# Existence of Solutions for a Class of Quasi-Linear Singular Integro-Differential Equations in a Sobolev Space 

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#### Abstract

An existence theorem is proved for a class of quasi-linear singular integro-differential equation with Cauchy kernel in a Sobolev Space.


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## 1. Introduction

Many classes of singular integral and integro-differential equations are currently used in many fields of engineering mechanics with applied character, like, elasticity, plasticity, viscoelasticity and fracture mechanics. The non-linear singular integrodifferential equations are further applied in other fields of engineering mechanics like structural analysis. Such structural analysis problems are reduced to the solution of a non-linear singular integro-differential equations connected with the behavior of the stress fields. These types of singular integral equations and singular integrodifferential equations form the latest high technology in the solution of very important problems of solid and fluid mechanics and, therefore, in recent years special attention is given to it $[1-3,10,11,15]$. We refer to Ladopoulos [8] for many applications of singular integral equations and singular integro-differential equations in engineering and science. Also, we refer to [4, 5, 12, 15-18] for the methods of solutions and many other applications.

The aim of this paper is to investigate the existence of solution of a class of quasi-linear singular integro-differential equation with Cauchy kernel in the form:

$$
\begin{equation*}
A(s, u(s)) u^{\prime}(s)-B(s, u(s)) \frac{1}{\pi i} \int_{\Gamma} \frac{u^{\prime}(\tau)}{\tau-s} d \tau=g(s, u(s)) \tag{1.1}
\end{equation*}
$$

under the initial condition

$$
\begin{equation*}
u(r)=0, \tag{1.2}
\end{equation*}
$$

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where $\Gamma$ is a closed smooth contour and $r$ is a fixed point on $\Gamma$. Our discussion is based on the applicability of Schauder's fixed point theorem [6, 7, 14, 15, 18] to our problem. Throughout the paper $L_{p}(\Gamma),(1 \leq p<\infty)$, means the Banach space of all measurable functions $f$ on $\Gamma$, with the norm:

$$
\|f(t)\|_{L_{p}}=\left(\int_{\Gamma}|f(t)|^{p} d t\right)^{p^{-1}}
$$

and $W_{p}^{1}(\Gamma),(1<p<\infty)$, means the Sobolev space of all functions $u \in L_{p}(\Gamma)$ which possess $u^{\prime} \in L_{p}(\Gamma),[9]$. We shall seek the solution of equation (1.1) in the Sobolev space $W_{p}^{1}$. It is worthwhile to notice that our problem (1.1) and (1.2) is a generalization of the problem that was discussed by Wolfersdorf in [18].

We shall discuss our problem through two steps. The first step, in Section 2, we find the function $u^{\prime}(s)$ in the form of an integral equation in the unknown function $u(s)$, and we discuss some properties of the kernel of a fixed point equation in $u(s)$. The second step, in Section 3, we prove an existence theorem for our problem (1.1) and (1.2).

## 2. Reduction to fixed point equation

In this section, we seek the solution of the problem (1.1), (1.2) and an estimation of the kernel of a fixed point equation.

Theorem 2.1. The quasi-linear singular integro-differential equation (1.1) with the initial condition (1.2) has the following solution:

$$
\begin{align*}
u^{\prime}(s)= & \frac{A(s, u(s)) g(s, u(s))}{f^{2}(s, u(s))} \\
& +\frac{B(s, u(s)) \exp (M(s))}{\pi i f(s, u(s))} \int_{\Gamma}\left(\frac{g(\nu, u(v))}{\mu(v, u(v))}\right)\left(\frac{1}{\nu-s}\right) d v \tag{2.1}
\end{align*}
$$

where

$$
\begin{align*}
\mu(v, u(v)) & =X^{+}(v)(A(v, u(v))-B(v, u(v))) \\
f(s, u(s)) & =\sqrt{A^{2}(s, u(s))-B^{2}(s, u(s))} \tag{2.2}
\end{align*}
$$

where

$$
\begin{aligned}
X^{+}(s) & =\sqrt{C(s)} \exp (M(s)) \\
C(s) & =\frac{A(s, u(s))+B(s, u(s))}{A(s, u(s))-B(s, u(s))}
\end{aligned}
$$

and

$$
M(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\ln C(\tau)}{\tau-z} d \tau
$$

together with the following conditions:
(i) The functions $A(s, u(s)), B(s, u(s))$ and $g(s, u(s))$ are continuous positive functions defined on the region: $D=\{(s, u): s \in \Gamma,|u|<\infty\}$. We assume

$$
\begin{equation*}
A(s, u(s))>B(s, u(s)),\left|A^{2}(s, u(s))-B^{2}(s, u(s))\right| \geq 1 \tag{2.3}
\end{equation*}
$$

for each $(s, u)$ belongs to $D$ and

$$
\begin{equation*}
\operatorname{ind}\left(\frac{A(s, u(s))+B(s, u(s))}{A(s, u(s))-B(s, u(s))}\right)=0 \tag{2.4}
\end{equation*}
$$

(ii) The functions $A(s, u(s)), B(s, u(s))$ and $g(s, u(s))$ satisfy the following HolderLipschtiz conditions:

$$
\begin{align*}
& \left|A\left(s_{2}, u_{2}\right)-A\left(s_{1}, u_{1}\right)\right| \leq \gamma_{1}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right)  \tag{2.5}\\
& \left|B\left(s_{2}, u_{2}\right)-B\left(s_{1}, u_{1}\right)\right| \leq \gamma_{2}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right) \tag{2.6}
\end{align*}
$$

and

$$
\begin{equation*}
\left|g\left(s_{2}, u_{2}\right)-g\left(s_{1}, u_{1}\right)\right| \leq \gamma_{3}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right) \tag{2.7}
\end{equation*}
$$

where $\gamma_{i},(i=1,2,3)$ are positive constants and $0<\delta<1$.
Proof. Let us consider the analytic function

$$
\begin{equation*}
\Phi(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{u^{\prime}(\sigma)}{\sigma-z} d \sigma, \Phi^{-}(\infty)=0 \tag{2.8}
\end{equation*}
$$

which has the following Sokhotski formulae [5]:

$$
\begin{equation*}
\Phi^{ \pm}(s)= \pm \frac{1}{2} u^{\prime}(s)+\frac{1}{2 \pi i} \int_{\Gamma} \frac{u^{\prime}(\tau)}{\tau-z} d \tau \tag{2.9}
\end{equation*}
$$

From which, we have

$$
\begin{equation*}
\Phi^{+}(s)+\Phi^{-}(s)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{u^{\prime}(\tau)}{\tau-z} d \tau, \quad \Phi^{+}(s)-\Phi^{-}(s)=u^{\prime}(s) . \tag{2.10}
\end{equation*}
$$

Substituting from equations (2.10) into (1.1), we get the following boundary value problem (BVP):

$$
\begin{equation*}
\Phi^{+}(s)=\left(\frac{A(s, u(s))+B(s, u(s))}{A(s, u(s))-B(s, u(s))}\right) \Phi^{-}(s)+\frac{g(s, u(s))}{A(s, u(s))-B(s, u(s))} . \tag{2.11}
\end{equation*}
$$

From the theory of linear singular integral equations, [5], and condition (2.4), we can put:

$$
\begin{equation*}
C(s) \frac{A(s, u(s))+B(s, u(s))}{A(s, u(s))-B(s, u(s))}=\frac{X^{+}}{X^{-}} \tag{2.12}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
X(z)=\exp (M(z)) \quad \text { with } \quad M(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\ln C(\tau)}{\tau-z} d \tau \tag{2.13}
\end{equation*}
$$

where

$$
\begin{equation*}
X^{+}(s)=\sqrt{C(s)} \exp \left(M(s) \quad \text { and } \quad X^{-}(s)=\frac{1}{\sqrt{C(s)}} \exp (M(s)\right. \tag{2.14}
\end{equation*}
$$

Substituting from equations (2.12) into (2.11), we get

$$
\begin{equation*}
\left(\frac{\Phi^{+}(s)}{X^{+}(s)}\right)=\left(\frac{\Phi^{-}(s)}{X^{-}(s)}\right)+\frac{g(s, u(s))}{(A(s, u(s))-B(s, u(s))) X^{+}} . \tag{2.15}
\end{equation*}
$$

From [5], the BVP (2.13) has the following solution:

$$
\Phi(z)=(X(z))\left(\frac{1}{2 \pi i} \int_{\Gamma}\left(\frac{g(\tau, u(\tau))}{[A(\tau, u(\tau))-B(\tau, u(\tau))] X^{+}(\tau)}\right)\left(\frac{1}{\tau-z}\right) d \tau\right)
$$

Since

$$
\begin{align*}
\Phi^{+}(s)= & \frac{g(s, u(s))}{2[A(s, u(s))-B(s, u(s))]} \\
& +\frac{X^{+}(s)}{2 \pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{[A(\nu, u(v))-B(\nu, u(\nu))](\nu-s) X^{+}(\nu)}\right) d \nu \tag{2.16}
\end{align*}
$$

and

$$
\begin{align*}
\Phi^{-}(s)= & \frac{-X^{-}(s) g(s, u(s))}{2 X^{+}(s)[A(s, u(s))-B(s, u(s))]} \\
& +\frac{X^{-}(s)}{2 \pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{[A(\nu, u(v))-B(\nu, u(\nu))](\nu-s) X^{+}(\nu)}\right) d \nu . \tag{2.17}
\end{align*}
$$

Then
$\Phi^{+}(s)-\Phi^{-}(s)=\frac{g(s, u(s))}{2[A(s, u(s))-B(s, u(s))]}\left(1+\frac{X^{-}(s)}{X^{+}(s)}\right)$

$$
\begin{equation*}
+\frac{\left(X^{+}(s)-X^{-}(s)\right)}{2 \pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{[A(\nu, u(v))-B(\nu, u(\nu))](\nu-s) X^{+}(\nu)}\right) d \nu . \tag{2.18}
\end{equation*}
$$

From (2.12) and (2.14), we have

$$
\begin{equation*}
X^{+}(s)-X^{-}(s)=\left(\frac{2 B(s, u(s))}{\sqrt{A^{2}(s, u(s))-B^{2}(s, u(s))}}\right) \exp (M(s)) \tag{2.19}
\end{equation*}
$$

From (2.10), (2.12), (2.18) and (2.19), we obtain

$$
\begin{equation*}
u^{\prime}(s)=\frac{A(s, u(s)) g(s, u(s))}{f^{2}(s, u(s))}+\frac{B(s, u(s)) \exp (M(s))}{\pi i f(s, u(s))} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu \tag{2.20}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu(\nu, u(\nu))=X^{+}(\nu)(A(\nu, u(\nu))-B(\nu, u(\nu)) \tag{2.21}
\end{equation*}
$$

$$
f(s, u(s))=\sqrt{A^{2}(s, u(s))-B^{2}(s, u(s))} .
$$

Integrating the expression (2.20), and applying the initial condition (1.2), we can see that our original problem turns to a fixed point equation for $u(s)$ as follows:

$$
u(s)=\int_{r}^{s}\left\{\frac{A(\sigma, u(\sigma)) g(\sigma, u(\sigma))}{f^{2}(\sigma, u(\sigma))}\right.
$$

$$
\left.+\frac{B(\sigma, u(\sigma)) \exp (M(\sigma))}{\pi i f(\sigma, u(\sigma))} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-\sigma}\right) d \nu\right\} d \sigma .
$$

The above integral equation can be written in the operator form:

$$
\begin{equation*}
S u=u, \tag{2.22}
\end{equation*}
$$

where the operator $S$ is defined as follows:

$$
\begin{equation*}
(S u)(s)=\int_{r}^{s} T(\sigma, u(\sigma)) d \sigma \tag{2.23}
\end{equation*}
$$

with the kernel function

$$
\begin{align*}
T(s, u(s))= & \frac{A(s, u(s)) g(s, u(s))}{f^{2}(s, u(s))} \\
& +\frac{B(s, u(s)) \exp (M(s))}{\pi i f(s, u(s))} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu \tag{2.24}
\end{align*}
$$

where $s \in \Gamma$.
Lemma 2.1. The kernel function given by (2.24) is bounded for $p>1$ under the following conditions:
(i) $A^{2}(s, u(s))-B^{2}(s, u(s)) \geq 1$ for each $(s, u(s)) \in D$,
(ii) The functions $A(s, u(s)), B(s, u(s))$ and $g(s, u(s))$ satisfy Hölder-Lipischiz conditions stated in (2.5)-(2.7),
where $M(s)$ and $\mu(s, u(s))$ are defined by (2.2) and (2.13), respectively.
Proof. We shall estimate the kernel $T(s, u(s))$ for $p>1$, as follows:

$$
\begin{equation*}
\|T(s, u(s))\|_{p} \leq\left\|N_{1}(s)\right\|_{p}+\left\|N_{2}(s)\right\|_{p} \tag{2.25}
\end{equation*}
$$

where

$$
\begin{equation*}
N_{1}(s)=\frac{A(s, u(s)) g(s, u(s))}{f^{2}(s, u(s))} \tag{2.26}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{2}(s)=\frac{B(s, u(s)) \exp (M(s))}{\pi i f(s, u(s))} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu . \tag{2.27}
\end{equation*}
$$

By using condition (2.3) and from [7, 13], we have

$$
\begin{equation*}
\left\|N_{1}(s)\right\|_{p} \leq\|A(s, u(s))\|_{P_{1}}\|g(s, u(s))\|_{p_{2}}, \tag{2.28}
\end{equation*}
$$

where

$$
\begin{gather*}
p_{1}^{-1}+p_{2}^{-1}=p^{-1} \\
\|A(s, u(s))\|_{P_{1}}=\left(\int_{\Gamma}|A(s, u(s))|^{p_{1}} d s\right)^{p_{1}^{-1}} \tag{2.29}
\end{gather*}
$$

and

$$
\begin{equation*}
\|g(s, u(s))\|_{p_{2}}=\left(\int_{\Gamma}|g(s, u(s))|^{p_{2}} d s\right)^{p_{2}^{-1}} \tag{2.30}
\end{equation*}
$$

By using the conditions (2.5) and (2.7), we obtain

$$
\begin{equation*}
|A(s, u(s))| \leq \eta_{1} \text { with } \eta_{1}=\gamma_{1} R+\sigma_{1}, \sigma_{1}=\max _{s \in \Gamma}|A(s, 0)| \tag{2.31}
\end{equation*}
$$

Therefore, from (2.29) and (2.31), we have

$$
\begin{equation*}
\|A(s, u(s))\|_{P_{1}} \leq \eta_{1} L^{p_{1}^{-1}} \tag{2.32}
\end{equation*}
$$

where $L=|\Gamma|=\int_{\Gamma} d s$, is the length of $\Gamma$. Similarly,

$$
\begin{equation*}
|g(s, u(s))| \leq \eta_{2} \text { with } \eta_{2}=\gamma_{3} R+\sigma_{2}, \quad \sigma_{2}=\max _{s \in \Gamma}|g(s, 0)| . \tag{2.33}
\end{equation*}
$$

Therefore, from (2.30) and (2.33), we have

$$
\begin{equation*}
\|g(s, u(s))\|_{p_{2}} \leq \eta_{2} L^{p_{2}^{-1}} \tag{2.34}
\end{equation*}
$$

Substituting from (2.32) and (2.34) into (2.28), we have

$$
\begin{equation*}
\left\|N_{1}(s)\right\|_{p} \leq Q_{1} \equiv \text { const } \tag{2.35}
\end{equation*}
$$

where

$$
Q_{1}=\eta_{1} \eta_{2} L^{p^{-1}}
$$

Also, by using condition (2.3) and [13], we obtain

$$
\begin{align*}
\left\|N_{2}(s)\right\|_{p} & \leq\left\|B(s, u(s)) \exp (M(s))\left(\frac{1}{\pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu\right)\right\|_{p} \\
& \leq\|B(s, u(s))\|_{q_{1}}\|\exp (M(s))\|_{q_{2}}\left\|\frac{1}{\pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu\right\|_{q_{3}} \tag{2.36}
\end{align*}
$$

where $q_{1}^{-1}+q_{2}^{-1}+q_{3}^{-1}=p^{-1}$. To estimate a bound for the norm given in (2.36), we carry out the investigation in many steps:
(a) Estimation of $\|B(s, u(s))\|_{q_{1}}$. As in (2.31) it is easy to see that

$$
\begin{equation*}
|B(s, u(s))| \leq \eta_{3}, \text { with } \eta_{3}=\gamma_{2} R+\sigma_{3}, \quad \sigma_{3}=\max _{s \in \Gamma}|B(s, 0)| \tag{2.37}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\|B(s, u(s))\|_{q_{1}} \leq \eta_{3} L^{q_{1}^{-1}} \tag{2.38}
\end{equation*}
$$

(b) Estimation of $\|\exp (M(s))\|_{q_{2}}$. Since [4],

$$
\|\exp (M(s))\|_{q_{2}} \leq\left(1+\|M(s)\|_{q_{2}}\right) \exp \left(\|M(s)\|_{q_{2}}\right)
$$

where from (2.13) and [5],

$$
\begin{equation*}
\|M(s)\|_{q_{2}}=\left\|\frac{1}{2 \pi i} \int_{\Gamma} \frac{\ln C(\tau)}{\tau-z} d \tau\right\|_{q_{2}} \leq \rho_{1}\|C(\tau)\|_{q_{2}} \leq \rho_{1}\left(\eta_{1}+\eta_{3}\right)^{2} L^{q_{2}^{-1}} \tag{2.39}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\|\exp (M(s))\|_{q_{2}} \leq\left(1+\rho_{1} \Lambda L^{q_{2}^{-1}}\right) \exp \left(\rho_{1} \Lambda L^{q_{2}^{-1}}\right) \tag{2.40}
\end{equation*}
$$

where

$$
\Lambda=\left(\eta_{1}+\eta_{3}\right)^{2}
$$

(c) Estimation of

$$
\left\|\frac{1}{\pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu\right\|_{q_{3}} .
$$

By using equations (2.14), (2.21) and condition (2.3), we get

$$
\begin{aligned}
\left\|\frac{1}{\pi i} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu\right\|_{q_{3}} & \leq \rho_{2} \frac{\|g(s, u(s))\|_{q_{3}}}{\|\exp (M(s))\|_{q_{3}}} \\
& \leq \rho_{2} \eta_{2} L^{q_{3}^{-1}}\left(\frac{1}{\|\exp (M(s))\|_{q_{3}}}\right) \\
& \leq \rho_{2} \eta_{2} L^{q_{3}^{-1}}
\end{aligned}
$$

From (2.38), (2.40) and (2.41) into (2.36), we obtain

$$
\begin{equation*}
\left\|N_{2}(s)\right\|_{p} \leq \Omega L^{p^{-1}}=Q_{2} \tag{2.42}
\end{equation*}
$$

where

$$
\Omega=\left[\rho_{2} \eta_{2} \eta_{3}\left(L^{-q_{2}^{-1}}+\rho_{1} \Lambda\right) \exp \left(\rho_{1} \Lambda L^{q_{2}^{-1}}\right)\right]
$$

Finally, substituting from (2.35) and (2.42) into (2.25), we have

$$
\begin{equation*}
\|T(s, u(s))\|_{p} \leq Q \tag{2.43}
\end{equation*}
$$

where $Q=Q_{1}+Q_{2} \equiv$ const.

## 3. Existence theorem

For non-negative constants $R$ and $\alpha_{u}$, let us define the following compact set:

$$
\begin{equation*}
K_{R, \alpha_{u}}^{0, \delta}=\left\{u \in C_{0}(\Gamma),|u| \leq R,\left|u\left(s_{2}\right)-u\left(s_{1}\right)\right| \leq \alpha_{u}\left|s_{2}-s_{1}\right|^{\delta}, s_{1}, s_{2} \in \Gamma\right\} \tag{3.1}
\end{equation*}
$$

where $C_{0}(\Gamma)$ is the space of all continuous functions $u(s)$ that are defined on $\Gamma$ such that $u(r)=0$. We are going to prove some assertions about the set $K_{R, \alpha_{u}}^{0, \delta}$, its image under the operator $S$, i.e., $S\left(K_{R, \alpha_{u}}^{0, \delta}\right)$ and the continuity of the operator $S$ defined in (2.23).

It is easy to see that the set $K_{R, \alpha_{u}}^{0, \delta}$ is a convex set. In fact, we are willing to show that the operator $S$ defined by (2.23) transforms the function $u$ to a function belongs to the class $K_{R, \alpha_{u}}^{0, \delta}$. Therefore, for each $u \in K_{R, \alpha_{u}}^{0, \delta}$, let us have

$$
\begin{equation*}
(S u)(s)=v(s), \quad v(s) \in C_{0}(\Gamma) \tag{3.2}
\end{equation*}
$$

where

$$
(S u)(s)=\int_{r}^{s} T(\sigma, u(\sigma)) d \sigma .
$$

Then, for any $s \in \Gamma$, we have from (2.23) and (2.43) that

$$
\begin{equation*}
|v(s)| \leq \int_{\Gamma}|T(\sigma, u(\sigma))| d \sigma \leq\|T(s, u(s))\|_{p} L^{k^{-1}} \leq R \tag{3.3}
\end{equation*}
$$

where $R=Q, L^{k^{-1}}$ and $p^{-1}+k^{-1}=1$. Now, we evaluate $\left|v\left(s_{2}\right)-v\left(s_{1}\right)\right|$ for $p>1$, $p^{-1}+k^{-1}=1, k<\delta^{-1}, s_{1}, s_{2} \in \Gamma$, we have

$$
\begin{aligned}
\left|v\left(s_{2}\right)-v\left(s_{1}\right)\right| & \leq\left|\int_{r}^{s_{1}} T(\sigma, u(\sigma)) d \sigma-\int_{r}^{s_{2}} T(\sigma, u(\sigma)) d \sigma\right| \\
& \leq\left|\int_{s_{1}}^{s_{2}} T(\sigma, u(\sigma)) d \sigma\right| \leq\|T(s, u)\|_{p}\left|s_{2}-s_{1}\right|^{k^{-1}}
\end{aligned}
$$

If $\|T(s, u)\|_{p} \leq Q \leq \alpha$, then all the transformed functions $v(s)$ belong to the set $K_{R, \alpha_{u}}^{0, \delta}$. Hence, the following lemma is valid.
Lemma 3.1. Let the functions $A(s, u(s)), B(s, u(s))$ and $g(s, u(s))$ satisfy the conditions (2.3)-(2.7). Then, for arbitrary $u \in K_{R, \alpha_{u}}^{0, \delta}$ and $Q \leq \alpha$, the transformed points $(S u)(s)=v(s)$ belong to the set $K_{R, \alpha_{u}}^{0, \delta}$.

In the following we will prove the continuity of the operator $S$ over $K_{R, \alpha_{u}}^{0, \delta}$.
Lemma 3.2. The operator $S$, defined in (3.2), which transforms the set $K_{R, \alpha_{u}}^{0, \delta}$ into itself is continuous.
Proof. Let $\left\{u_{n}\right\}_{n=1}^{\infty}$ be a sequence of elements of the set $K_{R, \alpha_{u}}^{0, \delta}$ converges uniformly to the element $u \in K_{R, \alpha_{u}}^{0, \delta}$. Let us assume that

$$
\begin{aligned}
& C_{n}(s)=C\left(s, u_{n}(s)\right)=\frac{A\left(s, u_{n}(s)\right)+B\left(s, u_{n}(s)\right)}{A\left(s, u_{n}(s)\right)-B\left(s, u_{n}(s)\right)} \\
& M_{n}(s)=M\left(s, u_{n}(s)\right)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{\ln C\left(\tau, u_{n}(\tau)\right)}{\tau-s} d \tau \\
& X_{n}^{ \pm}(s)=X^{ \pm}\left(s, u_{n}(s)\right), \mu\left(\nu, u_{n}(\nu)\right)=X_{n}^{+}(\nu)\left[A\left(\nu, u_{n}(\nu)\right)-B\left(\nu, u_{n}(\nu)\right)\right]
\end{aligned}
$$

and

$$
f\left(s, u_{n}(s)\right)=\sqrt{A^{2}\left(s, u_{n}(s)\right)-B^{2}\left(s, u_{n}(s)\right)} .
$$

We consider the following difference:

$$
\left|v_{n}(s)-v(s)\right| \leq \int_{r}^{s}\left|T\left(\sigma, u_{n}(\sigma)\right)-T(\sigma, u(\sigma))\right| d \sigma
$$

where

$$
\left|T\left(s, u_{n}(s)\right)-T(s, u(s))\right| \leq\left|\left(N_{n}^{1}\right)(s)\right|+\left|\left(N_{n}^{2}\right)(s)\right|
$$

such that

$$
\left(N_{n}^{1}\right)(s)=\frac{A\left(s, u_{n}(s)\right) g\left(s, u_{n}(s)\right)}{f^{2}\left(s, u_{n}(s)\right)}-\frac{A(s, u(s)) g(s, u(s))}{f^{2}(s, u(s))}
$$

and

$$
\begin{aligned}
\left(N_{n}^{2}\right)(s)= & \frac{B\left(s, u_{n}(s)\right) \exp \left(M_{n}(s)\right)}{\pi i f\left(s, u_{n}(s)\right)} \int_{\Gamma}\left(\frac{g\left(\nu, u_{n}(\nu)\right)}{\mu\left(\nu, u_{n}(\nu)\right)}\right)\left(\frac{1}{\nu-s}\right) d \nu \\
& -\frac{B(s, u(s)) \exp (M(s))}{\pi i f(s, u(s))} \int_{\Gamma}\left(\frac{g(\nu, u(\nu))}{\mu(\nu, u(\nu))}\right)\left(\frac{1}{\nu-s}\right) d \nu
\end{aligned}
$$

Now, we show that $\lim _{n \rightarrow \infty}\left|v_{n}(s)-v(s)\right|=0$. Since $\left\|u_{n}\right\|$ is uniformly bounded, using the conditions (2.3)-(2.7), we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty}\left|A\left(s, u_{n}(s)\right)-A(s, u(s))\right|=0 \\
& \lim _{n \rightarrow \infty}\left|B\left(s, u_{n}(s)\right)-B(s, u(s))\right|=0,
\end{aligned}
$$

and

$$
\lim _{n \rightarrow \infty}\left|g\left(s, u_{n}(s)\right)-g(s, u(s))\right|=0 .
$$

Therefore

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|\left(N_{n}^{1}\right)(s)\right|=0 \tag{3.4}
\end{equation*}
$$

In the following, we estimate $\left(N_{n}^{2}\right)(s)$, to carry out our investigation we consider

$$
\left|I_{n}\right|=\left|M\left(s, u_{n}(s)\right)-M(s, u(s))\right|,
$$

$$
\left|I_{n}\right| \leq \frac{1}{2}\left|\ln C\left(s, u_{n}(s)\right)-\ln C(s, u(s))\right|
$$

$$
+\left\lvert\, \frac{1}{2 \pi i} \int_{\Gamma}\left(\frac{\ln C\left(\tau, u_{n}(\tau)\right)-\ln C\left(s, u_{n}(s)\right)}{\tau-s}\right.\right.
$$

$$
\begin{equation*}
\left.-\frac{\ln C(\tau, u(\tau))-\ln C(s, u(s))}{\tau-s}\right) d \tau \mid . \tag{3.5}
\end{equation*}
$$

Since $A(s, u(s))$ and $B(s, u(s))$ are uniformly continuous, then

$$
\lim _{n \rightarrow \infty}\left|C\left(s, u_{n}(s)\right)-C(s, u(s))\right|=0 .
$$

Therefore,

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left|\sqrt{C\left(s, u_{n}(s)\right)}-\sqrt{C(s, u(s))}\right|=0,  \tag{3.6}\\
& \lim _{n \rightarrow \infty}\left|\ln C\left(s, u_{n}(s)\right)-\ln C(s, u(s))\right|=0, \tag{3.7}
\end{align*}
$$

and

$$
\lim _{n \rightarrow \infty}\left|f\left(s, u_{n}(s)\right)-f(s, u(s))\right|=0
$$

The difference in the right hand side of the inequality (3.5) tends uniformly to zero as $n$ tends to $\infty$.

To determine the integral in the same inequality (3.5), we draw a circle of center $z$ and radius $\omega$, so small that inside the circle lies the single arc of the curve $\Gamma$.

We decompose this integral into two parts $I_{n}^{\ell}(s)+I_{n}^{\Gamma-\ell}(s)$ where $\ell$ is the part of $\Gamma$ inside the circle and $\Gamma-\ell$ is the remaining part.

By using equation (2.12) and condition (2.3), we have

$$
|C(s, u(s))|=\left|\frac{[A(s, u(s))+B(s, u(s))]^{2}}{A^{2}(s, u(s))-B^{2}(s, u(s))}\right| \leq[A(s, u(s))+B(s, u(s))]^{2}
$$

Due to the continuity of the functions $A(s, u(s)), B(s, u(s))$ and from the inequalities (2.5), (2.6), we obtain

$$
\left|C\left(\tau, u_{n}(\tau)\right)-C\left(s, u_{n}(s)\right)\right| \leq \widetilde{\gamma}\left(1+\alpha_{u}\right)|\tau-s|^{\delta}
$$

where $\widetilde{\gamma}=2\left(\eta_{1}+\eta_{2}\right)\left(\gamma_{1}+\gamma_{2}\right)$.

Therefore, we get the following inequality

$$
\begin{aligned}
\left|\frac{1}{2 \pi i} \int_{\ell}\left(\frac{\ln C\left(\tau, u_{n}(\tau)\right)-\ln C\left(s, u_{n}(s)\right)}{\tau-s}\right) d \tau\right| & \leq \widetilde{\gamma}\left(1+\alpha_{u}\right) \int_{\ell}|\tau-s|^{\delta-1}|d \tau| \\
& \leq 2 \widetilde{\gamma} m\left(1+\alpha_{u}\right) \int_{0}^{\omega} t^{\delta-1} d t<\frac{\varepsilon}{3}
\end{aligned}
$$

where $\varepsilon$ is a small positive number, $t=|\tau-s|,|d \tau|=m d t, m$ is a positive constant (see [5]), and $\omega \leq\left(\delta \varepsilon / 6 \widetilde{\gamma} m\left(1+\alpha_{u}\right)\right)^{\delta^{-1}}$. Consequently,

$$
\left|\frac{1}{2 \pi i} \int_{\ell}\left(\frac{\ln C(\tau, u(\tau))-\ln C(s, u(s))}{\tau-s}\right) d \tau\right|<\frac{\varepsilon}{3} .
$$

Then

$$
\left|I_{n}^{\ell}(s)\right|<\frac{2 \varepsilon}{3}
$$

Since the point $s \in \ell$ lies outside the arc of integration $\Gamma-\ell$ and $I_{n}^{\Gamma-\ell}(s)$ are continuous, then we can select a positive integer $n$ such that the inequality

$$
\left|I_{n}^{\Gamma-\ell}(s)\right|<\frac{\varepsilon}{3}
$$

holds for each $n>J_{\varepsilon}$. Hence it follows that

$$
\left|I_{n}\right|<\varepsilon \text { for } n>J_{\varepsilon}
$$

and

$$
\lim _{n \rightarrow \infty}\left|M\left(s, u_{n}(s)\right)-M(s, u(s))\right|=0
$$

Therefore,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|\exp M\left(s, u_{n}(s)\right)-\exp M(s, u(s))\right|=0 \tag{3.8}
\end{equation*}
$$

From (2.14), (3.6) and (3.8), we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|X_{n}^{+}(s)-X^{+}(s)\right|=0 \tag{3.9}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|\mu\left(s, u_{n}(s)\right)-\mu(s, u(s))\right|=0 \tag{3.10}
\end{equation*}
$$

By using the properties of the convergent sequence, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mid\left(N_{n}^{2}(s) \mid=0\right. \tag{3.11}
\end{equation*}
$$

Hence, by using (3.4) and (3.11), we obtain

$$
\lim _{n \rightarrow \infty}\left|v_{n}(s)-v(s)\right|=0
$$

Then, the operator $S$ is continuous.
From the preceding lemmas and Arzela-Ascoli theorem [9], the image of $K_{R, \alpha_{u}}^{0, \delta}$ is compact, therefore we can use Schauder's fixed point theorem to show that the operator $S$ has at least one fixed point. Thus, we can state the main theorem as follows:

Theorem 3.1. The quasi-linear singular integro-differential equation with Cauchy kernel

$$
A(s, u(s)) u^{\prime}(s)-B(s, u(s)) \frac{1}{\pi i} \int_{\Gamma} \frac{u^{\prime}(\tau)}{\tau-s} d \tau=g(s, u(s))
$$

under the initial condition $u(r)=0$ where $\Gamma$ is a closed smooth contour and $r$ is a fixed point on $\Gamma$, has at least one solution $u$ in the space $W_{p}^{1}$, under the following conditions:
(a) Assume that the functions $A(s, u(s)), B(s, u(s))$ and $g(s, u(s))$ are continuous and positive on the region: $D=\{(s, u): s \in \Gamma,|u|<\infty\}$.
(b) $A^{2}(s, u(s))-B^{2}(s, u(s)) \geq 1$ for each $(s, u(s)) \in D$.
(c) (i) $\left|A\left(s_{2}, u_{2}\right)-A\left(s_{1}, u_{1}\right)\right| \leq \gamma_{1}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right)$,
(ii) $\left|B\left(s_{2}, u_{2}\right)-B\left(s_{1}, u_{1}\right)\right| \leq \gamma_{2}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right)$
(iii) $\left|g\left(s_{2}, u_{2}\right)-g\left(s_{1}, u_{1}\right)\right| \leq \gamma_{3}\left(\left|s_{1}-s_{2}\right|^{\delta}+\left|u_{1}-u_{2}\right|\right)$ where $\gamma_{i},(i=1,2,3)$ are positive constants and $0<\delta<1$.

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