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Some Aspects of 2-Fuzzy 2-Normed Linear Spaces

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Abstract. This paper defines the concept of 2-fuzzy 2-normed linear space. In the 2-fuzzy 2-normed space, α-2-norm is defined corresponding to the fuzzy 2-norm. Standard results in fuzzy 2-normed linear spaces are extended to 2-fuzzy 2-normed linear spaces. The famous closed graph theorem and the Reisz theorem is proved in the realm of 2-fuzzy 2-normed linear spaces.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [7] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler in [2]. The concept of fuzzy norm and α-norm were introduced by Bag and Samanta in [1] Jialuzhang [3] has defined fuzzy linear space in a different way.

In the present paper, we introduce the concept of 2-fuzzy 2-normed linear space or fuzzy 2-normed linear space of the set of all fuzzy sets of a set. As an analogue of Bag and Samanta [1], we introduce the notion of α-2-norm on a linear space corresponding to the 2-fuzzy 2-norm. Saadati and Vaezpour [5] have proved closed graph theorem on a fuzzy Banach space using fuzzy norm. We have generalized this concept to a 2-fuzzy 2-normed linear space. Finally we have proved Riesz theorem in 2-fuzzy 2-normed linear spaces.

Convergence and completeness in fuzzy 2-normed space in terms of set of all fuzzy points was discussed by Meenakshi [4], we generalize it for 2-fuzzy sets in terms of α-2-norms.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gähler [2], Bag and Samanta [1] and Jialuzhang [3].

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Definition 2.1. *Let X be a real vector space of dimension greater than 1 and let $||.,.||$ be a real valued function on $X \times X$ satisfying the following conditions:*

(1) $||x, y|| = 0$ *if and only if x and y are linearly dependent,*
(2) $||x, y|| = ||y, x||$,
(3) $||x, y|| = |\alpha|||x, y||$, *where α is real,*
(4) $||x, y + z|| \leq ||x, y|| + ||x, z||$.

$||.,.||$ is called a 2-norm on X and the pair $(X, ||.,.||)$ is called a linear 2-normed space.

Definition 2.2. *Let X be a linear space over K (field of real or complex numbers). A fuzzy subset N of $X \times R$ (R, the set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in K$.*

(N1) *For all $t \in R$ with $t \leq 0$, $N(x, t) = 0$,*
(N2) *for all $t \in R$ with $t > 0$, $N(x, t) = 1$, if and only if $x = 0$,*
(N3) *for all $t \in R$ with $t > 0$, $N(cx, t) = N(x, t/|c|)$, if $c \neq 0$,*
(N4) *for all $s, t \in R$, $x, u \in X, N(x + u, s + t) \geq \min\{N(x, s), N(u, t)\}$,*
(N5) *$N(x, .)$ is a non decreasing function of R and $\lim N(x, t) = 1$.*

The pair (X, N) will be referred to as a fuzzy normed linear space.

Theorem 2.1. *Let (X, N) be a fuzzy normed linear space. Assume further that*

$$\text{(N6) } \mathrm{N(x,t)} > 0 \textit{ for all } \mathrm{t} > 0 \textit{ implies } \mathrm{x} = 0.$$

Define $||x||_\alpha = \inf\{t : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$. Then $\{||.||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.

Definition 2.3. *Let X be any non-empty set and $F(X)$ be the set of all fuzzy sets on X. For $U, V \in F(X)$ and $k \in K$ the field of real numbers, define*

$$U + V = \{(x + y, \lambda \wedge \mu)|(x, \lambda) \in U, (y, \mu) \in V\},$$

and $kU = \{(kx, \lambda)|(x, \lambda) \in U\}$.

Definition 2.4. *A fuzzy linear space $\widetilde{X} = X \times (0, 1]$ over the number field K where the addition and scalar multiplication operation on X are defined by $(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu)$, $k(x, \lambda) = (kx, \lambda)$ is a fuzzy normed space if to every $(x, \lambda) \in \widetilde{X}$ there is associated a non-negative real number, $||(x, \lambda)||$, called the fuzzy norm of (x, λ), in such a way that*

(1) $||(x, \lambda)|| = 0$ *iff $x = 0$ the zero element of X, $\lambda \in (0, 1]$,*
(2) $||k(x, \lambda)|| = |k|||(x, \lambda)||$ *for all $(x, \lambda) \in \widetilde{X}$ and all $k \in K$,*
(3) $||(x, \lambda) + (y, \mu)|| \leq ||(x, \lambda \wedge \mu)|| + ||(y, \lambda \wedge \mu)||$ *for all (x, λ), $(y, \mu) \in \widetilde{X}$,*
(4) $||(x, \vee_t \lambda_t|| = \wedge_t ||(x, \lambda_t|| for \lambda_t \in (0, 1]$.

3. 2-fuzzy 2-norm

In this section at first we define 2-fuzzy 2-normed linear space and α-2-norms on the set of all fuzzy sets of a non-empty set.

Definition 3.1. *Let X be a non-empty and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{(x,\mu)/x \in X$ and $\mu \in (0,1]\}$. Clearly f is a bounded function for $|f(x)| \leq 1$. Let K be the space of real numbers, then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by*

$$f + g = \{(x,\mu) + (y,\eta)\} = \{(x + y, \mu \wedge \eta)/(x,\mu) \in f \text{ and } (y,\eta) \in g\}$$

and

$$kf = (kf, \mu)/(x,\mu) \in f$$

where $k \in K$.

The linear space $F(X)$ is said to be normed space if for every $f \in F(X)$, there is associated a non-negative real number $||f||$ called the norm of f in such a way that

(1) $||f|| = 0$ *if and only if* $f = 0$. *For*
$||f|| = 0 \iff \{||(x,\mu)||/(x,\mu) \in f\} = 0 \iff x = 0, \mu \in (0,1] \iff f = 0.$

(2) $||kf|| = |k|||f||$, $k \in K$. *For*

$$||kf|| = \{||k(x,\mu)||/(x,\mu)f, k \in K\} = \{|k|||x,\mu||/(x,\mu) \in f\} = |k|||f||.$$

(3) $||f + g|| \leq ||f|| + ||g||$ *for every* $f, g \in F(X)$. *For*

$$\begin{aligned} ||f + g|| &= \{||(x,\mu) + (y,\eta)||/x, y \in X, \mu, \eta \in (0,1]\} \\ &= \{||(x + y), (\mu \wedge \eta)||/x, y \in X, \mu, \eta \in (0,1]\} \\ &\leq \{||(x, \mu \wedge \eta|| + ||(y, \mu \wedge \eta||/(x,\mu) \in f \text{ and } (y,\eta) \in g\} \\ &= ||f|| + ||g||. \end{aligned}$$

Then $(F(X), ||.||)$ is a normed linear space.

Definition 3.2. *A 2-fuzzy set on X is a fuzzy set on $F(X)$.*

Definition 3.3. *Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$. (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on F(X)) if and only if,*

(N1) *for all $t \in R$ with $t \leq 0$ $N(f_1, f_2, t) = 0$,*
(N2) *for all $t \in R$ with $t \geq 0$, $N(f_1, f_2, t) = 1$, if and only if f_1 and f_2 are linearly dependent,*
(N3) *$N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2,*
(N4) *for all $t \in R$, with $t \geq 0$, $N(f_1, cf_2, t) = N(f_1, f_2, t/|c|)$ if $c \neq 0$, $c \in K$ (field),*
(N5) *for all $s, t \in R$, $N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$,*
(N6) *$N(f_1, f_2, .) : (0, \infty) \to [0,1]$ is continuous,*
(N7) $\lim_{t\to\infty} N(f_1, f_2, t) = 1$.

Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Theorem 3.1. *Let N be 2-fuzzy norm on X then*
(i) $N(f_1, f_2, t)$ is a non decreasing function with respect to $'t'$ for each $x \in X$.
(ii) $N(f_1, f_2, f_3, t) = N(f_1, f_3 - f_2, t)$.

Proof. Let $t < s$ then $k = s - t > 0$ and we have

$$\begin{aligned}(f_1, f_2, t) &= \min\{N(f_1, f_2, t), 1\} \\ &= \min\{N(f_1, f_2, t).N(f, 0, k)\} \\ &= \min\{N(f_1, f_2, t), N(f, 0, s-t)\} \\ &\leq N(f_1, f_2 + 0, t + s - t) \\ &= N(f_1, f_2, s).\end{aligned}$$

Hence for $t < s, N(f_1, f_2, t) \leq N(f_1, f_2, s)$. This proves the (i). To prove (ii) we have

$$\begin{aligned}N(f_1, f_2 - f_3, t) &= N(f_1, (-1)(f_3 - f_2), t) \\ &= N\left(f_1, f_3 - f_2, \frac{t}{|-1|}\right) \\ &= N(f_1, f_3 - f_2, t).\end{aligned}$$

∎

Theorem 3.2. *Let $(F(X), N)$ be a fuzzy 2-normed linear space. Assume that*

$$\text{(N8) } \mathrm{N}(\mathrm{f}_1, \mathrm{f}_2, \mathrm{t}) > 0$$

for all $t > 0$ implies f_1 and f_2 are linearly dependent, define $||f_1, f_2||_\alpha = \inf\{t : N(f_1, f_2, t) \geq \alpha \in (0, 1)\}$. Then $\{||., .||_\alpha : \alpha \in [0, 1)\}$ is an ascending family of 2-norms on $F(X)$. These 2-norms are called α-2-norms on $F(X)$ corresponding to the fuzzy 2-norms.

Proof.

(1) Let $||f_1, f_2||_\alpha = 0$. This implies
  (i) $\inf\{t : N(f_1, f_2, t)) \geq \alpha\} = 0$.
  (ii) For all $t \in R, t > 0, N(f_1, f_2, t) \geq \alpha > 0, \alpha \in (0, 1)$.
  (iii) f_1, f_2 are linearly dependent from (N8).
  Conversely, assume that f_1, f_2 are linearly dependent. This implies
  (i) $N(f_1, f_2, t) = 1$ for all $t > 0$ by (N2),
  (ii) for all $\alpha \in (0, 1), \inf\{t : N(f_1, f_2, t) \geq \alpha\} = 0$,
  (iii) $||f_1, f_2||_\alpha = 0$.

(2) As $N(f_1, f_2, t)$ is invariant under any permutation, it follows that $||f_1, f_2||_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$, then

$$\begin{aligned}||f_1, cf_2||_\alpha &= \inf\{s : N(f_1, cf_2, s) \geq \alpha\} \\ &= \inf\{s : N(f1, f2, \frac{s}{|c|}) \geq \alpha\}.\end{aligned}$$

Let $t = s/|c|$, then

$$\begin{aligned}||f_1, cf_2||_\alpha &= \inf\{|c|t : N(f_1, f_2, t) \geq \alpha\} \\ &= |c| \inf\{t : N(f_1, f_2, t) \geq \alpha\} \\ &= |c|||f1, f2||_\alpha.\end{aligned}$$

If $c = 0$, then

$$\begin{aligned}||f_1, cf_2|| &= ||f_1, o||_\alpha \\ &= 0 = 0.||f, f_2|| \\ &= |c|||f_1, f_2||, \text{ for every } c \in K.\end{aligned}$$

(4) We have

$$\begin{aligned}&||f_1, f_2||_\alpha + ||f_1, f_3||_\alpha \\ &= \inf\{t : N(f_1, f_2, t) \geq \alpha\} + \inf\{s : N(f_1, f_3, s) \geq \alpha\} \\ &= \inf\{t + s : N(f_1, f_2, t) \geq \alpha, N(f_1, f_3, s) \geq \alpha\} \\ &\geq \inf\{t + s : N(f_1, f_2 + f_3, t + s) \geq \alpha\} \\ &\geq \inf\{r : N(f_1, f_2 + f_3, r) \geq \alpha\}, r = t + s \\ &= ||f_1, f_2 + f_3||_\alpha.\end{aligned}$$

Therefore

$$||f_1, f_2 + f_3||_\alpha \leq \|f_1, f_2||_\alpha + ||f_1, f_3||_\alpha.$$

Thus $\{||.,.||_\alpha : \alpha \in (0,1)\}$ is an α-2-norm on $F(X)$.

Let $0 < \alpha_1 < \alpha_2$, then

$$||f_1, f_2||_{\alpha_1} = \inf\{t : N(f_1, f_2, t) \geq \alpha_1\},$$
$$||f_1, f_2||_{\alpha_2} = \inf\{t : N(f_1, f_2, t) \geq \alpha_2\}.$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(f_1, f_2, t) \geq \alpha_2\} \subset \{t : N(f_1, f_2, t) \geq \alpha_1\}$$

implies

$$\inf\{t : N(f_1, f_2, t) \geq \alpha_2\} \subset \inf\{t : N(f_1, f_2, t) \geq \alpha_1\}.$$

Therefore, $||f_1, f_2||_{\alpha_2} \geq ||f_1, f_2||_{\alpha_1}$. Hence $\{||.,.||_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-2-norms on $F(X)$ corresponding to the fuzzy 2-norm on $F(X)$. ∎

Hereafter we use the notion fuzzy 2-norm on $F(X)$ instead of 2-fuzzy 2-normed linear space on X.

Definition 3.4. *A sequence $\{fn\}$ in a fuzzy 2-normed linear space $(F(X), N)$ is called a cauchy sequence with respect to α-2-norm if there exists $g, h \in F(X)$ which are linearly independent such that $\lim_{n\to\infty} ||f_n - f_m, g||_\alpha = 0$ and $\lim_{n\to\infty} ||f_n - f_m, h||_\alpha = 0$.*

Theorem 3.3. *Let $(F(X), N)$ be a fuzzy 2-normed linear space and let $f, g, h \in F(X)$.*

(i) *If $\{f_n\}$ is a cauchy sequence in $(F(X), N)$ with respect to α-2-norm then $\{||f_n, f||_\alpha\}$ and $\{||f_n, g||_\alpha\}$ are real cauchy sequences.*

(ii) *If $\{f_n\}$ and $\{g_n\}$ are cauchy sequences in $(F(X), N)$ with respect to α-2-norm and $\{\alpha_n\}$ is a real cauchy sequence then $\{f_n + g_n\}$ and $\{\alpha_n f_n\}$ are cauchy sequences in $(F(X), N)$ with respect to α-2-norm where $\alpha_n \in [0, 1]$.*

Proof.

(i) $||f_n, f||_\alpha = ||f_n - f_m + f_m, f||_\alpha$. Therefore,

$$||f_n, f||_\alpha \le ||f_n - f_m, f||_\alpha + ||f_m, f||_\alpha.$$

Also

$$||f_n, f||_\alpha - ||f_m, f||_\alpha \le ||f_n - f_m, f||_\alpha,$$

that is

$$|||f_n, f||_\alpha - ||f_m, f||_\alpha| \le ||f_n - f_m, f||_\alpha.$$

Therefore $\{||f_n, f||_\alpha\}$ is a real cauchy sequence, since the

$$\lim ||f_n - f_m, f||_\alpha = 0.$$

Similarly, $\{||f_n, g||\alpha\}$ is also a real cauchy sequence.

(ii) $||(f_n + g_n) - (f_m + g_m), f||_\alpha = ||(f_n - f_m) + (g_n - g_m), f||_\alpha$
$\le ||f_n - f_m, f|| + ||g_n - g_m, f||_\alpha \longrightarrow 0$.
Similarly $||(f_n + g_n) - (f_m + g_m), g||_\alpha \longrightarrow 0$. Therefore $\{||f_n + g_n||_\alpha\}$ is a cauchy sequence on $(F(X), N)$ with respect to α-2-norm. Also,

$$\begin{aligned}||\alpha_n f_n - \alpha_m f_m, f||_\alpha &= ||\alpha_n f_n - \alpha_n f_m + \alpha_n f_m - \alpha_m f_m, f||_\alpha \\ &= ||\alpha_n(f_n - f_m) + (\alpha_n - \alpha_m) f_m, f||_\alpha \\ &\le ||\alpha_n(f_n - f_m), f||_\alpha + ||(\alpha_n - \alpha_m) f_n, f||_\alpha \\ &= |\alpha_n| ||f_n - f_m, f||_\alpha + |\alpha_n - \alpha_m| ||f_m, f||_\alpha \\ &\longrightarrow 0.\end{aligned}$$

Since $\{\alpha_n\}$ and $\{||f_n, f||_\alpha\}$ are real cauchy sequences. Similarly $||\alpha_n f_n - \alpha_m f_m, g||_\alpha \longrightarrow 0$. Therefore $\{\alpha_n f_n\}$ is a cauchy sequences in $(F(X), N)$ with respect α-2-norm. ∎

4. Convergence and completeness for 2-fuzzy 2-normed linear spaces

In [6], a detailed study on convergence, completeness of sequences in a 2-normed linear spaces has been made by White. In this section we shall discuss these concepts for sequence in a fuzzy 2-normed linear space with respect to α-2-norm in $F(X)$.

Definition 4.1. *A sequence $\{fn\}$ in fuzzy 2- normed linear space $(F(X), N)$ is said to converge to f if $||f_n - f, g||_\alpha \longrightarrow 0$ as $n \longrightarrow \infty$ with respect to α-2-norm for all $g \in F(X)$.*

Theorem 4.1. *In the fuzzy 2-normed linear space $(F(X), N)$,*

(i) if $f_n \longrightarrow f$ and $g_n \longrightarrow g$, then $f_n + g_n \longrightarrow f + g$,

(ii) if $f_n \longrightarrow f$ and $\alpha_n \longrightarrow \alpha$, then $\alpha_n f_n \longrightarrow \alpha_n f$,

(iii) if $\dim(\mathrm{F(X), N}) \ge 2$, $f_n \longrightarrow f$ and $f_n \longrightarrow g$ then $f = g$ convergence is with respect to α-2-norm.

Proof.

(i) $||(f_n + g_n) - (f + g), h||_\alpha = ||(f_n - f) + (g_n - g), h||_\alpha$
$\le ||f_n - f, h||_\alpha + ||g_n - g, h||_\alpha \longrightarrow 0$. Therefore $f_n + g_n \longrightarrow f + g$.

(ii) If $h \in F(X)$,

$$\begin{aligned}||\alpha_n f_n - \alpha f, h||_\alpha &= ||\alpha_n f_n - \alpha_n f + \alpha_n f - \alpha f, h||_\alpha \\ &\leq ||\alpha_n f_n - \alpha_n f, h||_\alpha + ||\alpha_n f - \alpha f, h||_\alpha \\ &= |\alpha|||f_n - f, h||_\alpha + |\alpha_n - \alpha|||f, h||_\alpha \longrightarrow 0.\end{aligned}$$

Since $||f_n - f, h||_\alpha \longrightarrow 0$ and $|\alpha_n - \alpha \mid \longrightarrow 0$, it follows that $\alpha_n f_n \longrightarrow \alpha f$.

(iii) For any $h \in F(X)$,

$$\begin{aligned}||f - g, h||_\alpha &= ||f_n - f_n + f - g, h||_\alpha \\ &\leq ||f_n - g, h||_\alpha + || - (f_n - f)h||_\alpha \\ &= ||f_n - g, h||_\alpha + ||f_n - f, h||_\alpha \longrightarrow 0,\end{aligned}$$

since $f_n \longrightarrow f$ and $f_n \longrightarrow g$. Hence $f - g$, and h are linearly dependent for all $h \in F(X)$. Since $\dim(F(X), N) \geq 2$, the possibility is $f - g$ can be linearly dependent for all $h \in F(X)$ implies that $f - g = 0$ which implies that $f = g$. ∎

Theorem 4.2. *Let $(F(X), N)$ be fuzzy 2-normed linear space. If $\lim ||f_n - f_m, f||_\alpha = 0$, then $\{||f_n - g, f||_\alpha\}$ is a convergent sequence for each $g \in F(X)$.*

Proof.

$$||f_n - g, f||_\alpha = ||f_n - f_m + f_m - g, f||_\alpha \leq ||f_n - fm, f||_\alpha + ||f_m - g, f||_\alpha.$$

Therefore,

$$||f_n - g, f||_\alpha - ||f_m - g, f||_\alpha \leq ||f_n - f_m, f||_\alpha,$$

also

$$|||f_n - g, f||_\alpha - ||f_m - g, f||_\alpha| \leq ||f_n - f_m, f||_\alpha.$$

Hence $\{||f_n - g, f||_\alpha - ||f_m - g, f||_\alpha\}$ is a convergent sequence since $\lim ||f_n - f_m, f||_\alpha = 0$. ∎

Theorem 4.3. *If $\lim ||f_n - f, g|| = 0$ then $\lim ||f_n - g||_\alpha = ||f, g||_\alpha$.*

Proof. The

$$\lim |||f_n, g||_\alpha - ||f, g||_\alpha| \leq \lim ||f_n - f, g||_\alpha,$$

therefore $\lim ||f_n, g||_\alpha = ||f, g||_\alpha$. Also $||f_n, g||_\alpha = ||f, g||_\alpha = 0$ if the hypothesis holds for $f = g$. ∎

5. Fuzzy 2-Banach spaces

Definition 5.1. *The fuzzy 2-normed linear space $(F(X), N)$ in which every cauchy sequence converges is called a complete fuzzy 2-normed linear space. The fuzzy 2-normed linear space $(F(X), N)$ is a fuzzy 2-Banach space with respect to α-2-norm for it is a complete fuzzy 2-normed linear with respect to α-2-norm.*

Note 5.1. Let $\{f_n\}$ be a cauchy sequence in $(F(X), N)$ with respect to α-2-norm if there exists g and h in $F(X)$ which are linearly independent such that $\lim ||f_n - f_m, g||_\alpha = 0$ and $\lim ||f_n - f_m, h||_\alpha = 0$. That is, $\inf\{t : N(f_n - f_m, g, t) \geq \alpha\} = 0$ and $\inf\{t : N(f_n - f_m, h, t) \geq \alpha\} = 0$ which yields a sequence in real numbers that automatically converges. Therefore $(F(X), N)$ under α-2-norm is a fuzzy 2-Banach space.

Definition 5.2. *A fuzzy 2-linear functional F is a real valued function on $A \times B$ where A and B are subspaces of $(F(X), N)$ such that,*

(1) $F(f+h, g+h') = F(f,g) + F(f,h') + F(h,g) + F(h,h')$.

(2) $F(\alpha f, \beta g\} = \alpha\beta F(f,g), \alpha\beta \in [0,1]$. *$F$ is said to be bounded with respect to α-2-norm if there exists a constant $k \in [0,1]$ such that $|F(f,g)| \leq k||(f,g)||_\alpha$, for every $(f,g) \in A \times B$. If F is bounded then define*

$$\|F|| = \text{glb}\{k : |F(f,g)| \leq k||(f,g)||_\alpha \text{ for every } (f,g) \in A \times B\}.$$

Definition 5.3. *A fuzzy 2-linear operator T is a function from $A \times B$ to $C \times D$ where A, B are subspaces of fuzzy 2-normed linear space $(F(X), N_1)$ and C, D are subspaces of fuzzy 2-normed linear space $(F(Y), N_2)$ such that*

$$T(f+h, g+h') = T(f,g) + T(f,h') + T(h,g) + T(h,h')$$

and

$$T(\alpha f, \beta g) = \alpha\beta T(f,g), \text{ where } \alpha, \beta \in [0,1].$$

Theorem 5.1. *(Closed graph theorem) Let T be a fuzzy 2-linear operator from fuzzy 2-Banach space $(F(X), N_1)$ to fuzzy 2-Banach space $(F(Y), N_2)$. Suppose for every $\{f_n, f_n'\} \in (F(X), N_1)$ such that*

$$(f_n, f_n') \longrightarrow (f, f') \text{ and } (Tf_n, Tf_n') \longrightarrow (g, g')$$

for some $f, f' \in F(X)$, $g, g' \in F(Y)$, it follows $T(f,f') = (g,g')$. Then T is continuous.

Proof. The fuzzy 2-norm N on $(F(X), N_1) \times (F(Y), N_2)$ is given by

$$\begin{aligned} N((f_1,f_2),(g_1,g_2),t) &= \min\{N_1(f_1,f_2,t), N_2(g_1,g_2,t_1)\} \\ &= N_1(f_1,f_2,t) * N_2(g_1,g_2,t) \end{aligned}$$

where $*$ is the usual continuous t-norm. With this norm, $(F(X), N_1) \times (F(Y), N_2)$ is a complete fuzzy 2-normed linear space. For each $(f_1,f_2),(f_1',f_2') \in F(X)$ and $(g_1,g_2),(g_1',g_2') \in F(Y)$ and $t, s > 0$ it follows that

$$\begin{aligned} &N((f_1,f_2),(g_1,g_2),t) * N(f_1',f_2'),(g_1',g_2'),s) \\ &= [N_1(f_1,f_2,t) * N_2(g_1,g_2,t)] * [N_1(f_1',f_2',s) * N_2(g_1',g_2',s)] \\ &= [N_1(f_1,f_2,t) * N_1(f_1',f_2',s)] * [N_2(g_1,g_2,t) * N_2(g_1',g_2',s)] \\ &\leq N_1(f_1+f_1', f_2+f_2', s+t) * N_2(g_1+g_1', g_2+g_2', t+s) \\ &= N((f_1+f_1', f_2+f_2'),(g_1+g_1', g_2+g_2'), s+t). \end{aligned}$$

Now if $\{(f_n,f_n'),(g_n,g_n')\}$ is a cauchy sequence in $(F(X) \times F(X)) \times (F(Y) \times F(Y), N)$ then there exists $n_0 \in N$ such that,

$$N((f_n,f_n'),(g_n,g_n')) - ((f_m,f_m'),(g_m,g_m'),t) > 1-\epsilon$$

for every $\epsilon > 0$ and $t > 0$. So for $m, n > n_0$,

$$\begin{aligned} &N_1(f_n - f_m, f_n' - f_m', t) * N_2(g_n - g_m, g_n' - g_m', t) \\ &= N((f_n - f_m, f_n' - f_m'),(g_n - g_m, g_n' - g_m'), t) \\ &= N((f_n,f_n'),(g_n,g_n')((f_m,f_m'),(g_m,g_m'),t) > 1-\epsilon. \end{aligned}$$

Therefore $\{(f_n, f'_n)\}$ and $\{(g_n, g'_n)\}$ are cauchy sequences in $(F(X), N_1)$ and $(F(X), N_2)$ respectively and there exists $f \in F(X)$ and $g \in F(Y)$ such that $(f_n, f'_n) \longrightarrow (f, f')$ and $(g_n, g'_n) \longrightarrow (g, g')$ and consequently $\{(f_n, f'_n), (g_n, g'_n)\}$ converges to $((f, f'), (g, g'))$. Hence $((F(X)\times F(X)\times F(Y)\times F(Y), N)$ is a complete fuzzy 2-normed linear space. Here let $G = \{(f_n, f_n), T(f_n, f_n)$ for every $(f_n, f'_n) \in F(X) \times F(X)\}$, be the graph of the fuzzy 2-linear operator T.

Suppose $(f_n, f'_n) \longrightarrow (f, f')$ and $T(f_n, f'_n) \longrightarrow (g, g')$. Then from the previous argument $\{(f_n, f'_n n), (Tf_n, Tf'_n)\}$ converges to $((f, f'), (g, g'))$ which belongs to G. Therefore $T(f, f') = (g, g')$. Thus T is continuous. ∎

6. The Reisz theorem in fuzzy 2-normed linear spaces

Definition 6.1. *A subset Y of $F(X)$ is said to be a fuzzy 2-compact subset with respect to α-2-norm if for every sequence (g_n) in Y there exists a subsequence (g_{n_k}) of (g_n) which converges to an element $g \in Y$, that is $||g_{n_k} - g, f||_\alpha \longrightarrow 0$ as $n \longrightarrow \infty$ for every $f \in F(X)$.*

Lemma 6.1. *Let $F(X)$ be the fuzzy 2 -normed linear space and Y a fuzzy 2-compact subspace of $F(X)$. For $f_1, f_2 \in F(X)$, $\inf ||f_1 - g, f_2 - g||_\alpha = 0$ then there exists an element $g_0 \in Y$ such that $||f_1 - g_0, f_2 - g_0||_\alpha = 0$.*

Proof. For each integer k there exists an element $g_k \in Y$ such that $||f_1 - g_k, f_2 - g_k||_\alpha < 1/k$. Since $\{g_k\}$ is a sequence in a fuzzy 2-compact space Y, we can consider that $\{g_k\}$ is a convergent sequence in Y without loss of generality. Let $g_k \longrightarrow g_0$ as $k \longrightarrow 0$ for some $g_0 \in Y$.

For every $\epsilon < 0$ there exists a positive integer K with $1/K < \epsilon/3$ such that $k < K$ implies $||g_k - g_0, h||_\alpha < \epsilon/3$ for all $h \in F(X)$.

Consider

$$\begin{aligned}
||f_1 - g_0; f_2 - g_0||_\alpha &= ||g_k - g_0 + f_1 - g_k, f_2 - g_0||_\alpha \\
&\le ||g_k - g_0, f_2 - g_0|| + ||f_1 - g_k, f_2 - g_0||_\alpha \\
&= ||g_k - g_0, f_2 - g_0||_\alpha + ||f_1 - g_k, f_2 - g_k + g_k - g_0||_\alpha \\
&\le ||g_k - g_0, f_2 - g_0||_\alpha + ||f_1 - g_k, f_2 - g_k||_\alpha + ||f_1 - g_k, g_k - g_0||_\alpha \\
&= ||g_k - g_0, f_2 - g_0||_\alpha + ||f_1 - g_k, g_k - g_0||_\alpha + ||f_1 - g_k, f_2 - g_k||_\alpha \\
&< 2.\epsilon/3 + 1/k \\
&< 2.\epsilon/3 + 1/K \\
&< 2.\epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $||f_1 - g_0, f_2 - g_0||_\alpha = 0$. ∎

Theorem 6.1. *Let Y and Z be subspaces of fuzzy 2-normed linear spaces $(F(X), N)$ and Y be a fuzzy 2-compact proper subset of Z with dimension greater than one. For each $\theta \in (0, 1)$ there exist an element $(f_1, f_2) \in Z \times Z$ such that $||f_1, f_2||_\alpha = 1$, $||f_1 - g, f_2 - g||_\alpha \ge \theta$, for all $g \in Y$.*

Proof. Let $f_1, f_2 \in Z$ be linearly independent. Let $a = \inf_{g\in Y} ||f_1 - g, f_2 - g||_\alpha$. Assume $a = 0$, then by Lemma 6.1, there exists an element $g_o \in Y$ such that

$$||f_1 - g_0, f_2 - g_0||_\alpha = 0. \tag{6.1}$$

If $g_0 = 0$ then $||f_1, f_2||_\alpha = 0$. This implies $\inf\{t : N(f_1, f_2, t) \geq \alpha\} = 0$ which implies that f_1 and f_2 are linearly dependent this leads to a contradiction.

So g_0 is non-zero. Hence f_1, f_2, g_0 are linearly independent. But from definition and from (6.1) it follows that $f_1 - g_0, f_2 - g_0$ are linearly dependent. Thus there exists real numbers α_1, α_2 not all zero such that $\alpha_1(f_1 - g_0) + \alpha_2(f_2 - g_0) = 0$. Thus we have, $\alpha_1 f_1 + \alpha_2 f_2 + (-1)(\alpha_1 + \alpha_2)g_0 = 0$. Then f_1, f_2, g_0 are linearly dependent, which is a contradiction. Hence $a > 0$. For each $\theta \in (0, 1)$ there exists an element $g_0 \in Y$ such that $a \leq ||f_1 - g_0, f_2 - g_0||_\alpha \leq a/\theta$.

For each $j = 1, 2$, let

$$h_j = \frac{f_j - g_0}{||f_1 - g_0, f_2 - g_0||_\alpha^{1/2}}.$$

Then

$$\begin{aligned} ||h_1, h_2||_\alpha &= \left|\left| \frac{f_1 - g_0}{||f_1 - g_0, f_2 - g_0||_\alpha^{1/2}}, \frac{f_2 - g_0}{||f_1 - g_0, f_2 - g_0||_\alpha^{1/2}} \right|\right|_\alpha \\ &= \frac{||f_1 - g_0, f_2 - g_0||_\alpha}{||f_1 - g_0, f_2 - g_0||_\alpha} = 1. \end{aligned}$$

$$\begin{aligned} ||h_1 - g, h_2 - g||_\alpha &= \left|\left| \frac{f_1 - g_0}{||f_1 - g_0, f_2 - g_0||_\alpha^{1/2}} - g, \frac{f_2 - g_0}{||f_1 - g_0, f_2 - g_0||_\alpha^{1/2}} - g \right|\right|_\alpha \\ &= \frac{1}{||f_1 - g_0, f_2 - g_0||_\alpha} ||f_1 - (g_0 + g||f_1 - g_0, f_2 - g_0||_\alpha)^{1/2}, \\ &\quad f_2 - (g_0 + g||f_1 - g_0, f_2 - g_0||_\alpha)^{1/2}|| \\ &\geq \frac{a}{||f_1 - g_0, f_2 - g_0||_\alpha} \geq \frac{a}{a/\theta} = \theta \end{aligned}$$

for all $g \in Y$. ∎

Definition 6.2. *A subset Y of the fuzzy 2-normed linear space $(F(X), N)$ is called a fuzzy partially 2-closed subset if for linearly independent elements $f_1, f_2 \in F(X)$ if there exists a sequence $\{g_k\}$ in Y such that $||f_1 - g_k, f_2 - g_k||_\alpha \longrightarrow 0$ as $k \longrightarrow \infty$ then $f_j \in Y$ for some j.*

Theorem 6.2. *Let Y, Z be subspace of $F(X)$ the fuzzy 2-normed linear space and Y a fuzzy partially 2-closed subset of Z. Assume* dimZ ≥ 2. *For each $\theta \in (0, 1)$ there exists an element $(h_1, h_2) \in Z^2$ such that $||h_1, h_2||_\alpha = 1$, $||h_1 - g, h_2 - g||_\alpha \geq \theta$ for all $g \in Y$.*

Proof. Let $f_1, f_2 \in Z - Y$ be linearly independent and let $a = \inf ||f_1 - g, f_2 - g||_\alpha$. Assume that $a = 0$. Then there is a sequence $\{g_k\} \in Y$ such that $||f_1 - g_k, f_2 - g_k||_\alpha \longrightarrow 0$ as $k \to \infty$ and since Y is fuzzy partially 2-closed, $f_j \in Y$, for some j, which is a contradiction. Hence $a > 0$. Rest of the proof is the same as in proof of Theorem 6.1. ∎

7. Conclusion

The notion of α-2-norm on $F(X)$, the fuzzy power set of X is introduced. With the help of it, notions of Cauchy, convergent sequences and so completeness is studied.

Still more the idea is extended to frame Banach space in which Closed graph theorem is proved. Fuzzy 2-compact subset of $F(X)$ is defined and Riesz theorem in fuzzy 2-normed linear spaces is proved. Further the notions of the best approximation sets, strict 2-convexity using α-2-norm can be developed. Also the concept of product 2-fuzzy normed linear spaces using α-2-norm can be established.

References

[1] T. Bag and S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.* **11** (2003), no. 3, 687–705.

[2] S. Gähler, Lineare 2-normierte Räume, *Math. Nachr.* **28** (1964), 1–43.

[3] J. Zhang, The continuity and boundedness of fuzzy linear operators in fuzzy normed space, *J. Fuzzy Math.* **13** (2005), no. 3, 519–536.

[4] A. R. Meenakshi and R. Cokilavany, On fuzzy 2-normed linear spaces, *J. Fuzzy Math.* **9** (2001), no. 2, 345–351.

[5] R. Saadati and S. M. Vaezpour, Some results on fuzzy Banach spaces, *J. Appl. Math. Comput.* **17** (2005), no. 1-2, 475–484.

[6] A. G. White, Jr., 2-Banach spaces, *Math. Nachr.* **42** (1969), 43–60.

[7] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.