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Fuzzy Subcoalgebras and Duality

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Abstract. In this paper, we apply the concepts of fuzzy subsets to coalgebras and define fuzzy subcoalgebras and fuzzy left (right) coideals. Considering the applications of fuzzy subcoalgebras and fuzzy left (right) coideals, we discuss their properties under homomorphisms of coalgebras and the relationship among them. A fuzzy subcoalgebra and its duality as well as a fuzzy subalgebra and its duality is also discussed.

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1. Introduction

Since Rosenfeld [10] introduced the fuzzy subsets in the realm of group theory, many mathematicians have been involved in extending the concepts and results of abstract algebra to the broader framework of the fuzzy setting. The concept of a fuzzy subspace of a vector space was introduced by Katsaras and Liu in [5]. Abdukhalikov [2] defined a fuzzy subalgebra of an algebra on the basis of fuzzy subspace and classified fuzzy subalgebras of low-dimensional simple algebras. Moreover, Mordeson and Malik gave an up to date version of fuzzy commutative algebra in their book [9]. Recently, various new algebraic structures such as $BCH/BCI/BCK$-algebra, QS-algebra in the fuzzy setting have also been considered by many authors [7, 11]. On the other hand, Abdukhalikov [1] defined the concept of the dual subspace of a fuzzy subspace and investigated its properties. This work applied good idea for us. We can discuss the dual of fuzzy subalgebras.

In this paper, we apply the concepts of fuzzy subsets to coalgebras. We proceed as follows. In Section 2, we recall some basic definitions and notations which will be used in the sequel. In Section 3, we give the definitions of fuzzy subcoalgebras and fuzzy left (right) coideals, and characterize the sufficient and necessary condition that a fuzzy subset μ becomes fuzzy subcoalgebra (resp. fuzzy left (right) coideal). In Section 4, we study the images as well as preimages of fuzzy subcoalgebras and fuzzy left (right) coideals under homomorphism. In Section 5, we discuss the relationship

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between a fuzzy subcoalgebra and its duality as well as between a fuzzy subalgebra and its duality, and fuzzy maps are also discussed.

2. Preliminaries

In this section, some relevant definitions and notations are reproduced.

Definition 2.1. [3] *A k-coalgebra is a triple (C, Δ, ε), where C is a k-vector space, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are morphisms of k-vector spaces such that the following diagrams are commutative:*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "I\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes I"] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[dd, "\Delta"] & \\
C \otimes k & & k \otimes C \\
 & C \otimes C \arrow[ul, "I\otimes\varepsilon"] \arrow[ur, "\varepsilon\otimes I"'] &
\end{tikzcd}$$

Remark 2.1. If we change the direction of the morphisms in the above diagrams, we can get the definition of a k-algebra.

Remark 2.2. Let (C, Δ, ε) be a coalgebra. For an element $c \in C$, we denote $\Delta(c) = \sum c_1 \otimes c_2$. With the usual summation conventions we should have written

$$\Delta(c) = \sum_{i=1,n} c_{i1} \otimes c_{i2}.$$

The sigma notation suppresses the index "i". It is a way to emphasize the form of $\Delta(c)$, and it is very useful for writing long compositions involving the comultiplication in a compressed way. In this paper, we require that $\{c_{i1}\}$, $\{c_{i2}\}$ are linearly independent respectively. In this case, the decomposition of $\Delta(c)$ is unique.

Let (C, Δ, ε) be a k-coalgebra. A mapping $\mu : C \to [0, 1]$ is called *fuzzy subset* of C.

Definition 2.2. *Let C, C' be any two k-coalgebras and $f : C \to C'$ be a k-map. For any fuzzy subset μ of C, we define the fuzzy subset μ' of C' by*

$$\mu'(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \{\mu(x)\} & y \in f(C) \\ 0 & y \notin f(C) \end{cases}$$

μ' is called the image *of μ under f and denoted by $f(\mu)$. For any fuzzy subset μ' of C', we define the fuzzy subset μ of C by $\mu(x) = \mu'(f(x))$ for all $x \in C$, μ is called the* preimage *of μ' under f and denoted by $f^{-1}(\mu')$.*

Definition 2.3. [2] *Let μ be a fuzzy subset of k-algebras A. For any $x, y \in A$ and $\alpha, \beta \in k$, if it satisfies the following conditions:*

(1) $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

then μ is called a fuzzy subalgebra *of A.*

Definition 2.4. *Let μ be a fuzzy subset of k-algebra A. For any $x, y \in A$ and $\alpha, \beta \in k$, if it satisfies the following conditions:*

(1) $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$,

then μ is called a fuzzy ideal *of A.*

3. Fuzzy subcoalgebras and fuzzy left (right) coideals

In this section, we define the fuzzy subcoalgebra and fuzzy left (right) coideal of a coalgebra. Here, we don't suppress the index "i" in the decomposition of $\Delta(c)$.

Definition 3.1. [3] *Let (C, Δ, ε) be a coalgebra. A k-subspace D of C is called a* subcoalgebra *if $\Delta(D) \subseteq D \otimes D$.*

Definition 3.2. *Let μ be a fuzzy subset of C. For any $x \in C$,*

$$\Delta(x) = \sum_{i=1,n} x_{i1} \otimes x_{i2}.$$

Then μ is called a fuzzy subcoalgebra *of C, if it satisfies the following conditions:*

(1) $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\}$, *for any $x, y \in C$ and $\alpha, \beta \in k$,*
(2) $\mu(x) \leq \min\{\mu(x_{i1}), \mu(x_{i2})\}$, *for any $x \in C$ and all i.*

Example 3.1. Let C be a vector space with basis $\{g_i, d_i | i \in N^*\}$. We define $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ by

$$\Delta(g_i) = g_i \otimes g_i, \quad \Delta(d_i) = g_i \otimes d_i + d_i \otimes g_{i+1}, \tag{3.1}$$

$$\varepsilon(g_i) = 1, \qquad \varepsilon(d_i) = 0. \tag{3.2}$$

Then (C, Δ, ε) is a coalgebra [3].

Let $x \neq 0$. Then

$$x = \sum a_i g_i + \sum b_i d_i$$

where $a_i, b_i \neq 0$. We define $\mu(x) = (\wedge\mu(g_i)) \wedge (\wedge\mu(d_i))$ and define

$$\mu(g_i) = \frac{i}{i+1}, \quad \mu(d_i) = \frac{1}{i+1}.$$

If $x = 0$, we define $\mu(0) = 1$. Then μ is a fuzzy subcoalgebra of C. Indeed, let $x = \sum a_i g_i + \sum b_i d_i$ and $y = \sum k_i g_i + l_i d_i$, where $a_i, b_i, k_i, l_i \neq 0$. Then for $\alpha, \beta \in k$,

$$\alpha x + \beta y = \sum(\alpha a_i + \beta k_i) g_i + \sum(\alpha b_i + \beta l_i) d_i,$$

we have

$$\mu(\alpha x + \beta y) = (\wedge\mu(g_i)) \wedge (\wedge\mu(d_i)) \geq \min\{\mu(x), \mu(y)\}.$$

Since

$$\begin{aligned}\Delta(x) &= \sum a_i \Delta(g_i) + \sum b_i \Delta(d_i)\\ &= \sum a_i (g_i \otimes g_i) + \sum b_i (g_i \otimes d_i + d_i \otimes g_{i+1}),\end{aligned}$$

we have

$$\begin{aligned}\mu(x) &= (\wedge\mu(g_i)) \wedge (\wedge\mu(d_i)) \leq \min\{\mu(g_i), \mu(g_i)\},\\ \mu(x) &= (\wedge\mu(g_i)) \wedge (\wedge\mu(d_i)) \leq \min\{\mu(g_i), \mu(d_i)\},\\ \mu(x) &= (\wedge\mu(g_i)) \wedge (\wedge\mu(d_i)) \leq \min\{\mu(d_i), \mu(g_{i+1})\}.\end{aligned}$$

So $\mu(x) \leq \min\{\mu(x_{i1}), \mu(x_{i2})\}$. Hence μ is a fuzzy subcoalgebra of C.

Theorem 3.1. *Let μ be a fuzzy subset of k-coalgebra C with its attained upper bound $\delta > 0$. Then the following statements are equivalent:*

(1) μ *is a fuzzy subcoalgebra of* C,

(2) $\mu_t = \{x \in C|\mu(x) \geq t\}$ *is a subcoalgebra of* C, *for every* $t \in [0, \delta]$,

(3) $\mu_t = \{x \in C|\mu(x) \geq t\}$ *is a subcoalgebra of* C, *for every* $t \in Im\ \mu$,

(4) $\mu_{\hat{t}} = \{x \in C|\mu(x) > t\}$ *is a subcoalgebra of* C, *for every* $t \in [0, \delta)$.

Proof.
(1) $\Longrightarrow$(2) Let μ be a fuzzy subcoalgebra of C. We know that $\mu(0) \geq \mu(x)$ for any $x \in C$. In this case, $\delta = \mu(0)$. Let $x, y \in \mu_t$ and $\alpha, \beta \in k$. Then $\mu(x) \geq t, \mu(y) \geq t$ and $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\} \geq t$. Therefore, $\alpha x + \beta y \in \mu_t$. Let $x \in \mu_t$. Because $\min\{\mu(x_{i1}), \mu(x_{i2})\} \geq \mu(x) \geq t$, we have $\mu(x_{i1}) \geq t$ and $\mu(x_{i2}) \geq t$. Hence, $\Delta(\mu_t) \subseteq \mu_t \otimes \mu_t$.
(2)$\Longrightarrow$(3) Obviously.
(3)$\Longrightarrow$(4) Let $0 \leq t < \delta$. Then $\mu_{\hat{t}} = \bigcup\{\mu_s : t < s \leq \delta, s \in \text{Im } \mu\}$, which is a subcoalgebra of C.
(4)$\Longrightarrow$(1) For any $x, y \in C$ and $\alpha, \beta \in k$, we want to show $\mu(\alpha x+\beta y) \geq \min\{\mu(x),\mu(y)\}$. If not, there exist x, y such that $t = \mu(\alpha x + \beta y) < \min\{\mu(x), \mu(y)\}$, Then $0 \leq t < \delta$ and $x, y \in \mu_{\hat{t}}$. Since $\mu_{\hat{t}}$ is a subspace of C, $\alpha x + \beta y \in \mu_{\hat{t}}$, which implies that $t = \mu(\alpha x + \beta y) > t$. This is a contradiction. Next, it remains to prove $\mu(x) \leq \min\{\mu(x_{i1}), \mu(x_{i2})\}$, for $x \in C$. Suppose that there exist x, x_{i1}, x_{i2} such that $\mu(x) > \min\{\mu(x_{i1}), \mu(x_{i2})\} = t$. Then $x \in \mu_{\hat{t}}$. Since $\mu_{\hat{t}}$ is subcoalgebra of C, we have $\Delta(\mu_{\hat{t}}) \subseteq \mu_{\hat{t}} \otimes \mu_{\hat{t}}$, then $\mu(x_{i1}) > t$ and $\mu(x_{i2}) > t$. This is a contradiction. ∎

Example 3.2. In Example 3.1, we may assume that the basis is $\{g_1, g_2, d_1\}$. Following from that definition, we know that $\mu(0) = 1$, $\mu(g_1) = 1/2$, $\mu(g_2) = 2/3$, $\mu(d_1) = 1/2$. So we can find the upper bound $\mu(0)$ of the fuzzy subset μ. Since μ is a fuzzy subcoalgebra of C, we can make use of the Theorem 3.1 to get some subcoalgebras of C. Here, $\mu_{\frac{1}{2}}, \mu_{\frac{2}{3}}, \mu_{\hat{\frac{1}{2}}}, \mu_{\hat{\frac{2}{3}}}$ are subcoalgebras of C. On the other hand, we define $\mu : C \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \langle g_1 \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mu_1 = \langle g_1 \rangle$ is a subcoalgebra of C, by Theorem 3.1, we know that μ is a fuzzy subcoalgebra of C.

Definition 3.3. [3] *Let (C, Δ, ε) be a coalgebra. A k-subspace I of C is called a* left (right) coideal *if $\Delta(I) \subseteq C \otimes I$ (respectively $\Delta(I) \subseteq I \otimes C$).*

Definition 3.4. *Let μ be a fuzzy subset of C. For any $x \in C$,*

$$\Delta(x) = \sum_{i=1,n} x_{i1} \otimes x_{i2}.$$

If it satisfies the following conditions:

(1) $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\}$ *for any* $x, y \in C$ *and* $\alpha, \beta \in k$,

(2) $\mu(x) \leq \mu(x_{i2})$ $(\mu(x) \leq \mu(x_{i1}))$ *for any* $x \in C$ *and all* i,

then μ is called a fuzzy left (right) coideal *of C.*

Remark 3.1.

(1) A fuzzy subcoalgebra is fuzzy left (right) coideal.

(2) If μ is a fuzzy left and right coideal, by the definition, we have $\mu(x) \leq \mu(x_{i2})$ and $\mu(x) \leq \mu(x_{i1})$, so $\mu(x) \leq \min\{\mu(x_{i1}), \mu(x_{i2})\}$. Hence μ is a fuzzy subcoalgebra.

Example 3.3. By Remark 3.1, we know that the fuzzy subcoalgebras μ in Example 3.1 and 3.2 are fuzzy left (right) coideals.

Example 3.4. The field k is a k-coalgebra with comultiplication $\Delta : k \to k \otimes k$ the canonical isomorphism, $\Delta(x) = 1 \otimes x$ and counit $\varepsilon : k \to k$ the identity map [3].

We define

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in k \setminus \{0, 1\} \\ 0.5 & \text{if } x = 1 \\ 1 & \text{if } x = 0. \end{cases}$$

Then μ is a fuzzy left coideal of C. However, if $x \in k \setminus \{1\}$, for $\mu(x) > \mu(1)$, we have that μ is not a fuzzy right coideal of C.

Theorem 3.2. *A fuzzy subset μ of C is fuzzy left (right) coideal if and only if the level sets $\mu_t = \{x \in C : \mu(x) \geq t$ for $0 \leq t \leq \mu(0)\}$ are left (right) coideals of C.*

Proof. ($\Longrightarrow$) Let $0 \leq t \leq \mu(0)$. Then $\mu_t \neq \emptyset$. Let $x, y \in \mu_t$ and $\alpha, \beta \in k$. We have $\mu(\alpha x + \beta y) \geq \min\{\mu(x), \mu(y)\} \geq t$, so $\alpha x + \beta y \in \mu_t$. Let $x \in \mu_t \subseteq C$. Since $\mu(x_{i2}) \geq \mu(x) \geq t$, we have $\Delta(\mu_t) \subseteq C \otimes \mu_t$. Thus μ_t are left coideals of C.

($\Longleftarrow$) We only prove $\mu(x) \leq \mu(x_{i2})$. Let $x \in C$. We may assume that $\mu(x) = t$, then $x \in \mu_t$. Since $\Delta(\mu_t) \subseteq C \otimes \mu_t$, we have $x_{i2} \in \mu_t$. So $\mu(x_{i2}) \geq t = \mu(x)$. ∎

Example 3.5. In Example 3.3, the fuzzy subsets μ are fuzzy left codieals, by Theorem 3.2, we know that $\mu_{\frac{1}{2}}, \mu_{\frac{2}{3}}$ are left coideals of C and define $\mu : C \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \langle g_1 \rangle \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mu_1 = \langle g_1 \rangle$ is a left coideal of C, we can obtain that μ is a fuzzy left coideal of C using Theorem 3.2.

Example 3.6. In Example 3.4, the fuzzy subset μ is a fuzzy left coideal of C. So we obtain that $\mu_{0.5}, \mu_{0.7}, \mu_1$ are left coideals of C.

Proposition 3.1. *Let $(\mu_i)_{i \in N}$ be a family of fuzzy subcoalgebras (fuzzy left/right coideals) of C. Then $\bigcap_{i \in N} \mu_i$ is a fuzzy subcoalgebra (fuzzy left/right coideal).*

Proof. The result follows from $(\bigcap_{i \in N} \mu_i)(x) = \bigwedge_{i \in N} (\mu_i(x))$, for $x \in C$. ∎

However, the union of two fuzzy subcoalgebras (fuzzy left (right) coideals) can not be fuzzified. Let μ and λ be fuzzy subcoalgebras (fuzzy left (right) coideals) of C. Then $\mu \cup \lambda$ can not be a fuzzy subcoalgebra (fuzzy left (right) coideal) of C. In fact fuzzy subcoalgebras and fuzzy left (right) coideal are fuzzy subspaces of coalgebra C, whereas the union of fuzzy subspaces is not a fuzzy subspace in general. The counterexample is Example 3.7.

Example 3.7. Here, we also use the coalgebra in Example 3.1 and assume the basis is $\{g_1, g_2, d_1\}$. Let $x \neq 0$. Then $x = \sum a_i g_i + b_1 d_1$ where $a_i, b_1 \neq 0$. We define $\mu(x) = (\wedge\mu(g_i)) \wedge \mu(d_1)$ where $\mu(g_1) = 1$, $\mu(g_2) = \frac{1}{3}$, $\mu(d_1) = \frac{1}{4}$. We also define $\lambda(x) = (\wedge\lambda(g_i)) \wedge \lambda(d_1)$ where $\lambda(g_1) = \frac{1}{2}$, $\lambda(g_2) = 1$, $\lambda(d_1) = \frac{1}{5}$. If $x = 0$, we define $\mu(0) = 1$ and $\lambda(0) = 1$. Then μ, λ are fuzzy subcoalgebras of C. We have

$$\begin{aligned}(\mu \cup \lambda)(g_1 + g_2) &= \mu(g_1 + g_2) \vee \lambda(g_1 + g_2) \\ &= \min\{\mu(g_1), \mu(g_2)\} \vee \min\{\lambda(g_1), \lambda(g_2)\} \\ &= \frac{1}{3} \vee \frac{1}{2} = \frac{1}{2},\end{aligned}$$

however

$$\begin{aligned}\min\{(\mu \cup \lambda)(g_1), (\mu \cup \lambda)(g_2)\} &= \min\{\mu(g_1) \vee \lambda(g_1), \mu(g_2) \vee \lambda(g_2)\} \\ &= \min\{1, 1\} = 1,\end{aligned}$$

so

$$(\mu \cup \lambda)(g_1 + g_2) < \min\{(\mu \cup \lambda)(g_1), (\mu \cup \lambda)(g_2)\},$$

this shows that $\mu \cup \lambda$ is not a fuzzy subcoalgebra of C. By Remark 3.1, the cases of fuzzy left (right) coideals can easily get.

4. The homomorphisms of coalgebras

In this section, some fundamental concepts of fuzzy subcoalgebras and fuzzy left (right) coideals under homomorphisms are discussed.

Definition 4.1. [3] *Let C and D be two coalgebras. The k-linear map $f : C \to D$ is a* morphism of coalgebra*, if the following diagrams are commutative:*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"'] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & k & \end{tikzcd}$$

Proposition 4.1. *Let μ be a fuzzy subcoalgebra (resp. fuzzy left/right coideal) of D and $f : C \to D$ be a morphism of coalgebra. Then $f^{-1}(\mu)$ is a fuzzy subcoalgebra (resp. fuzzy left /right coideal) of C.*

Proof. Let $x, y \in C$ and $\alpha, \beta \in k$. Then

$$\begin{aligned}f^{-1}(\mu)(\alpha x + \beta y) &= \mu(f(\alpha x + \beta y)) \\ &= \mu(\alpha f(x) + \beta f(y)) \geq \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}.\end{aligned}$$

And

$$\Delta_D(f(x)) = \sum_{i=1,n} f(x)_{i1} \otimes f(x)_{i2} = \sum_{j=1,n} f(x_{j1}) \otimes f(x_{j2}),$$

so

$$\begin{aligned}f^{-1}(\mu)(x) = \mu(f(x)) &\leq \min\{\mu(f(x)_{i1}), \mu(f(x)_{i2})\} \\ &= \min\{\mu(f(x_{j1})), \mu(f(x_{j2}))\}\end{aligned}$$

$$= \min\{f^{-1}(\mu)(x_{j1}), f^{-1}(\mu)(x_{j2})\}.$$

So $f^{-1}(\mu)$ is a fuzzy subcoalgebra of C. The case of fuzzy left (right) coideal can be proved by the similar method. ∎

Proposition 4.2. *Let μ be a fuzzy subcoalgebra (fuzzy left/right coideal) of C and suppose that $f : C \to D$ is a morphism of coalgebra. Then $f(\mu)$ is a fuzzy subcoalgebra (fuzzy left/right coideal) of D.*

Proof. Let $x, y \in D$. We want to show that $f(\mu)(x+y) \geq \min\{f(\mu)(x), f(\mu)(y)\}$. There exist m, n such that $f(m) = x, f(n) = y$ which implies $f(m+n) = x+y$, so $m+n \in f^{-1}(x+y)$. We get

$$\begin{aligned} f(\mu)(x+y) = \sup_{z\in f^{-1}(x+y)}\{\mu(z)\} &\geq \sup_{m+n\in f^{-1}(x+y)}\{\mu(m+n)\} \\ &\geq \sup_{m+n\in f^{-1}(x+y)}\{\min\{\mu(m), \mu(n)\}\} \\ &= \sup_{m\in f^{-1}(x)}\{\mu(m)\} \wedge \sup_{n\in f^{-1}(y)}\{\mu(n)\} \\ &= \min\{f(\mu)(x), f(\mu)(y)\}. \end{aligned}$$

Let $x \in D$ and $\alpha \in k$. It is easy to get $f(\mu)(\alpha x) \geq f(\mu)(x)$.

Let $x \in D$. There exists $m \in C$, such that $f(m) = x$. We have

$$\begin{aligned} \sum_{i=1,n} x_{i1} \otimes x_{i2} = \Delta_D(x) &= \Delta_D(f(m)) \\ &= (f \otimes f)\Delta_C(m) \\ &= \sum_{j=1,n} f(m_{j1}) \otimes f(m_{j2}), \end{aligned}$$

So $m_{j1} \in f^{-1}(x_{i1}), m_{j2} \in f^{-1}(x_{i2})$. Also

$$\begin{aligned} f(\mu)(x) &= \sup_{m\in f^{-1}(x)}\{\mu(m)\} \\ &\leq \sup_{m_{(j1)}\in f^{-1}(x_{(i1)}), m_{(j2)}\in f^{-1}(x_{(i2)})}\{\min\{\mu(m_{j1}), \mu(m_{j2})\}\} \\ &= \min\{\sup_{m_{j1}\in f^{-1}(x_{i1})}\{\mu(m_{j1})\}, \sup_{m_{j2}\in f^{-1}(x_{i2})}\{\mu(m_{j2})\}\} \\ &= \min\{f(\mu)(x_{i1}), f(\mu)(x_{i2})\}. \end{aligned}$$

Hence $f(\mu)$ is a fuzzy subcoalgebra of D. The case of fuzzy left (right) coideal is similar to prove. ∎

5. The duality between fuzzy subalgebras and fuzzy subcoalgebras

In this section, we investigate the relationship between a fuzzy subcoalgebra and its dual fuzzy subalgebra as well as between a fuzzy subalgebra and its dual fuzzy subcoalgebra.

Definition 5.1. [1] *Let μ be a fuzzy k-vector subspace of V. Define: $\mu^* : V^* \longrightarrow [0,1]$ by*

$$\mu^*(f) = \begin{cases} 1 - \sup\{\mu(x)|\, x \in V, f(x) \neq 0\} & \text{if } f \neq 0, \\ 1 - \inf\{\mu(x)|\, x \in V\} & \text{if } f = 0. \end{cases}.$$

Obviously, μ^* is the fuzzy subset of V^*, in which V^* denotes the dual space of V, that is, the vector space of all linear maps from V to k.

Lemma 5.1. [1] *The fuzzy subset μ^* is a fuzzy subspace of V^*.*

Let (C, Δ, ε) be a k-coalgebra. We define the maps $M : C^* \otimes C^* \to C^*, M = \Delta^*\rho$, where ρ is defined as $\rho : C^* \otimes C^* \to (C \otimes C)^*$ by $\rho(f \otimes g)(m \otimes n) = f(m)g(n)$ and $u : k \to C^*$ by $u = \varepsilon^*\phi$, where $\phi : k \to k^*$ is the canonical isomorphism. By Proposition 1.3.6 of [3], we have that (C^*, M, u) is an algebra.

Proposition 5.1.

(1) *Let μ be a fuzzy subcoalgebra of C. Then μ^* is a fuzzy ideal of C^*.*

(2) *Let μ be a fuzzy left (right) coideal of C. Then μ^* is a fuzzy left (right) ideal of C^*.*

Proof. (1) By Lemma 5.1, we know that μ^* is a fuzzy subspace of C^*. It remains to show $\mu^*(f * g) \geq \max\{\mu^*(f), \mu^*(g)\}$, for any $f, g \in C^*$.

Let $f \neq 0$ and $g \neq 0$. We have

$$\begin{aligned} \mu^*(f * g) &= 1 - \sup\{\mu(x)|x \in C, (f * g)(x) \neq 0\} \\ &\geq 1 - \sup\{\mu(x_1) \wedge \mu(x_2)|x_1, x_2 \in C, \sum f(x_1)g(x_2) \neq 0\} \\ &= 1 - \sup\{\mu(x_1) \wedge \mu(x_2)|x_1, x_2 \in C, f(x_1)g(x_2) \neq 0\} \\ &= 1 - \sup\{\mu(x_1) \wedge \mu(x_2)|x_1, x_2 \in C, f(x_1) \neq 0 \text{ and } g(x_2) \neq 0\} \\ &= 1 - \sup\{\mu(x_1)|x_1 \in C, f(x_1) \neq 0\} \wedge \sup\{\mu(x_2)|x_2 \in C, g(x_2) \neq 0\} \\ &\geq 1 - \sup\{\mu(y)|y \in C, f(y) \neq 0\} \wedge \sup\{\mu(z)|z \in C, g(z) \neq 0\}. \end{aligned}$$

If

$$\sup\{\mu(y)|y \in C, f(y) \neq 0\} \geq \sup\{\mu(z)|z \in C, g(z) \neq 0\},$$

then $\mu^*(f) \leq \mu^*(g)$, so $\mu^*(f * g) \geq \mu^*(g)$. If

$$\sup\{\mu(y)|y \in C, f(y) \neq 0\} \leq \sup\{\mu(z)|z \in C, g(z) \neq 0\},$$

then $\mu^*(f) \geq \mu^*(g)$, so $\mu^*(f * g) \geq \mu^*(f)$. Hence, $\mu^*(f * g) \geq \max\{\mu^*(f), \mu^*(g)\}$.

Let $f = 0$ and $g \neq 0$. Then for any $x \in C$, we have

$$(f * g)(x) = \sum f(x_1)g(x_2) = 0.$$

So

$$\begin{aligned} \mu^*(f * g) = \mu^*(0) = \mu^*(f) \\ &= 1 - \inf\{\mu(x)|\, x \in C\} \\ &\geq 1 - \inf\{\mu(x)|\, x \in C, g(x) \neq 0\} \\ &\geq 1 - \sup\{\mu(x)|\, x \in C, g(x) \neq 0\} = \mu^*(g). \end{aligned}$$

We get $\mu^*(f * g) \geq \max\{\mu^*(f), \mu^*(g)\}$.

Let $g = 0$ and $f \neq 0$. We also have $\mu^*(f * g) \geq \max\{\mu^*(f), \mu^*(g)\}$. If $f = 0$ and $g = 0$, the result is obvious.

(2) We imitate (1) to prove. ∎

Let k-algebra (A, M, u) be finite dimension. We define the maps $\Delta : A^* \to A^* \otimes A^*$ and $\varepsilon : A^* \to k$ by $\Delta = \rho^{-1} M^*$ and $\varepsilon = \psi u^*$, where $\rho : A^* \otimes A^* \to (A \otimes A)^*$, $\rho(f \otimes g)(m \otimes n) = f(m)g(n)$ is bijective and $\psi : k^* \to k$ is the canonical isomorphism, $\psi(f) = f(1)$ for $f \in k^*$. Then $(A^*, \Delta, \varepsilon)$ is a coalgebra.

Proposition 5.2. (1) *Let μ be a fuzzy ideal of finite dimensional algebra A. Then μ^* is a fuzzy subcoalgebra of A^*.*
(2) *Let μ be a fuzzy left (right) ideal of finite dimensional algebra A. Then μ^* is a fuzzy left (right) coideal of A^*.*

Proof. (1) By Lemma 5.1, we know that μ^* is a fuzzy subspace of A^*. For any $f \in A^*$, we want to show $\mu^*(f) \leq \min\{\mu^*(f_i), \mu^*(g_i)\}$.

Let $f \neq 0$. We have

$$\begin{aligned}
\mu^*(f) &= 1 - \sup\{\mu(x) | x \in A, f(x) \neq 0\} \\
&\leq 1 - \sup\{\mu(ab) | ab \in A, f(ab) \neq 0\} \\
&\leq 1 - \sup\{\mu(a) \vee \mu(b) | a, b \in A, \sum f_i(a) g_i(b) \neq 0\} \\
&= 1 - \sup\{\mu(a) \vee \mu(b) | a, b \in A, f_i(a) g_i(b) \neq 0\} \\
&= 1 - \sup\{\mu(a) | a \in A, f_i(a) \neq 0\} \vee \sup\{\mu(b) | b \in A, g_i(b) \neq 0\}.
\end{aligned}$$

If

$$\sup\{\mu(a) | a \in A, f_i(a) \neq 0\} \geq \sup\{\mu(b) | b \in A, g_i(b) \neq 0\},$$

then $\mu^*(f_i) \leq \mu^*(g_i)$, so $\mu^*(f) \leq \mu^*(f_i)$. If

$$\sup\{\mu(a) | a \in A, f_i(a) \neq 0\} \leq \sup\{\mu(b) | b \in A, g_i(b) \neq 0\},$$

then $\mu^*(f_i) \geq \mu^*(g_i)$, so $\mu^*(f) \leq \mu^*(g_i)$. Hence $\mu^*(f) \leq \min\{\mu^*(f_i), \mu^*(g_i)\}$.

Let $f = 0$. Note that $f_i = g_i = 0$, so $\mu^*(f) \leq \min\{\mu^*(f_i), \mu^*(g_i)\}$.

(2) We can imitate (1) to prove. ∎

Let V be a finite dimensional vector space. Then the map $\theta_V : V \to V^{**}$, $\theta_V(v)(v^*) = v^*(v)$ for any $v \in V, v^* \in V^*$ is an isomorphism of vector spaces.

Let A be a finite dimensional algebra and C a finite dimensional coalgebra. Then

(1) $\theta_A : A \to A^{**}$ is an isomorphism of algebras.

(2) $\theta_C : C \to C^{**}$ is an isomorphism of coalgebras.

Definition 5.2. *Let f be a morphism from coalgebra C to coalgebra D, μ a fuzzy subset of C, ν a fuzzy subset of D. If f satisfies $\nu(f(x)) \geq \mu(x)$, then f is called a* fuzzy morphism of coalgebras, *denotes $\tilde{f}$. If f is epimorphism and $\nu(f(x)) = \mu(x)$, then μ is homomorphic to ν and we write $\mu \sim \nu$. In particular, if f is an isomorphism and $\nu(f(x)) = \mu(x)$, then μ is isomorphic to ν and we write $\mu \cong \nu$.*

Example 5.1. Let C be a coalgebra, I be a coideal and $p : C \to C/I$ be a canonical projection of k-vector spaces. We know that there exists a unique coalgebra structure on C/I such that p is a morphism of coalgebra [3].

We define $\mu : C \to [0,1]$ by $\mu(x) = \begin{cases} 1 & x = 0 \\ 0.3 & x \neq 0 \end{cases}$ and define $\nu : C/I \to [0,1]$ by $\nu(x) = \begin{cases} 1 & x = 0 \\ 0.5 & x \neq 0 \end{cases}$. Then μ, ν are fuzzy subsets of coalgebras C and C/I, respectively. Let $x \in C$. If $x = 0$, then $p(x) = 0$, so $\nu(p(x)) = 1 = \mu(x)$, if $0 \neq x \in I$, then $p(x) = 0$, we have $\nu(p(x)) = 1 > 0.3 = \mu(x)$, if $0 \neq x \in C \setminus I$, we also have $\nu(p(x)) = 0.5 > 0.3 = \mu(x)$. Hence for any $x \in C$, we have $\nu(p(x)) \geq \mu(x)$, which implies that p is a fuzzy morphism of coalgebras.

Corollary 5.1. *Let μ be a fuzzy subcoalgebra of finite dimensional coalgebra C and $\mu(0) = \sup\{\mu(C \setminus \{0\})\}$. Then $\mu \cong \mu^{**}$ as fuzzy subcoalgebras.*

Proof. Let $\theta : C \to C^{**}$ be the morphism of coalgebras. By Proposition 5.1 and Proposition 5.2, we know that μ^{**} is a fuzzy subcoalgebra of C^{**}. Then we get $\mu \cong \mu^{**}$ by [1, Theorem 4.3]. ∎

Any algebra A, we can associate in a natural way a coalgegra, where is not defined on the entire dual space A^*, but on a certain subspace of it.

Let (A, M, ε) be an algebra. Consider the set $A^\circ = \{f \in A^* | \ker(f)$ contains an ideal of finite codimension $\}$. We know that A° is a coalgebra, whose structure maps are $\Delta : A^\circ \to A^\circ \otimes A^\circ, \Delta = \psi^{-1} M^*$, where $\psi : A^* \otimes A^* \to (A \otimes A)^*$ is the canonical injection. $\varepsilon : A^\circ \to k, \varepsilon(a^*) = a^*(1)$. Remark

$$A^\circ = \{f \in A^* | \exists f_i, g_i \in A^* : f(xy) = \sum f_i(x) g_i(y) \quad \forall x, y \in A\}.$$

For more details, see [3].

Definition 5.3. *Let μ be a fuzzy subspace of A. Define:*

$$\mu^\circ(x) = \begin{cases} \mu^*(x) & x \in A^\circ, \\ 0 & x \notin A^\circ. \end{cases}$$

Following from Lemma 5.1, we know that μ^* is a fuzzy subspace of A^*. Obviously, $\mu^\circ \subseteq \mu^*$. Indeed, μ° is a fuzzy subspace of A°, for any $x, y \in A^\circ$ and $\alpha, \beta \in k$, we have $\mu^\circ(\alpha x + \beta y) = \mu^*(\alpha x + \beta y) \geq \min\{\mu^*(x), \mu^*(y)\} = \min\{\mu^\circ(x), \mu^\circ(y)\}$.

Example 5.2. Let $H = k[x]$, polynomials in x, where we assume x is primitive. Then $H^* \cong k[[f]]$, the formal power series in f, and $H^\circ = \{\sum a_{ij} f^j \phi_{\lambda_i} | a_{ij}, \lambda_i \in k\}$, where $\phi_{\lambda_i} = \sum_{n \geq 0} \lambda_i^n f^{(n)} \in \mathrm{Alg}(H, k)$ (See [8]).

Let $f(x) \in H$. We denote the degree of $f(x)$ by $\deg f(x)$ and define the fuzzy subset of H by

$$\mu(f(x)) = \begin{cases} 1 & f(x) = 0 \\ \frac{1}{\deg f(x) + 1} & f(x) \neq 0 \end{cases}.$$

Then notice that for $f(x), g(x) \in H$ and $\alpha, \beta \in k$,

$$\deg(\alpha f(x) + \beta g(x)) \leq \max\{\deg f(x), \deg g(x)\},$$

so μ is a fuzzy subspace of H. Let $\varphi \in H^\circ$. If $\varphi = 0$, we define $\mu^\circ(\varphi) = 1 - \inf\{\mu(f(x))\}$, if $\varphi \neq 0$, we define $\mu^\circ(\varphi) = 1 - \sup\{\mu(f(x)) | \varphi(f(x)) \neq 0\}$. Then μ° is a fuzzy subspace of H°.

Let A be algebra and A° be finite dual.

Proposition 5.3.

(1) *Let μ be a fuzzy ideal of A. Then μ° is a fuzzy subcoalgebra of A°.*

(2) *Let μ be a fuzzy left (right) ideal of A. Then μ° is a fuzzy left (right) coideal of A°.*

Proof. (1) We know that μ° is a fuzzy subspace of A°. It remains to show $\mu^\circ(f) \geq \min\{\mu^\circ(f_i), \mu^\circ(g_i)\}$, for any $f \in A^\circ$.

Let $f \neq 0$. We have

$$\begin{aligned}\mu^\circ(f) &= 1 - \sup\{\mu(x)|x \in A, f(x) \neq 0\}\\ &\leq 1 - \sup\{\mu(ab)|ab \in A, f(ab) \neq 0\}\\ &\leq 1 - \sup\{\mu(a) \vee \mu(b)|a, b \in A, \sum f_i(a)g_i(b) \neq 0\}\\ &= 1 - \sup\{\mu(a) \vee \mu(b)|a, b \in A, f_i(a)g_i(b) \neq 0\}\\ &= 1 - \sup\{\mu(a)|a \in A, f_i(a) \neq 0\} \vee \sup\{\mu(b)|b \in A, g_i(b) \neq 0\}\end{aligned}$$

If

$$\sup\{\mu(a)|\, a \in A, f_i(a) \neq 0\} \geq \sup\{\mu(b)|b \in A, g_i(b) \neq 0\},$$

then $\mu^\circ(f_i) \leq \mu^\circ(g_i)$, so $\mu^\circ(f) \leq \mu^\circ(f_i)$. If

$$\sup\{\mu(a)|\, a \in A, f_i(a) \neq 0\} \leq \sup\{\mu(b)|b \in A, g_i(b) \neq 0\},$$

then $\mu^\circ(f_i) \geq \mu^\circ(g_i)$, so $\mu^\circ(f) \leq \mu^\circ(g_i)$. Hence $\mu^\circ(f) \leq \min\{\mu^\circ(f_i), \mu^\circ(g_i)\}$.

Let $f = 0$. We note that $f_i = g_i = 0$, so $\mu^\circ(f) \leq \min\{\mu^\circ(f_i), \mu^\circ(g_i)\}$.

(2) We can imitate (1) to prove it. ∎

Definition 5.4. [3] *A coalgebra C is called* coreflexive *if $\phi_C : C \to C^{*\circ}$ is an isomorphism.*

Corollary 5.2. *Let μ be a fuzzy subcoalgebra of coreflexive coalgebra C and $\mu(0) = \sup\{\mu(C \setminus \{0\})\}$. Then $\mu \cong \mu^{*\circ}$ as fuzzy subcoalgebras.*

Proof. The method is similar to Corollary 5.1. ∎

Proposition 5.4.

(1) *Let $\widetilde{\varphi} : \mu_C \to \nu_D$ be a fuzzy morphism of coalgebras and $\inf\{\nu(D)\} \geq \inf\{\mu(C)\}$. Then $\widetilde{\varphi^*} : \nu^*_{D^*} \to \mu^*_{C^*}$ is a fuzzy morphism of algebras.*

(2) *Let $\widetilde{\varphi} : \mu_A \to \nu_B$ be a fuzzy morphism of finite dimensional algebras and $\inf\{\nu(B)\} \geq \inf\{\mu(A)\}$. Then $\widetilde{\varphi^*} : \nu^*_{B^*} \to \mu^*_{A^*}$ is a fuzzy morphism of coalgebras.*

Proof. (1) Let $\varphi : C \to D$ be a morphism of coalgebras. Then $\varphi^* : D^* \to C^*$ is a morphism of algebras by [3]. For $f \in D^*$, it remains to show $\mu^*(\varphi^*(f)) \geq \nu^*(f)$. For any $x \in C$, we have $\nu(\varphi(x)) \geq \mu(x)$. So if $\varphi^*(f) \neq 0$,

$$\begin{aligned}\mu^*(\varphi^*(f)) &= 1 - \sup\{\mu(x)|x \in C, \varphi^*(f(x)) = f(\varphi(x)) \neq 0\}\\ &\geq 1 - \sup\{\nu(\varphi(x))|x \in C, f(\varphi(x)) \neq 0\}\\ &= 1 - \sup\{\nu(y)|\varphi(x) = y, f(y) \neq 0\}\\ &\geq 1 - \sup\{\nu(z)|z \in D, f(z) \neq 0\} = \nu^*(f)\end{aligned}$$

If $\varphi^*(f) = 0$, then

$$\mu^*(\varphi^*(f)) = \mu^*(0) = 1 - \inf_{x \in C}\{\mu(x)\} \geq 1 - \inf_{y \in D}\{\nu(y)\} = \nu^*(0) \geq \nu^*(f).$$

(2) Note that if $\varphi : A \to B$ is a morphism of finite dimensional algebras, then $\varphi^* : B^* \to A^*$ is a morphism of coalgebras. The proof of remainder is similar to (1). ∎

Proposition 5.5. *Let* $\widetilde{\varphi} : \mu_A \to \nu_B$ *be a fuzzy morphism of algebras and* $\inf\{\nu(B)\} \geq \inf\{\mu(A)\}$. *Then* $\widetilde{\varphi^\circ} : \nu^\circ_{B^\circ} \to \mu^\circ_{A^\circ}$ *is a fuzzy morphism of coalgebras.*

Proof. From [3, Proposition 1.5.4], if $\varphi : A \to B$ is a morphism of algebras, then the induced map $\varphi^\circ : B^\circ \to A^\circ$ is a morphism of coalgebras. The method is similar to Proposition 5.4. ∎

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