some special cases: If we choose the first n_2-1 columns of the second block of the parity-check matrix H as the binary representation of the numbers $1,2,4,8,16,32,\ldots$; $7,14,28,\ldots$; $5,10,20,\ldots$; $11,22,44,\ldots$; $13,26,52,\ldots$, successively (whenever the lesser columns are required, we shall stop there at; then the last column can be chosen such that its linear combination with (n_2-1)-th column is different from syndromes of all single and double error patterns of the preceding columns. It is by this technique that we can write down the parity-check matrices of the codes for the cases other than those whose parity-check matrices are given in the previous section. In fact we notice that this technique is applicable when

(i) the number of parity checks is 3,
(ii) the number of parity checks is 4,
(iii) the number of parity checks is 5 except in the examples 4.4 and 4.5,
(iv) the number of parity checks is 6 except when $n_1 < n_2$.

We now propose some open problems:

4.3. Open problems

Open Problem 4.3.1. The existence of such codes has been shown only when the code length does not exceed 50. We conjecture that such codes exists for all possible values of the parameters satisfying (4.2).

Open Problem 4.3.2. Can there be a systematic way of constructing these codes for which the above mentioned rule is not applicable?

Remark 4.1. It may be remarked that if we interchange the two blocks i.e. if we consider (n_2+n_1,k) linear codes, these will form a class of $(2,1)$-optimal codes.

5. Non-binary $(1,2)$-optimal codes

5.1. Ternary $(1,2)$-optimal linear codes

In this section, we explore the possibility of the existence of $(1,2)$-optimal codes in the ternary case i.e. over GF(3) (see Buccimazza, Dass and Jain [2]).

Let us take $q = 3$ and $b_1 = 1$, $b_2 = 2$ in (4.1). We get

$$3^{n_1+n_2-k} = 2n_1 + 6n_2 - 5\,. \tag{5.1}$$

We now examine the values of n_1, n_2 and k satisfying (5.1). For this, we shall assign values to n_1 as $1, 2, 3, \ldots$ successively and find out the corresponding values of n_2 and k.

(i) Let $n_1 = 1$

Then (5.1) reduces to

$$3^{n_2-k} = 2n_2 - 1 \tag{5.2}$$

The various values of n_2 and k satisfying (5.2) with $n_2 \geq 2$ (because $b_2 = 2$) are

$$\begin{aligned}(n_2, k) \quad = \quad &\{(2,1), (5,3), (14,11), (41,37), (122,177),\\ &(365,359), \ldots \text{ and so on}\}\,.\end{aligned}$$

These values show the possibility of the existence of $(1+2,1)$, $(1+5,3)$, $(1+14,11)$, $(1+41,37)$, $(1+122,117)$, $(1+365,359)\ldots$ ternary $(1,2)$-optimal codes.

In the following, we show that corresponding to the first set of values, such a code exists i.e. $(1+2,1)$ ternary $(1,2)$-optimal code exists.

Example 5.1. Consider the following parity-check matrix for $(1+2,1)$ ternary code:

$$H = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}.$$

The code which is the null space of H can correct all bursts of length 1 in the first sub block of length 1 and all bursts of length 2(fix) in the second block of length 2 and no others. We list below all the error vectors and their corresponding syndromes which can be seen to be distinct altogether and exhaustive.

Table 7

Error vectors	Syndromes
First sub block	
1 0 0	1 0
2 0 0	2 0
Second sub block	
0 1 0	0 2
0 1 1	2 2
0 1 2	1 2
0 2 0	0 1
0 2 1	2 1
0 2 2	1 1

(ii) Let $n_1 = 2$.

Then (5.1) reduces to

$$3^{n_2-k+2} = 6n_2 - 1\,.$$

It is clear that the right hand side of above equation is never a power of 3. Therefore, this equation has no integer solution for n_2 and k and hence such a $(1,2)$-optimal code cannot exist in this case.

(iii) Let $n_1 = 3$.

Then (5.1) reduces to

$$3^{n_2-k+3} = 6n_2 + 1\,.$$

Once again, it is evident that this equation has no integer solution for n_2 and k and thus such a $(1,2)$-optimal code can not exist in this case as well.

(iv) Let $n_1 = 4$.

Then (5.1) reduces to

$$3^{n_2-k+3} = 2n_2 + 1\,. \tag{5.3}$$

The following are the values of n_2 and k satisfying (5.3) with $n_2 \geq 2$:

$$\begin{aligned}(n_2,k) \quad &= \quad \{(4,5),(13,13),(40,39),(121,119),\\ &\qquad (364,361)\ldots \text{ and so on}\}\end{aligned}$$

This shows the possibility of the existence of $(4+4,5)$, $(4+13,13)$, $(4+40,39)$, $(4+121,119)$, $(4+364,361),\ldots$ ternary $(1,2)$-optimal codes.

(v) Let $n_1 = 5$.

Then the equation (5.1) becomes

$$3^{n_2-k+5} = 6n_2 + 5\,.$$

This equation clearly has no integer solution for n_2 and k and thus no $(1,2)$-optimal code exists in this case.

(vi) Let $n_1 = 6$.

Then the equation (5.1) becomes

$$3^{n_2-k+6} = 6n_2 + 7\,.$$

Once again the case of no integer solution and hence of no $(1,2)$-optimal code in this case.

(vii) Let $n_1 = 7$.

Then equation (5.1) reduces to

$$3^{n_2-k+6} = 2n_2 + 3\,. \tag{5.4}$$

The values of n_2 and k satisfying (5.4) with $n_2 \geq 2$ are as follows:

$$\begin{aligned}(n_2,k) \quad &= \quad \{(3,7),(12,15),(39,41),(120,121),\\ &\qquad (363,363),\ldots \text{and so on}\}\,.\end{aligned}$$

This show the possibility of the existence of $(7+3,7)$, $(7+12,15)$, $(7+39,41)$, $(7+120,121)$, $(7+363,363),\ldots$ ternary $(1,2)$-optimal codes.

(viii) Let $n_1 = 8$.
Then the equation (5.1) gives

$$3^{n_2-k+8} = 6n_2 + 11\,.$$

This equation has no integer solution for n_2 and k and thus no (1, 2)-optimal code exists in this case.

(ix) Let $n_1 = 9$.
Then equation (5.1) gives

$$3^{n_2-k+9} = 6n_2 + 13$$

which is again a case of no integer solution and no $(1,2)$-optimal code.

(x) Let $n_1 = 10$.
Then equation (5.1) gives

$$3^{n_2-k+9} = 2n_2 + 5\,. \tag{5.5}$$

The values of n_2 and k satisfying (5.5) with $n_2 \geq 2$ are:

$$\begin{aligned}(n_2,k) \quad = \quad &\{(2,9),(11,17),(38,43),(119,123),\\ &(362,365)\ldots \text{ and so on}\},\end{aligned}$$

which once again shows the possibility of the existence of $(10+2,9)$, $(10+11,17)$, $(10+38,43)$, $(10+119,123)$, $(10+362,365),\ldots$ ternary $(1,2)$-optimal codes.

5.1.1. Conclusions and open problem

Solutions of equation (5.1) for $n_1 = 1,2,\ldots,10$ have been investigated. It is seen that equation (5.1) has solutions for $n_1 = 1,4,7$ and 10. It is clear that the equation (5.1) has integer solutions for $n_1 = 13,16,19,\ldots$ and no integer solutions for $n_1 = 11,12,14,15,17,18,\ldots$.

The authors viz. Buccimazza, Dass and Jain [2] have been able to obtain code corresponding to one of the solutions only. This justifies that such a $(1,2)$-optimal ternary code exists. However in view of the existence of other solutions of equation (5.1), the existence of corresponding codes is an open problem.

5.2. $(1,2)$-optimal codes over GF(5)

In this section, the possibility of existence of $(1,2)$-optimal codes in the 5-ary case i.e. over GF(5) have been explored (see Dass, Iembo and Jain [7]).

Let us take $q = 5$, and $b_1 = 1$, $b_2 = 2$ in (4.1). We get

$$5^{n_1+n_2-k} = 4n_1 + 20n_2 - 19\,. \tag{5.6}$$

We now examine the values of n_1, n_2 and k satisfying (5.6). For this, we shall assign values to n_1 as $1, 2, 3, \ldots$ successively and find out the corresponding values of n_2 and k.

(i) Let $n_1 = 1$.
Then (5.6) reduces to

$$5^{n_2-k} = 4n_2 - 3\,. \tag{5.7}$$

The various values of n_2 and k satisfying (5.11) with $n_2 \geq 2$ (because $b_2 = 2$) are

$$\begin{aligned}(n_2, k) &= \{(2,1),(7,5),(32,39),(157,153),\\ &\quad (782,777) \text{ and so on}\}\,.\end{aligned}$$

These values show the possibility of the existence of $(1+2,1)$, $(1+7,5)$, $(1+32,29)$, $(1+157,153)$, $(1+782,777),\ldots$, $(1,2)$-optimal codes over GF(5).

In the following, we show that corresponding to the first set of values, such a code exists i.e. $(1+2,1)$ 5-ary $(1,2)$-optimal code exists.

Example 5.2. Consider the following parity-check matrix for $(1+2,1)$ 5-ary code:

$$H = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

The code which is the null space of H can correct all bursts of length 1 in the first sub block of length 1 and all bursts of length 2(fix) in the second block of length 2 and no others. As shown in Table 7, it can be shown that the syndromes of all the error vectors are distinct altogether and exhaustive.

(ii) Let $n_1 = 2$.
Then (5.6) reduces to

$$5^{n_2-k+2} = 20n_2 - 11\,.$$

It is clear that R.H.S. of above equation is never power of 5. Therefore, this equation has no integer solution for n_2 and k and hence such a $(1,2)$-optimal code can not exist in this case.

(iii) Let $n_1 = 3$.
Then (5.6) reduces to

$$5^{n_2-k+3} = 20n_2 - 7\,.$$

Once again, it is evident that this equation has no integer solution for n_2 and k and thus such a $(1,2)$-optimal code cannot exist in this case as well.

(iv) Let $n_1 = 4$.
Then (5.6) reduces to

$$5^{n_2-k+4} = 20n_2 - 3\,.$$

This equation clearly has no integer solution for n_2 and k and thus no $(1, 2)$-optimal code exists in this case.

(v) Let $n_1 = 5$.
Then the equation (5.6) reduced to

$$5^{n_2-k+5} = 20n_2 + 1\,.$$

Once again the case of no integer solution and hence of no $(1, 2)$-optimal code in this case.

(vi) Let $n_1 = 6$.
Then (5.6) reduces to

$$5^{n_2-k+5} = 4n_2 + 1\,. \tag{5.8}$$

The following are the values of n_2 and k satisfying (5.8) with $n_2 \geq 2$:

$$(n_2, k) = \{(31, 34), (156, 158), (781, 782), \ldots \text{ and so on}\}.$$

This shows the possibility of the existence of $(6 + 31, 34)$, $(6 + 156, 158)$, $(6 + 781, 782), \ldots$ 5-ary $(1, 2)$-optimal codes.

(vii) Let $n_1 = 7$.
Then the equation (5.6) reduces to

$$5^{n_2-k+7} = 20n_2 + 9\,.$$

This equation has no integer solution for n_2 and k and thus no (1, 2)-optimal code exists in this case.

(viii) Let $n_1 = 8$.
Then the equation (5.6) gives

$$5^{n_2-k+8} = 20n_2 + 13\,,$$

which is again a case of no integer solution and no $(1, 2)$-optimal code.

(ix) Let $n_1 = 9$.
Then the equation (5.6) gives

$$5^{n_2-k+9} = 20n_2 + 17\,,$$

a case of no integer solution and no $(1, 2)$-optimal code.

(x) Let $n_1 = 10$.
Then equation (5.6) reduces to

$$5^{n_2-k+10} = 20n_2 + 21\,,$$

again a case of no integer solution and no optimal code.

(xi) Let $n_1 = 11$.

Then the equation (5.6) gives

$$5^{n_2-k+10} = 4n_2 + 5\,. \tag{5.9}$$

The values of n_2 and k satisfying (5.9) with $n_2 \geq 2$ are :

$$(n_2, k) = \{(5, 13), (30, 37), (155, 161), (780, 785), \text{ and so on}\},$$

which once again shows the possibility of the existence of $(11 + 5, 13)$, $(11 + 30, 37)$, $(11 + 155, 101)$, $(11 + 78, 785), \ldots$ 5-ary $(1, 2)$-optimal codes.

5.2.1. Conclusions and open problem

In this section, we have investigated solutions of equation (5.6) for $n_1 = 1, 2, \ldots, 11$. It is seen that equation (5.6) has solutions for $n_1 = 1, 6$ and 11. It is clear that equation (5.6) has integer solutions for $n_1 = 16, 21, \ldots$ and no integer solutions for $n_1 = 12, 13, 14, 15, 17, 18, 19, 20, \ldots$. It is seen that it has been possible to obtain code corresponding to one of the solutions only. This justifies that such a $(1, 2)$-optimal 5-ary code exists. However, in view of the existence of other solutions of equation (5.6), the existence of corresponding code is an open problem.

5.3. $(1, 2)$-optimal codes over GF(7)

In this section, we explore the possibility of existence of such codes in the 7-ary case i.e. over GF(7) (see Dass, Iembo and Jain [8]).

Let us take $q = 7$ and $b_1 = 1$, $b_2 = 2$ in (4.1). We get

$$7^{n_1+n_2-k} = 6n_1 + 42n_2 - 41\,. \tag{5.10}$$

We now examine the values of n_1, n_2 and k satisfying (5.10). For this, we shall assign values to n_1 as $1, 2, 3, \ldots$ successively and find out the corresponding values of n_2 and k.

(i) Let $n_1 = 1$.

Then (5.10) reduces to

$$7^{n_2-k} = 6n_2 - 5. \tag{5.11}$$

Various values of n_2 and k satisfying (5.11) with $n_2 \geq 2$ (because $b_2 = 2$) are:

$$(n_2, k) = \{(2, 1), (9, 7), (58, 55), (401, 397), \ldots \text{ and so on}\}\,.$$

These values show the possibility of the existence of $(1 + 2, 1)$, $(1 + 9, 7)$, $(1 + 58, 55)$, $(1 + 401, 397)$, …, 7-ary optimal codes.

In the following, we show that corresponding to the first set of values, such a code exists i.e. $(1 + 2, 1)$ 7-ary $(1, 2)$-optimal code exists.

Example 5.3. Consider the following parity-check matrix for $(1 + 2, 1)$ 7-ary code:

$$H = \begin{bmatrix} 5 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix}.$$

The code which is the null space of H can correct all bursts of length 1 in the first sub block of length 1 and all bursts of length 2(fixed) in the second block of length

2 and no others. It can be verified from the error vectors and their corresponding syndromes.

(ii) Let $n_1 = 2$.
Then (5.10) reduces to

$$7^{n_2-k+2} = 42n_2 - 29\,.$$

It is clear that the right hand side of above equation is never a power of 7. Therefore, this equation has no integer solution for n_2 and k and hence such a $(1,2)$-optimal code can not exist in this case.

(iii) Let $n_1 = 3$.
Then (5.10) reduces to

$$7^{n_2-k+3} = 42n_2 - 23\,.$$

Once again, it is evident that this equation has no integer solution for n_2 and k and thus such a $(1,2)$-optimal code can not exist in this case as well.

(iv) Let $n_1 = 4$.
Then (5.10) becomes

$$7^{n_2-k+4} = 42n_2 - 17\,.$$

This equation clearly has no integer solution for n_2 and k and thus no $(1,2)$-optimal code exists in this case.

(v) Let $n_1 = 5$.
Then the equation (5.10) reduces to

$$7^{n_2-k+5} = 42n_2 - 11\,.$$

Once again the case of no integer solution and hence of no $(1,2)$-optimal code in this case.

(vi) Let $n_1 = 6$.
Then equation (5.10) gives

$$7^{n_2-k+6} = 42n_2 - 5\,.$$

This equation has no integer solution for n_2 and k and thus no $(1,2)$-optimal code exists in this case.

(vii) Let $n_1 = 7$.
Then equation (5.10) reduces to

$$7^{n_2-k+7} = 42n_2 + 1$$

which is again a case of no integer solution and no $(1,2)$-optimal code.

(viii) Let $n_1 = 8$.

Then the equation (5.10) gives

$$7^{n_2-k+7} = 6n_2 + 1 \,. \tag{5.12}$$

The values of n_2 and k satisfying (5.12) with $n_2 \geq 2$ are as follows:

$$(n_2, k) = \{(8, 13), (57, 61), (400, 403) \text{ and so on}\}.$$

This shows the possibility of the existence of $(8+8, 13)$, $(8+57, 61)$, $(8+400, 403), \ldots$, 7-ary $(1,2)$-optimal codes.

5.3.1. Conclusions and open problem

In this section, solutions of equation (5.10) for $n_1 = 1, 2, \ldots, 8$ have been investigated . It is seen that equation (5.10) has solutions for $n_1 = 1$ and 8. It is clear that equation (5.10) has integer solutions for $n_1 = 15, 22, \ldots$ and no integer solution for $n_1 = 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, \ldots$. Dass, Iembo and Jain [7] have been able to obtain code corresponding to one of the solutions only. This justifies that such a $(1,2)$-optimal 7-ary code exists. However, in view of the existence of other solutions of equation (5.10), the existence of corresponding codes is an open problem.

Acknowledgements. Part of this paper has been submitted by the second author as a partial fulfilment for the award of M.Phil degree to the University of Delhi in February, 2005. The authors wish to thank the referees for their critical remarks which have helped to submit the paper in the present form.

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