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An Iterative Approximation Method for a Countable Family of Nonexpansive Mappings in Hilbert Spaces

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Abstract. In this paper, to find a common element of the fixed point set of common fixed points of a countable family of nonexpansive mappings and the solution set of the variational inequality for α-inverse-strongly monotone, we introduce an iterative approximation method in a real Hilbert space. Then the strong convergence theorem is proved under some appropriate conditions imposed on the parameters. This result extended and improved the corresponding results of Yao and Yao [15] and many others.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. Let $B : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(B,C)$, is to find $x^* \in C$ such that

$$\langle Bx^*, v - x^*\rangle \geq 0$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [14, 16] and the references therein. We recall that a mapping $B : C \longrightarrow H$ is said to be:

(1) Monotone if $\langle Bu - Bv, u - v\rangle \geq 0, \quad \forall u, v \in C$.
(2) L-Lipschitz if there exists a constant $L > 0$ such that

$$\|Bu - Bv\| \leq L\|u - v\|, \quad \forall u, v \in C.$$

(3) α-inverse-strongly monotone [2, 5] if there exists a positive real number α such that

$$\langle Bu - Bv, u - v\rangle \geq \alpha\|Bu - Bv\|^2, \quad \forall u, v \in C.$$

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It is obvious that any α-inverse-strongly monotone mapping B is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S. An operator A is strongly positive on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \qquad \forall x \in H. \tag{1.1}$$

A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle u - v, w\rangle \leq 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [10]. For finding an element of $F(S) \cap VI(B, C)$, Takahashi and Toyoda [12] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Bx_n) \tag{1.2}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C, \{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(B, C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in F(S) \cap VI(B, C)$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping B of C into $\mathbb{R}^n$ is monotone and k-Lipschitz continuous and $VI(B, C)$ is nonempty, Korpelevich [4] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = u \in C \\ y_n = P_C(x_n - \lambda Bx_n) \\ x_{n+1} = P_C(x_n - \lambda By_n), \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\lambda \in (0, 1/k)$. He proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative process converge to the same point $z \in VI(B, C)$. Recently, Nadezhkina and Takahashi [8], Zeng and Yao [17] proposed some new iterative schemes for finding elements in $F(S) \cap VI(B, C)$. Recently, Iiduka and Takahashi [3] proposed another iterative scheme as following

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Bx_n), \quad n \geq 1 \end{cases} \tag{1.4}$$

where B is an α-cocoerceive map, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\lambda_n\} \subseteq (0, 2\alpha)$ satisfy some parameters controlling conditions. They showed that, if $F(S) \cap VI(B, C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to some $z \in F(S) \cap VI(B, C)$. By using this idea, Yao and Yao [15] gave the iterative scheme (1.5) below for finding an element of $F(S) \cap VI(B,C)$ under the assumption that a set $C \subseteq H$ is nonempty, closed and convex, a mapping $S : C \longrightarrow C$ is nonexpansive and a mapping $B : C \longrightarrow H$ is α-inverse-strongly-monotone:

$$\left\{ \begin{array}{l} x_1 = u \in C \\ y_n = P_C(x_n - \lambda_n B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n B y_n), \quad n \geq 1, \end{array} \right. \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that if $F(S) \cap VI(B,C) \neq \emptyset$ and and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to $q \in F(S) \cap VI(B,C)$.

On the other hand, Moudafi [6] introduced the viscosity approximation method for nonexpansive mappings (see [13] for further developments in both Hilbert and Banach spaces). Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n) T x_n + \sigma_n f(x_n), \quad n \geq 0, \tag{1.6}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved [6, 13] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.6) strongly converges to the unique solution q in C of the variational inequality

$$\langle (I - f) q, p - q \rangle \geq 0, p \in C.$$

Recently, Marino and Xu [7] introduced the following general iterative method:

$$x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0, x \in C \tag{1.8}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, to find a common fixed point of a countable family of nonexpansive mappings in Banach spaces, Aoyama *et al.* [1] introduced the following iterative sequence:

$$\left\{ \begin{array}{l} x_1 = x \in C \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) S_n x_n, \quad n \geq 1, \end{array} \right. \tag{1.9}$$

where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence of $[0,1]$, and $\{S_n\}$ is a sequence of nonexpansive mappings with some conditions. Then they proved that $\{x_n\}$ defined by (1.9) converges strongly to a common fixed point of $\{S_n\}$.

Inspired and motivated by the above research, we suggest and analyze a new iterative scheme for finding a common element of the fixed point set of common fixed points of a countable family of nonexpansive mappings and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in a real Hilbert space. Under some appropriate conditions imposed on the parameters, we obtain a strong convergence theorem for the sequence generated by the proposed method. The results of this paper extend and improve the results of Yao and Yao [15] and many others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\longrightarrow$, respectively.

A space X is said to satisfy Opials condition [9] if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \qquad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.2}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.3}$$

for all $x \in H, y \in C$. It is easy to see that the following is true:

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.4}$$

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.2. [11] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\longrightarrow\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then

$$\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0.$$

Lemma 2.3. [9] *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \longrightarrow C$ a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and if $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x = y$.*

Lemma 2.4. [13] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \quad n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$.
(2) $\limsup_{n\longrightarrow\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.5. [1, Lemma 3.2] *Let C be a nonempty closed subset of a Banach space and let $\{S_n\}$ be a sequence of mappings of C into itself. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_nz\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{S_ny\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by

$$Sy = \lim_{n\longrightarrow\infty} S_ny, \qquad \forall y \in C.$$

Then

$$\lim_{n\longrightarrow\infty} \sup\{\|Sz - S_nz\| : z \in C\} = 0.$$

Lemma 2.6. [7] *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

3. Main results

In this section, we prove the strong convergence theorem for a countable family of nonexpansive mappings in a real Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \longrightarrow C$ be a contraction with coefficient $\beta \in (0, 1)$, B an α-inverse-strongly monotone mapping of C into H and let $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $F := \cap_{n=1}^{\infty} F(S_n) \cap VI(B, C) \neq \emptyset$. Let A be a strongly bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \bar{\gamma}/\beta$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by*

$$\text{(3.1)} \qquad \begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda_n Bx_n) \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n \\ \qquad\qquad +((1 - \beta_n)I - \alpha_n A)S_nP_C(y_n - \lambda_n By_n), \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are the sequences in* $[0,1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *Suppose that* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* a, b *with* $0 < a < b < 2\alpha$ *satisfying*

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,

(iii) $\lim_{n\longrightarrow\infty} (\lambda_{n+1} - \lambda_n) = 0$.

Suppose that

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in D\} < \infty$$

for any bounded subset D *of* C. *Let* S *be a mapping of* C *into itself defined by* $Sy = \lim_{n\longrightarrow\infty} S_n y$ *for all* $y \in C$ *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges strongly to a point* $z \in F$ *which is the unique solution of the variational inequality*

$$\langle (A - \gamma f)z, z - x\rangle \leq 0, \quad x \in F. \tag{3.2}$$

Equivalently, we have $z = P_F(I - A + \gamma f)(z)$.

Proof. Note that from the condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.6, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. We will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. First, we show that $I - \lambda_n B$ is nonexpansive. For all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$,

$$\begin{aligned} \|(I - \lambda_n B)x - (I - \lambda_n B)y\|^2 &= \|(x - y) - \lambda_n (Bx - By)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n \langle x - y, Bx - By\rangle + \lambda_n^2 \|Bx - By\|^2 \\ &\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Bx - By\|^2, \end{aligned} \tag{3.3}$$

which implies that $I - \lambda_n B$ is nonexpansive. We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(S) \cap VI(B, C)$. Then $p = P_C(p - \lambda_n Bp)$. Setting $v_n = P_C(y_n - \lambda_n B y_n)$, we obtain from (3.3) that

$$\begin{aligned} \|v_n - p\| &= \|P_C(y_n - \lambda_n B y_n) - P_C(p - \lambda_n Bp)\| \\ &\leq \|(y_n - \lambda_n B y_n) - (p - \lambda_n Bp)\| \leq \|y_n - p\| \\ &= \|P_C(x_n - \lambda_n B x_n) - P_C(p - \lambda_n Bp)\| \\ &\leq \|(x_n - \lambda_n B x_n) - (p - \lambda_n Bp)\| \leq \|x_n - p\|. \end{aligned} \tag{3.4}$$

On the other hand, since A is a strongly positive bounded linear operator on C, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in C, \|x\| = 1\}.$$

For any x such that $\|x\| = 1$, we have

$$\begin{aligned} \langle ((1 - \beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n \langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n \|A\| \geq 0. \end{aligned}$$

This show that $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(1 - \beta_n)I - \alpha_n A\| &= \sup\{|\langle ((1 - \beta_n)I - \alpha_n A)x, x\rangle| : x \in C, \|x\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Ax, x\rangle : x \in C, \|x\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n \bar{\gamma}. \end{aligned}$$

It then follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(S_n v_n-p)\|\\ &\le (1-\beta_n-\alpha_n\bar{\gamma})\|v_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\|\\ &\le (1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma f(x_n)-Ap\|\\ &\le (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|\\ &\le (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\gamma\beta\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= (1-(\bar{\gamma}-\gamma\beta)\alpha_n)\|x_n-p\|+(\bar{\gamma}-\gamma\beta)\alpha_n\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\beta}.\end{aligned}$$

It follows from induction that

$$\|x_n-p\|\le \max\left\{\|x_1-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\beta}\right\},n\ge 1. \tag{3.5}$$

Hence $\{x_n\}$ is bounded, so are $\{v_n\},\{S_n v_n\}$, $\{f(x_n)\},\{By_n\}$ and $\{Bx_n\}$. Moreover, we observe that

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|P_C(y_{n+1}-\lambda_{n+1}By_{n+1})-P_C(y_n-\lambda_n By_n)\|\\ &\le \|(y_{n+1}-\lambda_{n+1}By_{n+1})-(y_n-\lambda_n By_n)\|\\ &= \|(y_{n+1}-\lambda_{n+1}By_{n+1})-(y_n-\lambda_{n+1}By_n)+(\lambda_n-\lambda_{n+1})By_n\|\\ &\le \|(y_{n+1}-\lambda_{n+1}By_{n+1})-(y_n-\lambda_{n+1}By_n)\|+|\lambda_n-\lambda_{n+1}|\|By_n\|\\ &\le \|y_{n+1}-y_n\|+|\lambda_n-\lambda_{n+1}|\|By_n\|\\ &= \|P_C(x_{n+1}-\lambda_{n+1}Bx_{n+1})-P_C(x_n-\lambda_n Bx_n)\|+|\lambda_n-\lambda_{n+1}|\|By_n\|\\ &\le \|(x_{n+1}-\lambda_{n+1}Bx_{n+1})-(x_n-\lambda_n Bx_n)\|+|\lambda_n-\lambda_{n+1}|\|By_n\|\\ &= \|(x_{n+1}-\lambda_{n+1}Bx_{n+1})-(x_n-\lambda_{n+1}Bx_n)+(\lambda_n-\lambda_{n+1})Bx_n\|\\ &\quad +|\lambda_n-\lambda_{n+1}|\|By_n\|\\ &\le \|(I-\lambda_{n+1}B)x_{n+1}-(I-\lambda_{n+1}B)x_n\|\\ &\quad +|\lambda_n-\lambda_{n+1}|(\|Bx_n\|+\|By_n\|)\\ &\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|(\|Bx_n\|+\|By_n\|). \qquad (3.6)\end{aligned}$$

Setting

$$z_n=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)S_n v_n}{1-\beta_n},$$

we have $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n, n\ge 1$. It follows that

$$\begin{aligned}z_{n+1}-z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)S_{n+1}v_{n+1}}{1-\beta_{n+1}}\\ &\quad -\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n A)S_n v_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)+S_{n+1}v_{n+1}-S_n v_n\\ &\quad +\frac{\alpha_n}{1-\beta_n}AS_n v_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}AS_{n+1}v_{n+1}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-AS_{n+1}v_{n+1})+\frac{\alpha_n}{1-\beta_n}(AS_n v_n-\gamma f(x_n))\end{aligned}$$

$$+\ S_{n+1}v_{n+1} - S_{n+1}v_n + S_{n+1}v_n - S_nv_n. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS_{n+1}v_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AS_nv_n\| + \|\gamma f(x_n)\|) \\
&\quad + \|S_{n+1}v_{n+1} - S_{n+1}v_n\| \\
&\quad + \|S_{n+1}v_n - S_nv_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS_{n+1}v_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AS_nv_n\| + \|\gamma f(x_n)\|) + \|v_{n+1} - v_n\| \\
&\quad + \|S_{n+1}v_n - S_nv_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS_{n+1}v_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AS_nv_n\| + \|\gamma f(x_n)\|) \\
&\quad + |\lambda_n - \lambda_{n+1}|(\|Bx_n\| + \|By_n\|) + \|S_{n+1}v_n - S_nv_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AS_{n+1}v_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|AS_nv_n\| + \|\gamma f(x_n)\|) \\
&\quad + |\lambda_n - \lambda_{n+1}|(\|Bx_n\| + \|By_n\|) \\
&\quad + \sup\{\|S_{n+1}v - S_nv\| : v \in \{v_n\}\}
\end{aligned} \tag{3.8}$$

which implies that (noting that (i), (ii), (iii))

$$\limsup_{n\longrightarrow\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 2.2, we obtain

$$\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0.$$

It then follows that

$$\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\| = \lim_{n\longrightarrow\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.9}$$

From (3.6) and (iii), we also have

$$\|v_{n+1} - v_n\| \longrightarrow 0$$

and

$$\begin{aligned}
\|y_{n+1} - y_n\| &\le \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_nAx_n)\| \\
&\le \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_{n+1} - Ax_n\| \longrightarrow 0.
\end{aligned}$$

Since

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(\gamma f(x_n) - Ax_n) + ((1-\beta_n)I - \alpha_nA)(S_nv_n - x_n)\| \\
&\le \alpha_n\|\gamma f(x_n) - Ax_n\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|S_nv_n - x_n\|,
\end{aligned}$$

this together with (3.9) implies that

$$\lim_{n\longrightarrow\infty}\|S_nv_n-x_n\|=0. \tag{3.10}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|((1-\beta_n)I-\alpha_nA)(S_nv_n-p)+\beta_n(x_n-p)+\alpha_n(\gamma f(x_n)-Ap)\|^2\\
&= \|((1-\beta_n)I-\alpha_nA)(S_nv_n-p)+\beta_n(x_n-p)\|^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad + 2\beta_n\alpha_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad + 2\alpha_n\langle((1-\beta_n)I-\alpha_nA)(S_nv_n-p),\gamma f(x_n)-Ap\rangle\\
&\le ((1-\beta_n-\alpha_n\bar{\gamma})\|S_nv_n-p\|+\beta_n\|x_n-p\|)^2+\alpha_n^2\|\gamma f(x_n)-Ap\|^2\\
&\quad + 2\beta_n\alpha_n\langle x_n-p,\gamma f(x_n)-Ap\rangle\\
&\quad + 2\alpha_n\langle((1-\beta_n)I-\alpha_nA)(S_nv_n-p),\gamma f(x_n)-Ap\rangle\\
&\le ((1-\beta_n-\alpha_n\bar{\gamma})\|v_n-p\|+\beta_n\|x_n-p\|)^2+c_n\\
&= (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|v_n-p\|\|x_n-p\|+c_n\\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad +(1-\beta_n-\alpha_n\bar{\gamma})\beta_n(\|v_n-p\|^2+\|x_n-p\|^2)+c_n\\
&= [(1-\alpha_n\bar{\gamma})^2-2(1-\alpha_n\bar{\gamma})\beta_n+\beta_n^2]\|v_n-p\|^2+\beta_n^2\|x_n-p\|^2\\
&\quad + ((1-\alpha_n\bar{\gamma})\beta_n-\beta_n^2)(\|v_n-p\|^2+\|x_n-p\|^2)+c_n\\
&= (1-\alpha_n\bar{\gamma})^2\|v_n-p\|^2-(1-\alpha_n\bar{\gamma})\beta_n\|v_n-p\|^2\\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n\\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n-p\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n,\\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})[\|(x_n-\lambda_nBx_n)-(p-\lambda_nBp)\|^2]\\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n,\\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})[\|x_n-p\|^2+\lambda_n(\lambda_n-2\alpha)\|Bx_n-Bp\|^2]\\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n-p\|^2+c_n,\\
&\le \|x_n-p\|^2+b(b-2\alpha)\|Bx_n-Bp\|^2+c_n \qquad (3.11)
\end{aligned}$$

where

$$\begin{aligned}
c_n &= \alpha_n^2\|\gamma f(x_n)-Av\|^2+2\beta_n\alpha_n\langle x_n-v,\gamma f(x_n)-Av\rangle\\
&\quad + 2\alpha_n\langle((1-\beta_n)I-\alpha_nA)(S_nv_n-v),\gamma f(x_n)-Av\rangle. \qquad (3.12)
\end{aligned}$$

This implies that

$$\begin{aligned}
-b(b-2\alpha)\|Bx_n-Bp\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n\\
&\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+c_n.
\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}c_n=0$ and from (3.9), we obtain

$$\lim_{n\longrightarrow\infty}\|Bx_n-Bp\|=0. \tag{3.13}$$

From (2.1), we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \|P_C(x_n - \lambda_n Bx_n) - P_C(p - \lambda_n Bp)\|^2 \\
&\le \langle (x_n - \lambda_n Bx_n) - (p - \lambda_n Bp), y_n - p\rangle \\
&= \frac{1}{2}\{\|(x_n - \lambda_n Bx_n) - (p - \lambda_n Bp)\|^2 + \|y_n - p\|^2 \\
&\quad - \|(x_n - \lambda_n Bx_n) - (p - \lambda_n Bp) - (y_n - p)\|^2\} \\
&\le \frac{1}{2}\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|(x_n - y_n) - \lambda_n(Bx_n - Bp)\|^2\} \\
&= \frac{1}{2}\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, Bx_n - Bp\rangle \\
&\quad - \lambda_n^2\|Bx_n - Bp\|^2\}.
\end{aligned}
$$

So, we obtain

$$
\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\langle x_n - y_n, Bx_n - Bp\rangle - \lambda_n^2\|Bx_n - Bp\|^2.
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - p\|^2 + (1 - \alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - p\|^2 + (1 - \alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})[\|x_n - p\|^2 - \|x_n - y_n\|^2 \\
&\quad + 2\lambda_n\langle x_n - y_n, Bx_n - Bp\rangle - \lambda_n^2\|Bx_n - Bp\|^2] \\
&\quad + (1 - \alpha_n\bar{\gamma})\beta_n\|x_n - p\|^2 + c_n \\
&\le (1 - \alpha_n\bar{\gamma})\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - y_n\|^2 \\
&\quad + 2\lambda_n(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - y_n\|\|Bx_n - Bp\| \\
&\quad - \lambda_n^2(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|Bx_n - Bp\|^2 + c_n,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - y_n\|^2 \\
&\qquad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad\quad + 2\lambda_n(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - y_n\|\|Bx_n - Bp\| \\
&\qquad\quad - \lambda_n^2(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|Bx_n - Bp\|^2 + c_n \\
&\qquad \le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad\quad + 2\lambda_n(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - y_n\|\|Bx_n - Bp\| \\
&\qquad\quad - \lambda_n^2(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|Bx_n - Bp\|^2 + c_n.
\end{aligned}
$$

Applying (3.9), (3.13) and $\lim_{n\longrightarrow\infty} c_n = 0$ to the last inequality, we obtain that

$$
\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0 \tag{3.14}
$$

It follows that

$$
\begin{aligned}
\|S_n v_n - v_n\| &\le \|S_n v_n - x_n\| + \|x_n - y_n\| + \|y_n - v_n\| \\
&= \|S_n v_n - x_n\| + \|x_n - y_n\| + \|P_C(x_n - \lambda_n Bx_n) - P_C(y_n - \lambda_n By_n)\|
\end{aligned}
$$

$$\leq \|S_n v_n - x_n\| + \|x_n - y_n\| + \|(x_n - \lambda_n B x_n) - (y_n - \lambda_n B y_n)\|$$

$$\leq \|S_n v_n - x_n\| + 2\|x_n - y_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.15}$$

This implies that

$$\|v_n - y_n\| \leq \|v_n - S_n v_n\| + \|S_n v_n - x_n\| + \|x_n - y_n\| \longrightarrow 0, n \longrightarrow \infty. \tag{3.16}$$

Applying Lemma 2.5 and (3.15), we have

$$\begin{aligned}\|Sv_n - v_n\| &\leq \|Sv_n - S_n v_n\| + \|S_n v_n - v_n\| \\ &\leq \sup\{\|Sv - S_n v\| : v \in \{v_n\}\} + \|S_n v_n - v_n\| \longrightarrow 0\end{aligned}$$

Observe that $P_F(I - A + \gamma f)$ is a contraction of C into itself. Indeed, for all $x, y \in C$, we have

$$\begin{aligned}&\|P_F(I - A + \gamma f)(x) - P_F(I - A + \gamma f)(y)\| \\ &\qquad \leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\ &\qquad \leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\ &\qquad \leq (1 - \bar{\gamma})\|x - y\| + \gamma\beta\|x - y\| \\ &\qquad = (1 - (\bar{\gamma} - \gamma\beta))\|x - y\|.\end{aligned}$$

Since H is complete, there exists a unique element $z \in C$ such that

$$z = P_F(I - A + \gamma f)(z).$$

Next, we show that

$$\limsup_{n \longrightarrow \infty} \langle (A - \gamma f)z, z - x_n \rangle \leq 0. \tag{3.17}$$

We choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\lim_{i \longrightarrow \infty} \langle (A - \gamma f)z, z - v_{n_i} \rangle = \limsup_{n \longrightarrow \infty} \langle (A - \gamma f)z, z - v_n \rangle.$$

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup w$. From $\|Sv_n - v_n\| \longrightarrow 0$, we obtain $Sv_{n_i} \rightharpoonup w$. Next, we show that $w \in F$. First, we show that

$$w \in F(S) = \cap_{n=1}^{\infty} F(S_n).$$

Assume $w \notin F(S)$. Since $v_{n_i} \rightharpoonup w$ and $w \neq Sw$, it follows by the Opial's condition that

$$\begin{aligned}\liminf_{i \longrightarrow \infty} \|v_{n_i} - w\| &< \liminf_{i \longrightarrow \infty} \|v_{n_i} - Sw\| \\ &\leq \liminf_{i \longrightarrow \infty} \{\|v_{n_i} - Sv_{n_i}\| + \|Sv_{n_i} - Sw\|\} \\ &< \liminf_{i \longrightarrow \infty} \|v_{n_i} - w\|\end{aligned}$$

which derives a contradiction. Thus, we have $w \in F(S) = \cap_{n=1}^{\infty} F(S_n)$. By the same argument as that in the proof of [8], we can show that $w \in VI(B, C)$. Hence $w \in F$. Since $z = P_F(I - A + \gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle (A-\gamma f)z, z-x_n\rangle &= \limsup_{n\longrightarrow\infty}\langle (A-\gamma f)z, z-v_n\rangle\\ &= \lim_{i\longrightarrow\infty}\langle (A-\gamma f)z, z-v_{n_i}\rangle\\ &= \langle (A-\gamma f)z, z-w\rangle \le 0.\end{aligned}\tag{3.18}$$

It follows from the last inequality, (3.10), (3.14) and (3.16) that

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(z)-Az, S_nv_n-z\rangle \le 0.\tag{3.19}$$

Finally, we prove $x_n \longrightarrow z$ as $n \longrightarrow \infty$. To this end, we calculate

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)S_nv_n-z\|^2\\ &= \|((1-\beta_n)I-\alpha_nA)(S_nv_n-z)+\beta_n(x_n-z)+\alpha_n(\gamma f(x_n)-Az)\|^2\\ &= \|((1-\beta_n)I-\alpha_nA)(S_nv_n-z)+\beta_n(x_n-z)\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\ &\quad+\ 2\beta_n\alpha_n\langle x_n-z, \gamma f(x_n)-Az\rangle\\ &\quad+\ 2\alpha_n\langle ((1-\beta_n)I-\alpha_nA)(S_nv_n-z), \gamma f(x_n)-Az\rangle\\ &\le ((1-\beta_n-\alpha_n\bar{\gamma})\|S_nv_n-z\|+\beta_n\|x_n-z\|)^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\ &\quad+\ 2\beta_n\alpha_n\gamma\langle x_n-z, f(x_n)-f(z)\rangle+2\beta_n\alpha_n\langle x_n-z, \gamma f(z)-Az\rangle\\ &\quad+\ 2(1-\beta)\gamma\alpha_n\langle S_nv_n-z, f(x_n)-f(z)\rangle\\ &\quad+\ 2(1-\beta)\alpha_n\langle S_nv_n-z, \gamma f(z)-Az\rangle\\ &\quad-\ 2\alpha_n^2\langle A(S_nv_n-z), \gamma f(z)-Az\rangle\\ &\le ((1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|)^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\ &\quad+\ 2\beta_n\alpha_n\gamma\beta\|x_n-z\|^2+2\beta_n\alpha_n\langle x_n-z, \gamma f(z)-Az\rangle\\ &\quad+\ 2(1-\beta_n)\gamma\alpha_n\beta\|x_n-z\|^2+2(1-\beta)\alpha_n\langle S_nv_n-z, \gamma f(z)-Az\rangle\\ &\quad-\ 2\alpha_n^2\langle A(S_nv_n-z), \gamma f(z)-Az\rangle\\ &= [(1-\alpha_n\bar{\gamma})^2+2\beta_n\alpha_n\gamma\beta+2(1-\beta_n)\gamma\alpha_n\beta]\|x_n-z\|^2\\ &\quad+\ \alpha_n^2\|\gamma f(x_n)-Az\|^2+\ 2\beta_n\alpha_n\langle x_n-z, \gamma f(z)-Az\rangle\\ &\quad+\ 2(1-\beta)\alpha_n\langle S_nv_n-z, \gamma f(z)-Az\rangle-\ 2\alpha_n^2\langle A(S_nv_n-z), \gamma f(z)-Az\rangle\\ &\le [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-z\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\ &\quad+\ 2\beta_n\alpha_n\langle x_n-z, \gamma f(z)-Az\rangle+2(1-\beta)\alpha_n\langle S_nv_n-z, \gamma f(z)-Az\rangle\\ &\quad+\ 2\alpha_n^2\|A(S_nv_n-z)\|\|\gamma f(z)-Az\|\\ &= [1-2(\bar{\gamma}-\alpha\gamma)\alpha_n]\|x_n-z\|^2+\alpha_n\{\alpha_n(\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2\\ &\quad+\ 2\|A(S_nv_n-z)\|\|\gamma f(z)-Az\|)+2\beta_n\langle x_n-z, \gamma f(z)-Az\rangle\\ &\quad+\ 2(1-\beta)\langle S_nv_n-z, \gamma f(z)-Az\rangle\}.\end{aligned}\tag{3.20}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{S_nv_n\}$ are bounded, we can take a constant $M>0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2+2\|A(S_nv_n-z)\|\gamma f(z)-Az\|\le M,$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - z\|^2 \leq [1 - 2(\bar{\gamma} - \alpha\gamma)\alpha_n]\|x_n - z\|^2 + \alpha_n \sigma_n, \tag{3.21}$$

where

$$\sigma_n = 2\beta_n \langle x_n - z, \gamma f(z) - Az\rangle + 2(1-\beta)\langle S_n v_n - z, \gamma f(z) - Az\rangle + \alpha_n M.$$

Using (i), (3.18) and (3.19), we get $\limsup_{n\longrightarrow\infty} \sigma_n \leq 0$. Now applying Lemma 2.4 to (3.21), we conclude that $x_n \longrightarrow z$. ∎

If $A = I, \gamma \equiv 1$, $\gamma_n = 1 - \alpha_n - \beta_n$, $S_n = S$ and $f := u$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.1. [15, Theorem 3.1]) *Let C be a closed convex subset of a real Hilbert space H. Let B be an $\alpha-$inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(B,C) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n B x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda_n B y_n), \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$,
(iv) $\lim_{n\longrightarrow\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(B,C)}u$.

Remark 3.1. As in [1, Theorem 4.1], we can generate a sequence $\{S_n\}$ of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z : z \in D\|\} < \infty$ for any bounded subset D of C by using convex combination of a general sequence $\{T_k\}$ of nonexpansive mappings with a common fixed point.

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