BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **32**(3) (2009), 307–311

Nonexistence of Nontrivial Periodic Solutions to a Class of Nonlinear Differential Equations of Eighth Order

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Abstract. By constructing a Lyapunov function, a new result is given, which guarantees the non-existence of nontrivial periodic solutions to nonlinear vector differential equation of eighth order:

$$X^{(8)}+AX^{(7)}+BX^{(6)}+CX^{(5)}+DX^{(4)}+E\dddot{X}+F(\dot{X})\ddot{X}+G(X)\dot{X}+H(X)=0.$$

An example is also established for the illustrations of topic. By this way, our findings raise a new result for the nonexistence of nontrivial periodic solutions related to this nonlinear vector differential equation of eighth order.

2000 Mathematics Subject Classification: 34D05, 34D20

Key words and phrases: Nonlinear differential equation, eighth order, periodic solution, Lyapunov's second (or direct) method.

1. Introduction

By now, in the relevant literature, the instability of solutions for various nonlinear scalar and vector differential equations of eighth order has been considered only by a few authors, see, Tunç [3], Tunç and Tunç [4] and the references registered in these papers. In addition, in 1996, Iyase [2] proved a result on the nonexistence of nontrivial periodic solutions to nonlinear eighth order scalar differential equation:

$$x^{(8)}+a_1x^{(7)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+a_5\dddot{x}+f_6(\dot{x})\ddot{x}+f_7(x)\dot{x}+f_8(x)=0. \tag{1.1}$$

Now, for basic information, we consider the linear constant coefficient differential equation of eighth order:

$$x^{(8)}+ \quad a_1x^{(7)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+a_5\dddot{x}+a_6\ddot{x}+a_7\dot{x}+a_8x=0. \tag{1.2}$$

It is well-known that the auxiliary equation of (1.2) is given by:

$$\psi(\lambda)\equiv\lambda^8+a_1\lambda^7+a_2\lambda^6+a_3\lambda^5+a_4\lambda^4+a_5\lambda^3+a_6\lambda^2+a_7\lambda+a_8=0.$$

If β is an arbitrary real number, then the real part of $\psi(i\beta)$ is given by

$$\phi(\beta)=\beta^8-a_2\beta^6+a_4\beta^4-a_6\beta^2+a_8.$$

Received: July 1, 2008; *Revised:* June 10, 2009.

It is also well-known that if

$$a_2 \le 0, \quad a_4 \ge 0, \quad a_6 \le 0, \quad a_8 > 0$$

in which case $\phi(\beta) > 0$, then the auxiliary equation cannot have any purely imaginary root whatever. It therefore follows from general theory that equation (1.2) does not have a periodic solution except $x = 0$. An analogous consideration of the imaginary part of $\psi(i\beta)$ also leads to conditions on a_1, a_3, a_5 and a_7 for the nonexistence of any periodic solution of (1.2) other than $x = 0$.

Now, in this paper, we consider instead of scalar nonlinear differential equation given by (1.1), its nonlinear vector differential equation form:

$$X^{(8)}+AX^{(7)}+BX^{(6)}+CX^{(5)}+DX^{(4)}+E\dddot{X}+F(\dot{X})\ddot{X}+G(X)\dot{X}+H(X) = 0 \tag{1.3}$$

in the real Euclidean space $\mathbb{R}^n$ (with the usual norm denoted in what follows by $\|.\|$), where A, B, C, D and E are constant $n \times n$- symmetric matrices; F and G are continuous $n \times n$- symmetric matrix functions depending in each case on the arguments shown; $H : \mathbb{R}^n \to \mathbb{R}^n$, H is continuous and $H(0) = 0$.

Throughout this paper, instead of equation (1.3), we consider its equivalent differential system:

$$\begin{aligned} &\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = S, \quad \dot{S} = T, \quad \dot{T} = U, \quad \dot{U} = V, \quad \dot{V} = W, \\ &\dot{W} = -AW - BV - CU - DT - ES - F(Y)Z - G(X)Y - H(X), \end{aligned} \tag{1.4}$$

which was obtained as usual by setting $\dot{X} = Y$,$\ddot{X} = Z$, $X = S$, $X^{(4)} = T$,$X^{(5)} = U$, $X^{(6)} = V$, $X^{(7)} = W$ from (1.3).

Let $J_G(X)$ and $J_F(Y)$ denote the Jacobian matrices corresponding to the matrix functions $G(X)$ and $F(Y)$, that is,

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), \quad J_F(Y) = \left(\frac{\partial f_i}{\partial y_j}\right), \quad (i,\ j = 1,\ 2, ...,\ n),$$

where $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n)$, $(g_1, g_2, ..., g_n)$ and $(f_1, f_2, ..., f_n)$ are the components of X, Y, G and F, respectively. In addition, it is assumed that the Jacobian matrices $J_G(X)$ and $J_F(Y)$ exist and are continuous.

The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$ and $\lambda_i(A)$, $(i = 1,\ 2, ...,\ n)$, are the eigenvalues of the $n \times n$-matrix A. It is also well-known that a real symmetric matrix $A = (a_{ij}), (i,\ j = 1,\ 2, ...,\ n)$, is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the transpose of X.

Obviously, for the case $n = 1$, equation (1.3) reduces to equation (1.1). It should also be noted that according to our observations in the literature, it is not founded any paper on the nonexistence of nontrivial periodic solutions of nonlinear vector differential equations of eighth order. The present study is the first attempt to obtain sufficient conditions related to the nonexistence of nontrivial periodic solutions of nonlinear vector differential equations of eighth order. The motivation for the present study has come, especially, from the paper of Iyase [2], Tunç [5], Tunç [6] and the papers mentioned above. Our aim is to accomplish the result of Iyase [Theorem 1, 2] to the nonlinear vector differential equation of eighth order, (1.3).

We shall use the following well-known algebraic result to prove our main result.

Lemma 1.1. [1] *Let A be a real symmetric $n \times n$ matrix and $a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1, 2, ..., n)$, where a', a are constants. Then*

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle .$$

2. Main result

The main result is the following theorem.

Theorem 2.1. *Suppose that there are constants a_1, a_2, a_4 and $n \times n$- symmetric constant matrices A, B, D, $n \times n$- symmetric matrix functions F, G and n-vector function H such that the following conditions hold:*

$$\begin{aligned}
&\lambda_i(A) \geq a_1 > 0, \quad \lambda_i(B) \leq a_2 \leq 0, \quad \lambda_i(D) \geq a_4 \geq 0, \\
&\lambda_i(F(Y)) \leq 0 \text{ for all } Y \in \mathbb{R}^n, \quad H(0) = 0, \\
&H(X) \neq 0 \;\text{ if } X \neq 0, \; \sum_{i=1}^{n} x_i h_i(X) > 0 \text{ for all } X \in \mathbb{R}^n,
\end{aligned}$$

where $H(X) = (h_1(X), ..., h_n(X))$, $(i = 1,\ 2, ...,\ n)$. Then equation (1.3) has no non-trivial periodic solutions.

Note that there are no restrictions on C, E and G.

Proof. To prove the theorem, we introduce a Lyapunov function

$$V_1 = V_1(X, Y, Z, S, T, U, V, W)$$

defined as follows:

$$\begin{aligned}
V_1 = &- \langle X, W \rangle - \langle X, AV \rangle - \langle X, BU \rangle - \langle X, CT \rangle - \langle X, DS \rangle \\
&- \langle X, EZ \rangle + \langle Y, V \rangle + \langle Y, AU \rangle + \langle Y, BT \rangle + \langle Y, CS \rangle + \langle Y, DZ \rangle \\
&+ \frac{1}{2} \langle Y, EY \rangle - \langle Z, U \rangle - \langle Z, AT \rangle - \langle Z, BS \rangle - \frac{1}{2} \langle Z, AZ \rangle \\
&+ \langle S, T \rangle + \frac{1}{2} \langle S, AS \rangle - \int_0^1 \langle F(\sigma Y)Y, X \rangle \, d\sigma - \int_0^1 \langle \sigma G(\sigma X)X, X \rangle \, d\sigma.
\end{aligned} \tag{2.1}$$

Clearly, we have $V_1(0, 0, 0, 0, 0, 0, 0, 0) = 0$. Next, from the assumptions of theorem and the above lemma it follows that:

$$V_1(0, 0, 0, \varepsilon, 0, 0, 0, 0) = \frac{1}{2} \langle \varepsilon, A\varepsilon \rangle \geq \frac{1}{2} \langle \varepsilon, a_1\varepsilon \rangle = \frac{1}{2} a_1 \|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \in \mathbb{R}^n$, $\varepsilon \neq 0$. Thus, in every neighborhood of $(0, 0, 0, 0, 0, 0, 0, 0)$, there exists a point $(\xi, \eta, \zeta, \mu, \tau, \omega, \rho, \lambda)$ such that $V_1(\xi, \eta, \zeta, \mu, \tau, \omega, \rho, \lambda) > 0$ for arbitrary ξ, η, ζ, μ, τ, ω, ρ, λ in $\mathbb{R}^n$.

Now, let

$$(X, Y, Z, S, T, U, V, W) = (X(t), Y(t), Z(t), S(t), T(t), U(t), V(t), W(t))$$

be an arbitrary periodic solution of system (1.4). Differentiating the Lyapunov function (2.1) along this solution, we obtain

$$
\begin{aligned}
\dot{V}_1 =& \frac{d}{dt} V_1(X,Y,Z,S,T,U,V,W) \\
=& \langle T,T\rangle - \langle S,BS\rangle + \langle Z,DZ\rangle + \langle X,H(X)\rangle + \langle G(X)Y,X\rangle + \langle F(Y)X,Z\rangle \\
& - \frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y,X\rangle \, d\sigma - \frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X,X\rangle \, d\sigma.
\end{aligned} \tag{2.2}
$$

Recall that

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 \langle \sigma G(\sigma X)X,X\rangle \, d\sigma &= \int_0^1 \langle \sigma G(\sigma X)Y,X\rangle \, d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \sigma G(\sigma X)Y,X\rangle \, d\sigma \\
&= \sigma^2 \langle G(\sigma X)Y,X\rangle \big|_0^1 = \langle G(X)Y,X\rangle
\end{aligned} \tag{2.3}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 \langle F(\sigma Y)Y,X\rangle \, d\sigma &= \frac{d}{dt}\int_0^1 \langle F(\sigma Y)X,Y\rangle \, d\sigma \\
&= \int_0^1 \langle F(\sigma Y)X,Z\rangle \, d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle F(\sigma Y)X,Z\rangle \, d\sigma \\
&\quad + \int_0^1 \langle F(\sigma Y)Y,Y\rangle \, d\sigma \\
&= \sigma \langle F(\sigma Y)X,Z\rangle \big|_0^1 + \int_0^1 \langle F(\sigma Y)Y,Y\rangle \, d\sigma \\
&= \langle F(Y)X,Z\rangle + \int_0^1 \langle F(\sigma Y)Y,Y\rangle \, d\sigma.
\end{aligned} \tag{2.4}
$$

Now, combining estimates (2.3) and (2.4) into (2.2), we get

$$
\dot{V}_1 = \langle T,T\rangle - \langle S,BS\rangle + \langle Z,DZ\rangle + \langle X,H(X)\rangle - \int_0^1 \langle F(\sigma Y)Y,Y\rangle \, d\sigma.
$$

Since $\lambda_i(B) \le a_2 \le 0$, $\lambda_i(D) \ge a_4 \ge 0$, $\sum\limits_{i=1}^{n} x_i h_i(X) > 0$ and $\lambda_i(F(Y)) \le 0$, then it is clear that $V_1(t) \ge 0$ for all $t \ge 0$, which implies that $V_1(t)$ is monotone in t. Since V_1 is continuous and X, Y, Z, S, T ,U, V, W are periodic in t, $V_1(t)$ is bounded. Hence $\lim\limits_{t\to\infty} V_1(t) = V_0$ (constant).

By using the fact $V_1(t) = V_1(t + m\omega)$ for arbitrary fixed t and arbitrary m, ω, we have $V_1(t) = V_0$ for all t. Thus $\dot{V}_1 = 0$ for all t. This case necessarily implies that $Y = 0$ for all t, and therefore also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $S = \ddot{Y} = 0$, $T = \dddot{Y} = 0$, $U = Y^{(4)} = 0$, $V = Y^{(5)} = 0$, $W = Y^{(6)} = 0$, $Y^{(7)} = \dot{W} = 0$ for t. Substituting the estimates

$$X = \xi, \quad Y = Z = S = T = U = V = W = 0$$

in (1.4), it follows that $H(\xi) = 0$ which necessarily implies that $\xi = 0$ because of $H(0) = 0$ and $H(X) \neq 0$ when $X \neq 0$. Hence

$$X = Y = Z = S = T = U = V = W = 0$$

for all t. ∎

Example 2.1. As a special case of system (1.4), for the case $n = 2$, let us choose A, B, D, F and H as follows:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H(X) = \begin{bmatrix} x_1^3 + x_1^5 \\ x_2^3 + x_2^5 \end{bmatrix}, \quad F(Y) = \begin{bmatrix} -y_1^2 - y_2^2 & 0 \\ 0 & -y_1^2 - y_2^2 \end{bmatrix}.$$

Then, respectively, we have

$$\lambda_1(A) = 2, \quad \lambda_2(A) = 4, \lambda_1(B) = 0, \quad \lambda_2(B) = -2, \quad \lambda_1(D) = 0, \lambda_2(D) = 6,$$

$$\sum_{i=1}^{2} x_i F_i(x) = x_1^4 + x_1^6 + x_2^4 + x_2^6 > 0$$

for all $x_1 \neq 0$ and $x_2 \neq 0$, and

$$\lambda_1(F(Y)) = \lambda_2(F(Y)) = -y_1^2 - y_2^2 \leq 0.$$

Thus, all the conditions of Theorem 2.1 hold.

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