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On Chromatic Uniqueness of a Family of K_4-Homeomorphs

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Abstract. In this paper, we study the chromatic uniqueness of one family of K_4-homeomorphs with girth 10.

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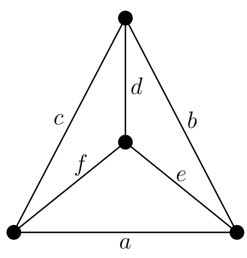
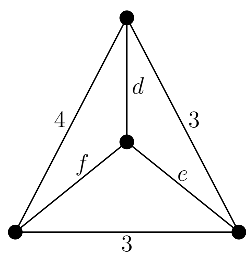
1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G, \lambda)$ (or simply $P(G)$) denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$ (or simply $P(G) = P(H)$). A graph G is chromatically unique (or simply χ-unique) if for any graph H such that $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. For examples of some chromatically unique graphs, the reader can refer to [12, 13, 14].



(a) $K_4(a, b, c, d, e, f)$



(b) $K_4(3, 3, 4, d, e, f)$

Figure 1. K_4-Homeomorphs

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A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of K_4 are replaced by the six paths of length a, b, c, d, e, f, respectively, as shown in Figure 1(a). So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 9$ has been studied by many authors (see [2, 4, 5, 6]). Recently, Peng in [7] has studied the chromaticity of one type of K_4-homeomorphs with girth 7, that is the chromaticity of $K_4(1, 3, 3, d, e, f)$. In the whole study of K_4-homeomorphs with girth 10, we need to consider 24 types of K_4-homeomorphs. In this paper, we discuss the chromaticity of one of these types, namely $K_4(3, 3, 4, d, e, f)$ (as shown in Figure 1(b)), where d, e, f are at least 3. The chromaticity of the other types of K_4-homeomorphs with girth 10 will be presented in other papers.

2. Preliminary results

In this section, we give some known results used in the sequel.

Lemma 2.1. *Assume that G and H are χ-equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (*see* [3])*;*
(2) *G and H have the same girth and same number of cycles with length equal to their girth* (*see* [11])*;*
(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph* (*see* [1])*;*
(4) *Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then*
  (i) $\min\{a, b, c, d, e, f\} = \min\{a', b', c', d', e', f'\}$ *and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$* (*see* [10])*;*
  (ii) *if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$* (*see* [4])*.*

Lemma 2.2. [8] *Let $G = K_4(a, b, c, d, e, f)$* (*see Figure 1*) *when exactly three of a, b, c, d, e, f are the same. Then G is not chromatically unique if and only if G is isomorphic to $K_4(r, r, r-2, 1, 2, r)$ or $K_4(r, r-2, r, 2r-2, 1, r)$ or $K_4(t, t, 1, 2t, t+2, t)$ or $K_4(t, t, 1, 2t, t-1, t)$ or $K_4(t, t+1, t, 2t+1, 1, t)$ or $K_4(1, t, 1, t+1, 3, 1)$ or $K_4(1, 1, t, 2, t+2, 1)$, where $r \geq 3$, $t \geq 2$.*

Lemma 2.3. [4, 9] *The graph $K_4(a, b, c, d, e, f)$ is χ-unique if exactly four numbers among a,b,c,d,e,f are the same.*

Lemma 2.4. [10] *The graph $K_4(a, b, c, d, e, f)$ is χ-unique if the positive integers a,b,c,d,e,f assume no more than two distinct values.*

3. Main result

Lemma 3.1. *Let $G \cong K_4(3, 3, 4, d, e, f)$ and $H \cong K_4(3, 3, 4, d', e', f')$, then*

(1) $P(G) = (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^5 - 3s^4 - 2s^3 + s^2 + 3s + 2 + R(G)]$, *where*

$$R(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+3} + s^{f+3} + s^{e+4} + s^{e+6} + s^{d+7} + s^{f+7} + s^{d+e+f},$$

$s = 1 - \lambda$, x is the number of the edges of G.

(2) *If* $P(G) = P(H)$, *then* $R(G) = R(H)$.

Proof. (1) Let $s = 1 - \lambda$. From [10], the chromatic polynomial of K_4-homeomorph $K_4(a,b,c,d,e,f)$ is as follows:

$$\begin{aligned} P(K_4(a,b,c,d,e,f) = (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2) \\ - (s+1)(s^a+s^b+s^c+s^d+s^e+s^f) \\ + (s^{a+d}+s^{b+f}+s^{c+e}+s^{a+b+e}+s^{b+d+c} \\ + s^{a+c+f}+s^{d+e+f}-s^{x-1})]. \end{aligned}$$

So when $a = 3$, $b = 3$ and $c = 4$, we have

$$\begin{aligned} P(K_4(3,3,4,d,e,f) &= (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2) \\ &\quad - (s+1)(s^3+s^3+s^4+s^d+s^e+s^f) \\ &\quad + (s^{d+3}+s^{f+3}+s^{e+4}+s^{e+6}+s^{d+7} \\ &\quad + s^{f+7}+s^{d+e+f}-s^{x-1})] \\ &= (-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^5-3s^4-2s^3 \\ &\quad + s^2+3s+2-s^d-s^e-s^f-s^{d+1}-s^{e+1}-s^{f+1} \\ &\quad + s^{d+3}+s^{f+3}+s^{e+4}+s^{e+6}+s^{d+7}+s^{f+7}+s^{d+e+f}] \\ &= (-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^5-3s^4-2s^3 \\ &\quad + s^2+3s+2+R(G)], \end{aligned}$$

where

$$\begin{aligned} R(G) = -s^d-s^e-s^f-s^{d+1}-s^{e+1}-s^{f+1}+s^{d+3}+s^{f+3}+s^{e+4} \\ + s^{e+6}+s^{d+7}+s^{f+7}+s^{d+e+f} \end{aligned}$$

as required.

(2) If $P(G) = P(H)$, then we can clearly see that $R(G) = R(H)$. ∎

Theorem 3.1. *K_4-homeomorph $K_4(3,3,4,d,e,f)$ with girth 10, where d,e,f are at least 3, is χ-unique.*

Proof. Let $G \cong K_4(3,3,4,d,e,f)$ with girth 10, that is, $d+e \geq 7$, $e+f \geq 7$, where $\min\{d,e,f\} \geq 3$. If there exists a graph H such that $P(G,\lambda) = P(H,\lambda)$, then from Lemma 2.1(3 and 4(ii)), H must be a K_4-homeomorph $K_4(a',b',c',d',e',f')$ with $\min\{a',b',c',d',e',f'\} = 3$ and the number of times that 3 occurs in the list $\{3,3,4,d,e,f\}$ is equal to the number of times that 3 occurs in the list $\{a',b',c',d',e',f'\}$. By Lemma 2.1(2), H has girth 10. Thus, H must be of the type $K_4(3,3,4,d',e',f')$, where $\min\{d',e',f'\} \geq 3$ and $d'+e' \geq 7$, $e'+f' \geq 7$. If one of $\{d',e',f'\}$ is equal to 3, then by Lemma 2.2, H is χ-unique. Since $G \sim H$, we have $G \cong H$. If two of $\{d',e',f'\}$ are equal to 3, then by Lemma 2.3, H is χ-unique. Since $G \sim H$, we have $G \cong H$. If all of $\{d',e',f'\}$ are equal to 3, then by Lemma 2.4, H is χ-unique. Since $G \sim H$, we have $G \cong H$. If two of $\{d',e',f'\}$ are equal to 4, then by Lemma 2.2, H is χ-unique. Since $G \sim H$, we have $G \cong H$. If all of $\{d',e',f'\}$ are equal to 4, then by Lemma 2.3 and Lemma 2.4, H is χ-unique. Since $G \sim H$, we have $G \cong H$. Let us find G and H which are not isomorphic, where $\min\{d',e',f'\} \geq 4$ and no two paths among $\{d',e',f'\}$ can have length equal to 4. Without loss of generality, let $d \leq f$ and $d' \leq f'$. Let us now solve the equation $R(G) = R(H)$. From Lemma 3.1, we have

$$R(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+3} + s^{f+3} + s^{e+4} + s^{e+6} + s^{d+7} + s^{f+7} + s^{d+e+f}$$

and

$$R(H) = -s^{d'} - s^{e'} - s^{f'} - s^{d'+1} - s^{e'+1} - s^{f'+1} + s^{d'+3} + s^{f'+3} + s^{e'+4} + s^{e'+6} + s^{d'+7} + s^{f'+7} + s^{d'+e'+f'}.$$

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1(1), $d+e+f = d'+e'+f'$. We obtain the following after simplification: (Note that our assumption in the following steps of the proof is $R_j(G) = R_j(H)$, where $1 \leq j \leq 6$).

$$R_1(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+3} + s^{f+3} + s^{e+4} + s^{e+6} + s^{d+7} + s^{f+7}$$

and

$$R_1(H) = -s^{d'} - s^{e'} - s^{f'} - s^{d'+1} - s^{e'+1} - s^{f'+1} + s^{d'+3} + s^{f'+3} + s^{e'+4} + s^{e'+6} + s^{d'+7} + s^{f'+7}.$$

Let us consider the h.r.p. in $R_1(G)$ and the h.r.p. in $R_1(H)$. We have

$$\max\{e+6, f+7, d+7\} = \max\{e'+6, f'+7, d'+7\}.$$

Without loss of generality, we will consider only the following six cases.

Case 1. If $\max\{e+6, f+7, d+7\} = e+6$ and $\max\{e'+6, f'+7, d'+7\} = e'+6$, then $e = e'$. Thus, we can cancel the following pairs of terms in $R_1(G)$ and $R_1(H)$: $-s^e$ with $-s^{e'}$, $-s^{e+1}$ with $-s^{e'+1}$, s^{e+4} with $s^{e'+4}$ and s^{e+6} with $s^{e'+6}$. Since $d \leq f$ and $d' \leq f'$ the l.r.p. in $R_1(G)$ is d and the l.r.p. in $R_1(H)$ is d'. So, $d = d'$ and we have $e = e'$. Since $d+e+f = d'+e'+f'$, $f = f'$. Thus, we have $\{d,e,f\} = \{d',e',f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 2. If $\max\{e+6, f+7, d+7\} = d+7$ and $\max\{e'+6, f'+7, d'+7\} = d'+7$, then $d = d'$. Since $d \leq f$ and $d' \leq f'$, this case is only possible if $d = f$ and $d' = f' > 4$. So, we have $d = f = d' = f' > 4$. From $d+e+f = d'+e'+f'$, we have $e = e'$. Thus, $\{d,e,f\} = \{d',e',f'\}$ as multisets. By Lemma 2.1(4(ii)), $G \cong H$.

Case 3. If $\max\{e+6, f+7, d+7\} = f+7$ and $\max\{e'+6, f'+7, d'+7\} = f'+7$, then $f = f'$. We can deal with this case in the same way as in Case 1. So, we have $G \cong H$.

Case 4. If $\max\{e+6, f+7, d+7\} = e+6$ and $\max\{e'+6, f'+7, d'+7\} = d'+7$, then $e = d'+1$. Since $d' \leq f'$, $\max\{e'+6, f'+7, d'+7\}$ can only be equal to $d'+7$ if $d' = f'$. Thus, $d' = f' > 4$. Therefore, we have

$$d' = f' = e - 1 \tag{3.1}$$

from $d+e+f=d'+e'+f'$, we have

$$d+f-e=e'-2. \tag{3.2}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), $\min\{d,e,f\}=\min\{d',e',f'\}$. The following subcases need to be considered.

Subcase 4.1. If $\min\{d,e,f\}=d$ and $\min\{d',e',f'\}=d'$, then $d=d'$. We can consider this case the same way as in Case 1. So, $G\cong H$.

Subcase 4.2. If $\min\{d,e,f\}=d$ and $\min\{d',e',f'\}=e'$, then $d=e'$. From (3.2), we have $f=e-2$. Note that $d'=f'=e-1=f+1$ from (3.1). We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_2(G)=-s^d-s^{f+2}-s^f-s^{d+1}-s^{f+2}-s^{f+1}+s^{d+3}+s^{f+3}+s^{f+6}+s^{f+8}+s^{d+7}+s^{f+7}$$

$$R_2(H)=-s^{f+1}-s^d-s^{f+1}-s^{f+2}-s^{d+1}-s^{f+2}+s^{f+4}+s^{f+4}+s^{d+4}+s^{d+6}+s^{f+8}+s^{f+8}.$$

After simplifying $R_2(G)$ and $R_2(H)$, we have

$$R_3(G)=-s^f+s^{d+3}+s^{f+3}+s^{f+6}+s^{d+7}+s^{f+7}$$

$$R_3(H)=-s^{f+1}+s^{f+4}+s^{f+4}+s^{d+4}+s^{d+6}+s^{f+8}.$$

Clearly, this contradicts $R_3(G)=R_3(H)$.

Subcase 4.3. If $\min\{d,e,f\}=d$ and $\min\{d',e',f'\}=f'$, then $d=f'$. From (3.1) we have $d=e-1$. From (3.2), we have $e'=f+1$. From $R_1(G)=R_1(H)$, we have

$$R_4(G)=-s^d-s^{d+1}-s^f-s^{d+1}-s^{d+2}-s^{f+1}+s^{d+3}+s^{f+3}+s^{d+5}+s^{d+7}+s^{d+7}+s^{f+7}$$

$$R_4(H)=-s^d-s^{f+1}-s^d-s^{d+1}-s^{f+2}-s^{d+1}+s^{d+3}+s^{d+3}+s^{f+5}+s^{f+7}+s^{d+7}+s^{d+7}.$$

After simplifying the equations we get $d=f$. Thus, $d=d'=f'=f$. We also have $d+1=e=f+1=e'$. Therefore, $e=e'$, $d=d'$ and $f=f'$. So, $G\cong H$. (Note that an alternative solution here is considering $d=d'$ since $d'=f'$. Then this subcase is the same as Case 1 and the same result can be obtained, i.e., $G\cong H$.)

Subcase 4.4. If $\min\{d,e,f\}=e$ and $\min\{d',e',f'\}=d'$, then $e=d'$. This contradicts (3.1), $d'=e-1$. Also note that $\max\{e+6,f+7,d+7\}=e+6$. So, $e+6\geq d+7$, that is $e\geq d+1$, similarly, $e\geq f+1$. Therefore, $\min\{d,e,f\}\neq e$. This is a contradiction.

Subcase 4.5. If $\min\{d,e,f\}=f$ and $\min\{d',e',f'\}=d'$, then $f=d'$. We have $d'=f'$, thus $f=f'$. From $d+e+f=d'+e'+f'$, we get $e=e'$. So, $G\cong H$.

Subcase 4.6. If $\min\{d,e,f\}=f$ and $\min\{d',e',f'\}=e'$, then $f=e'$. Since $d\leq f$, the only possibility for $\min\{d,e,f\}=f$ is when $d=f$. Thus, we have $f=e'=d$. From (3.2), we have $d=e-2$. From (3.1), we get $d'=f'=e-1=d+1$. Note that $f=e'=d=e-2$. Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_5(G)=-s^d-s^{d+2}-s^d-s^{d+1}-s^{d+3}-s^{d+1}+s^{d+3}+s^{d+3}+s^{d+6}+s^{d+8}+s^{d+7}+s^{d+7}$$

$R_5(H) = -s^{d+1} - s^d - s^{d+1} - s^{d+2} - s^{d+1} - s^{d+2} + s^{d+4} + s^{d+4} + s^{d+4} + s^{d+6} + s^{d+8} + s^{d+8}$.
After cancellation of like terms in $R_5(G)$ and $R_5(H)$, we have

$$-s^d + s^{d+3} + s^{d+7} + s^{d+7} = -s^{d+1} - s^{d+2} + s^{d+4} + s^{d+4} + s^{d+4} + s^{d+8}.$$

Clearly, $R_5(G) \neq R_5(H)$.

Case 5. If $\max\{e+6, f+7, d+7\} = d+7$ and $\max\{e'+6, f'+7, d'+7\} = f'+7$, then $f' = d$. Since $\max\{e+6, f+7, d+7\} = d+7$, we have $d+7 \geq f+7$, i.e., $d \geq f$. Since we have the assumption that $d \leq f$ then $d = f$. Thus, $f' = d = f$. We can cancel the following corresponding terms in $R_1(G)$ and $R_1(H)$: $-s^f$ and $-s^{f'}$, $-s^{f+1}$ with $-s^{f'+1}$, s^{f+3} with $s^{f'+3}$ and s^{f+7} with $s^{f'+7}$. After simplification, the l.r.p. in $R_1(G)$ is d or e and the l.r.p. in $R_1(H)$ is d' or e'. So, $d = d'$ or $d = e'$ or $e = e'$ or $e = d'$. We know that $d = f = f'$. Since $d+e+f = d'+e'+f'$ we have the two multisets $\{d, e, f\}$ and $\{d', e', f'\}$ the same. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 6. If $\max\{e+6, f+7, d+7\} = e+6$ and $\max\{e'+6, f'+7, d'+7\} = f'+7$, then we have

$$f' = e - 1. \tag{3.3}$$

From $d+e+f = d'+e'+f'$, we have

$$d + f = d' + e' - 1. \tag{3.4}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have $\min\{d, e, f\} = \min\{d', e', f'\}$. Without loss of generality, let $\min\{d, e, f\} = d$. We consider the following subcases.

Subcase 6.1. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = d'$, then $d = d'$. This case can be dealt with the same way as in Case 1. So, we get $G \cong H$.

Subcase 6.2. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = e'$, then $d = e'$. From (3.4) we have $d' = f + 1$. Note that from (3.3) we have $f' = e - 1$. From $R_1(G)$ and $R_1(H)$, we have

$$R_6(G) = -s^d - s^e - s^f - s^{d+1} - s^{e+1} - s^{f+1} + s^{d+3} + s^{f+3} + s^{e+4} + s^{e+6} + s^{d+7} + s^{f+7}$$

$$R_6(H) = -s^{f+1} - s^d - s^{e-1} - s^{f+2} - s^{d+1} - s^e + s^{f+4} + s^{e+2} + s^{d+4} + s^{d+6} + s^{f+8} + s^{e+6}.$$

After simplification, we have

$$\begin{aligned} &- s^f - s^{e+1} + s^{d+3} + s^{f+3} + s^{e+4} + s^{d+7} + s^{f+7} \\ &= -s^{e-1} - s^{f+2} + s^{f+4} + s^{e+2} + s^{d+4} + s^{d+6} + s^{f+8}. \end{aligned}$$

Since $\max\{e+6, f+7, d+7\} = e+6$, we have $e+6 \geq f+7$, i.e., $e+1 \geq f+2$. If $e+1 > f+2$ then we consider the following possibilities: If $e+1 = d+3$ then $e+5 = d+7 = f+8$. This gives us $e-1 = f+2$. The terms with such powers cannot be cancelled for they are of the same sign belonging to the same equation. If $e+1 = f+3$ then $e+2 = f+4$. By the same reason mentioned in the preceding statement this possibility is a contradiction. If $e+1 = d+7$ then the term with the power $f+7$ cannot be cancelled since $e+3=d+9=f+7$. Lastly, if $e+1=f+7$ then the term with the power $e-1$ cannot be cancelled since $e-1=f+5=d+1$, i.e., $e+4=f+10=d+6$. So, the above equation can only be satisfied if $e+1=f+2$ and $e+4=f+5=d+4$, i.e., $e=d=f+1$. This contradicts our assumption $d \le f$.

Subcase 6.3. If $\min\{d,e,f\}=d$ and $\min\{d',e',f'\}=f'$, then $d=f'$. From (3.3), we have $f'=e-1=d$. Since $\max\{e'+6,d'+7,f'+7\}=f'+7$, we have $f'+7 \ge d'+7$, that is $f' \ge d'$. This case is only possible if $d'=f'$. Thus, $d=f'=d'>4$. We can cancel the following pairs of terms in $R_1(G)$ and $R_1(G)$: $-s^d$ with $-s^{d'}$, $-s^{d+1}$ with $-s^{d'+1}$, s^{d+3} with $s^{d'+3}$ and s^{d+7} with $s^{d'+7}$. Since $\max\{e+6,f+7,d+7\}=e+6$, we have $e+6 \ge f+7$, i.e., $e>f$. Thus, after simplification, the l.r.p. in $R_1(G)$ is f and the l.r.p. in $R_1(H)$ is f' or e'. So, $f=f'$ or $f=e'$. We know that $d=f=f'$. Since $d+e+f=d'+e'+f'$ we have the two multisets $\{d,e,f\}$ and $\{d',e',f'\}$ the same. From Lemma 2.1 (4(ii)), $G \cong H$.

At this point, we have shown that given a K_4-homeomorph, $G=K_4(3,3,4,d,e,f)$, where $\min\{d,e,f\} \ge 3$ and $d+e \ge 7$, $f+e \ge 7$, whenever there exists a K_4-homeomorph, $H=K_4(3,3,4,d',e',f')$ such that $G \sim H$, $G \cong H$. Therefore, K_4-homeomorph, $G=K_4(3,3,4,d,e,f)$ with girth 10, where d,e,f are at least 3, is chromatically unique. The proof is now complete. ∎

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