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Quasirecognition by Prime Graph of ${}^2D_p(3)$ Where $p = 2^n + 1 \geq 5$ is a Prime

[1]A. BABAI, [2]B. KHOSRAVI AND [3]N. HASANI
[1,2,3]Department of Pure Mathematics, Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
424, Hafez Avenue, Tehran 15914, Iran
[2]School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746, Tehran, Iran
[2]khosravibbb@yahoo.com

Abstract. In this paper as the main result, we show that if G is a finite group such that $\Gamma(G) = \Gamma({}^2D_p(3))$, where $p = 2^n+1$, $(n \geq 2)$ is a prime number, then G has a unique non-abelian composition factor isomorphic to ${}^2D_p(3)$. We also show that if G is a finite group satisfying $|G| = |{}^2D_p(3)|$ and $\Gamma(G) = \Gamma({}^2D_p(3))$, then $G \cong {}^2D_p(3)$. As a consequence of our result we give a new proof for a conjecture of W. J. Shi and J. X. Bi [A characteristic property for each finite projective special linear group, in *Groups—Canberra 1989*, 171–180, Lecture Notes in Math., 1456, Springer, Berlin] for ${}^2D_p(3)$. Application of this result to the problem of recognition of finite simple groups by the set of element orders are also considered.

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1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: The vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, ..., \pi_{s(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose $2 \in \pi_1$.

The *spectrum* of a finite group G which is denoted by $\omega(G)$ is the set of its element orders. Obviously, $\omega(G)$ is partially ordered by divisibility.

A finite group G is said to be *recognizable by spectrum* if the equality $\omega(H) = \omega(G)$ implies that $H \cong G$. A finite simple non-abelian group G is called *quasirecognizable*

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by spectrum if each finite group H with $\omega(H) = \omega(G)$ has a composition factor isomorphic to G (see [1]).

A finite group G is said to be *recognizable by prime graph* if the equality $\Gamma(H) = \Gamma(G)$ implies that $H \cong G$. A non-abelian simple group P is said to be *quasirecognizable by prime graph* if every finite group whose prime graph is $\Gamma(P)$ has a unique non-abelian composition factor isomorphic to P (see [8]). Obviously recognition (quasirecognition) by prime graph implies recognition (quasirecognition) by spectrum, but the converse is not true in general. Also some methods of recognition by spectrum can not be used for recognition by prime graph.

In [1] it is proved that every finite simple group with at least three connected components (except A_6) are quasirecognizable by spectrum.

Hagie in [6] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In [11] and [14] finite groups with the same prime graph as a *CIT* simple group and $PSL(2, q)$ where $q = p^\alpha < 100$ are determined. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [8, 21]. Also in [12] it is proved that $PSL(2, p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$ is recognizable by prime graph and if $p \equiv 1 \pmod{12}$, then $PSL(2, p)$ is quasirecognizable by prime graph. In [10] and [13], finite groups with the same prime graph as $PSL(2, q)$ are determined.

In this paper as the main result, we show that if G is a finite group such that $\Gamma(G) = \Gamma({}^2D_p(3))$, where $p = 2^n + 1$, $(n \geq 2)$ is a prime number, then G has a unique non-abelian composition factor isomorphic to ${}^2D_p(3)$. We also show that if G is a finite group satisfying $|G| = |{}^2D_p(3)|$ and $\Gamma(G) = \Gamma({}^2D_p(3))$, then $G \cong {}^2D_p(3)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [2]. Also $[x]$, is the largest integer number, smaller or equal to x. Throughout the proof we use the classification of finite simple groups. The connected components of the prime graph of non-abelian simple groups with disconnected prime graph are listed in [19] and throughout this paper we use this list.

2. Preliminary results

Definition 2.1. [5] *A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.*

Lemma 2.1. [20, Theorem A] *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or 2-Frobenius group;*
(b) *a simple group;*
(c) *an extension of a π_1-group by a simple group;*
(d) *an extension of a simple group by a π_1-group;*
(e) *an extension of a π_1-group by a simple group by a π_1-group.*

The next lemma summarizes the basic structural properties of a Frobenius group [4, 5, 7] and a 2-Frobenius group [5]:

Lemma 2.2.

(a) *Let G be a Frobenius group of even order and let H, K be the Frobenius complement and Frobenius kernel of G, respectively. Then $s(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$.*

(b) *If G is a 2-Frobenius group, then $s(G) = 2$ and with the notations of Definition 2.1, we have $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$.*

Lemma 2.3. *If G is a finite group and $\Gamma(G) = \Gamma(^2D_p(3))$, where $p = 2^n + 1$ $(n \geq 2)$ is a prime number, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is trivial or a π_1-group, H is trivial or a nilpotent π_1-group and K/H is a non-abelian simple group with $s(K/H) \geq 3$ and $G/K \leq Out(K/H)$. Also if $j \in \{2, 3\}$, then there exists $i \geq 2$ such that $\pi_j(^2D_p(3)) = \pi_i(K/H)$.*

Proof. Since $s(G) \neq 2$, by using Lemma 2.2, it follows that G is neither a Frobenius group nor a 2-Frobenius group. Therefore by using Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, moreover H and G/K are trivial groups or π_1-groups. We note that a π_1-group is a group of even order with one connected component. Therefore it follows that $s(K/H) \geq s(G) = 3$. Also H is nilpotent. By assumption, $K/H \trianglelefteq G/H$ and hence $N_{G/H}(K/H) = G/H$. We claim that $C_{G/H}(K/H) = 1$. Otherwise let $xH \in C_{G/H}(K/H)$ and p_0 be a prime divisor of $o(xH)$. As we mentioned above, $\pi_2 \cup \pi_3 \subseteq \pi(K/H)$ and by the definition of prime graph it follows that every prime divisor of $|K/H|$ is connected to p_0, which is a contradiction, since $s(G) = 3$. Therefore $C_{G/H}(K/H) = 1$, and so G/H is isomorphic to a subgroup of $Aut(K/H)$. On the other hand, K/H is a nonabelian simple group and so $K/H \cong Inn(K/H)$, which implies that $G/K \leq Out(K/H)$. ∎

Lemma 2.4. [3, Remark 1] *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$ has only one solution, namely $3^2 - 2^3 = 1$.*

Lemma 2.5. [3, 9] *Except the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation*

$$p^m - 2q^n = \pm 1; \quad p, q \quad prime; \quad m, n > 1,$$

has exponents $m = n = 2$; i.e. it comes from a unit $p - q2^{1/2}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients p, q are primes.

Lemma 2.6. [22, Zsigmondy Theorem] *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$ (usually p' is denoted by r_n),*

(ii) *$p = 2$, $n = 1$ or 6,*

(iii) *p is a Mersenne prime and $n = 2$.*

Remark 2.1. [16] Let p be a prime number and $(a, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Then k is called *the order of a with respect to p* and we denote it by $ord_p(a)$.

Remark 2.2. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise nonadjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly, a subset of pairwise nonadjacent vertices of a graph containing the vertex 2 is called a 2-independent set. If $\rho(2,G)$ is some 2-independent set with the maximal number of vertices in $\Gamma(G)$, then $t(2,G) = |\rho(2,G)|$.

In [18, Tables 2-9], independent and 2-independent sets also independent and 2-independent numbers for all simple groups are listed and we use these results in the proof of the main theorem of this paper.

Remark 2.3. Let $G = {}^2D_p(3)$, where $p = 2^n + 1$ $(n \geq 2)$ is a prime number. By [19, Tables 1a-1c], we have $s(G) = 3$ and

$$\pi_1(G) = \pi(2 \times 3^{p(p-1)}(3^{p-1} - 1)\prod_{i=1}^{p-2}(3^{2i} - 1)).$$

Also the odd components of G are

$$\pi_2(G) = \pi((3^{2^n} + 1)/2)$$

and

$$\pi_3(G) = \pi((3^{2^n+1} + 1)/4).$$

We note that since $p \geq 5$ it follows that 5 and 7 belong to $\pi(\prod_{i=1}^{p-2}(3^{2i} - 1)) \subseteq \pi_1(G)$.

3. Main results

Theorem 3.1. *Let G be a finite group such that $\Gamma(G) = \Gamma({}^2D_p(3))$, where $p = 2^n+1$ $(n \geq 2)$ is a prime number. Then G has a unique non-abelian composition factor isomorphic to ${}^2D_p(3)$. In other words ${}^2D_p(3)$ is quasirecognizable by prime graph.*

Proof. By Lemma 2.3, if G is a finite group and $\Gamma(G) = \Gamma({}^2D_p(3))$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is trivial or a π_1-group, H is trivial or a nilpotent π_1-group and K/H is a non-abelian simple group with $s(K/H) \geq 3$. Now we use the classification of finite simple groups. According to [19, Tables 1a–1c], we consider each possibility for K/H, separately.

By using Remark 2.3, we conclude that K/H can not be isomorphic to $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, M_{11}, M_{22}, J_1 or HS.

Case 1. Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are prime numbers.

(1.1) If $\pi((3^{2^n}+1)/2) = \pi(p')$ and $\pi((3^{2^n+1}+1)/4) = \pi(p'-2)$, then $(3^{2^n}+1)/2 = {p'}^{\alpha}$, for some $\alpha \in \mathbb{N}$. Now by using Lemma 2.5, it follows that $\alpha = 1$ and so $p'-2 = (3^{2^n}-3)/2$. Therefore $p'-2 = 3$ and $n = 1$, which is a contradiction.

(1.2) If $\pi((3^{2^n}+1)/2) = \pi(p'-2)$ and $\pi((3^{2^n+1}+1)/4) = \pi(p')$, then similarly to (1.1) it follows that $p' = (3^{2^n} + 5)/2$. Also $\pi((3^{2^n+1} + 1)/4) = \pi(p')$, which implies that $\pi((3^{2^n+1} + 1)/4) = \pi((3^{2^n} + 5)/2)$. If $x \in \pi((3^{2^n+1} + 1)/4)$, then $x \in \pi((3^{2^n}+5)/2)$, which implies that $x = 7$ and this is a contradiction by Remark 2.3.

Case 2. Let $K/H \cong A_1(q)$, where $q = p'^{\alpha}$ and $q \equiv -1 \pmod 4$.

(2.1) If $\pi((3^{2^n}+1)/2) = \pi(q)$ and $\pi((3^{2^n+1}+1)/4) = \pi((q-1)/2)$, then similarly to (1.1) it follows that $(p'-1)/2 = (3^{2^n}-1)/4$. Since $(p'-1) \mid (q-1)$ it follows that $\pi((3^{2^n}-1)/4) \subseteq \pi((3^{2^n+1}+1)/4)$, and this is a contradiction since $\pi((3^{2^n}-1)/4) \cap \pi((3^{2^n+1}+1)/4) = \emptyset$.

(2.2) If $\pi((3^{2^n}+1)/2) = \pi((q-1)/2)$ and $\pi((3^{2^n+1}+1)/4) = \pi(q)$, then $(3^{2^n+1}+1)/4 = p'^{\beta}$, for some $\beta \in \mathbb{N}$. Since $\pi(p'-1) \subseteq \pi(p'^{\beta}-1) = \pi((3^{2^n+1}-3)/4)$ and $\pi(p'-1) \subseteq \pi((p'^{\alpha}-1)/2) = \pi((3^{2^n}+1)/2)$ we conclude that if $x \in \pi(p'-1)$, then $x = 2$, which is a contradiction.

Case 3. Let $K/H \cong A_1(q)$, where $q = 2^{\alpha}$. The odd components of K/H are $\pi(2^{\alpha}-1)$ and $\pi(2^{\alpha}+1)$. Since

$$(2^{\alpha}-1)(2^{\alpha}+1) = 2^{2\alpha}-1$$

and for every α, $3 \mid (2^{2\alpha}-1)$, we get a contradiction since $3 \in \pi_1({}^2D_p(3))$.

Case 4. Let $K/H \cong F_4(q)$, where $q = 2^f > 2$. The odd components of $F_4(q)$ are $\pi(q^4+1)$ and $\pi(q^4-q^2+1)$. By assumptions $\pi_1(F_4(q)) \subseteq \pi_1({}^2D_p(3))$ and by [18, Tables 5–9] we know that

$$t({}^2D_p(3)) = [(3p+3)/4], \quad \rho({}^2D_p(3)) = \{r_{2i} : [p/2] \le i \le p\} \cup \{r_i : [p/2] < i < p\},$$

$$\rho(2, {}^2D_p(3)) = \{2, r_{2p-2}, r_{2p}\}.$$

Also

$$\rho(F_4(q)) = \{r'_3, r'_4, r'_6, r'_8, r'_{12}\}, \quad \rho(2, F_4(q)) = \{2, r'_8, r'_{12}\},$$

where r_i and r'_i are primitive primes of 3^i-1 and q^i-1, respectively.

First, suppose that $p = 5$. If $\pi(2^{4f}+1) = \{41\}$, then by using Lemma 2.5, $2^{4f}+1 = 41$, which is a contradiction. If $\pi(2^{4f}+1) = \{61\}$, similarly we get a contradiction.

Therefore $p \ge 17$ since $p = 2^n+1$, and so $t({}^2D_p(3)) \ge 13$. By assumption r_{2p} and r_{2p-2} in $\rho({}^2D_p(3))$ belong to the odd components of $\Gamma({}^2D_p(3))$. Similarly, r'_8 and r'_{12} in $\rho(F_4(q))$ belong to the odd components of $\Gamma(F_4(q))$. Let A be a subset of $\rho({}^2D_p(3)) \cap \pi_1({}^2D_p(3))$ such that $|A| = 11$. We know that $t(F_4(q)) = 5$ and $s(F_4(q)) = 3$. Therefore $|A \cap \pi_1(F_4(q))| \le 3$ and so at least 8 elements of A do not belong to $\pi(K/H)$, where $K/H \cong F_4(q)$. Since H is a nilpotent group at most one element of A belongs to $\pi(H)$ and at least 7 elements of A belong to $\pi(G/K)$. We know that $G/K \le Out(K/H)$. By [2], $|Out(K/H)| = f$, where f is the order of the field automorphism. We know that a field automorphism centralizes the elements of $F_4(2)$. We know that

$$\pi(F_4(2)) = \{2, 3, 5, 13, 17\}.$$

So 17 is adjacent to all elements of A, which implies that 17 is adjacent to $r_{2(p-i)}$ for some $i \le 6$. Clearly 17 is a primitive prime of $3^{16}-1$ and $r_{2(p-i)}$ is a primitive prime of $2^{2(p-i)}-1$. So there exists a maximal torus T such that $17.r_{2(p-i)} \mid |T|$. By [18] every maximal torus T of ${}^2D_p(3)$ has order

$$\frac{1}{(4, 3^p+1)}.(3^{n_1}-1).(3^{n_2}-1).\cdots.(3^{n_k}-1).(3^{l_1}+1).(3^{l_2}+1).\cdots.(3^{l_m}+1)$$

for an appropriate partition $n_1 + n_2 + \ldots + n_k + l_1 + l_2 + \ldots + l_m = p$ of p, where m is odd. Moreover, for every partition, there exists a torus of the corresponding order. Now it follows that $p - i + 8 \leq p$, which is a contradiction.

Case 5. Let $K/H \cong {}^2G_2(q)$, where $q = 3^{2m+1}$. The odd components of K/H are $\pi(q - \sqrt{3q} + 1)$ and $\pi(q + \sqrt{3q} + 1)$. So

$$\pi((3^{p-1} + 1)/2) \cup \pi((3^p + 1)/4) = \pi(q - \sqrt{3q} + 1) \cup \pi(q + \sqrt{3q} + 1) \subseteq \pi(q^3 + 1)$$

If $x \in \pi((3^p + 1)/4)$, then $3^p \equiv -1 \pmod{x}$. Obviously, $x \neq 3$. Therefore, $3^{2p} \equiv 1 \pmod{x}$, which implies that $ord_x(3) \mid 2p$. Clearly, $ord_x(3) \neq 2$ and so $ord_x(3) = 2p$. Also, $x \in \pi(q^3 + 1)$ so $x \mid (3^{6m+3} + 1)$. So $3^{12m+6} \equiv 1 \pmod{x}$. Therefore $ord_x(3) \mid (12m + 6)$ and so $p \mid 3(2m + 1)$. Since $p \neq 3$ it follows that $p \mid (2m + 1)$. Therefore, $(3^p + 1) \mid (3^{(2m+1)} + 1)$, which implies that $x \mid (3^{(2m+1)} + 1)$. Since

$$x \in \pi(3^{2m+1} + 3^{m+1} + 1) \cup \pi(3^{2m+1} - 3^{m+1} + 1)$$

it follows that $x \mid 3^{m+1}$, which is a contradiction since $x \neq 3$.

Case 6. Let $K/H \cong G_2(q)$, where $q = 3^m$. The odd components of K/H are $\pi(3^{2m} + 3^m + 1)$ and $\pi(3^{2m} - 3^m + 1)$. Also

$$(3^{2m} + 3^m + 1)(3^{2m} - 3^m + 1) = (3^{4m} + 3^{2m} + 1),$$

so

$$\pi((3^{p-1} + 1)/2) \cup \pi((3^p + 1)/4) = \pi(3^{4m} + 3^{2m} + 1) \subseteq \pi(3^{6m} - 1).$$

If $x \in \pi((3^p + 1)/4)$, then $3^p \equiv -1 \pmod{x}$. Obviously $x \neq 3$. Therefore $3^{2p} \equiv 1 \pmod{x}$, which implies that $ord_x(3) = 2p$, so $2p \mid 6m$. Since $p \neq 3$, therefore $p \mid m$. If m is odd, then $(3^p + 1) \mid (3^m + 1)$ so $x \mid (3^m + 1)$, which implies that $x \mid (3^{4m} - 1)$. On the other hand, $x \mid (3^{4m} + 3^{2m} + 1)$ and so $x \mid (3^{2m} + 2)$. Since $x \mid (3^{2m} - 1)$ we get that $x = 3$, which is a contradiction. If m is even, then $(3^p + 1) \mid (3^m - 1)$. Similarly it follows that $x = 3$, which is a contradiction.

Case 7. Let $K/H \cong {}^2F_4(q)$, where $q = 2^{2f+1} > 2$. First suppose that $p = 5$. We know that $\pi_1({}^2D_5(3)) = \{3, 5, 7, 13\}$ and 3, 5 ,7 and 13 are primitive primes of $2^2 - 1$, $2^4 - 1$, $2^3 - 1$ and $2^{12} - 1$, respectively. Also r_6 is a primitive prime of $(q^6 - 1)$, so $r_6 \mid (2^{6(2f+1)} - 1)$ and so is a divisor of $(q^3 + 1)$. Since the order of the first component of ${}^2F_4(q)$ is $q^{12}(q^4 - 1)(q^3 + 1)(q^2 + 1)(q - 1)$, so r_6 is a divisor of $\pi_1({}^2F_4(q))$. Since $\pi_1({}^2F_4(q)) \subseteq \pi_1({}^2D_5(3))$, it follows that $6(2f + 1) \leq 12$, which is a contradiction.

Therefore $p \geq 17$, since $p = 2^n + 1$. Let A be a subset of $\rho({}^2D_p(3)) \cap \pi_1({}^2D_p(3))$ such that $|A| = 11$. Similarly to the Case 4, at least 7 elements of A belong to $\pi(G/K)$, and $|Out(K/H)| = 2f + 1$ where $2f + 1$ is the order of the field automorphism. We know that the automorphism group of a field is cyclic, so is abelian. Therefore all of the elements of A are pairwise adjacent, which is a contradiction.

Similarly it follows that K/H is not isomorphic to ${}^2B_2(q)$ where $q = 2^n + 1$, $E_8(q)$ and $A_1(q)$ where $q \equiv 1 \pmod{4}$.

Case 8. Let $K/H \cong E_7(2)$.

(8.1) If $\pi((3^{2^n}+1)/2) = \{73\}$ and $\pi((3^{2^n+1}+1)/4) = \{127\}$, then $(3^{2^n}+1)/2 = 73^\alpha$, for some $\alpha \in \mathbb{N}$. Now by using Lemma 2.5, it follows that $\alpha = 1$ and so $3^{2^n} = 145$ which is a contradiction.

(8.2) If $\pi((3^{2^n}+1)/2) = \{127\}$ and $\pi((3^{2^n+1}+1)/4) = \{73\}$, then similarly to (8.1) it follows that $(3^{2^n}+1)/2 = 127$ so $3^{2^n} = 253$ which is a contradiction. Similarly it follows that K/H can not be isomorphic to $E_7(3)$, ${}^2E_6(2)$ and any sporadic group.

Case 9. Let $K/H \cong {}^2D_{p'}(3)$, where $p' = 2^{n'}+1$ ($n' \geq 2$) is a prime. First suppose that $\pi((3^{p'}+1)/4) = \pi((3^{p-1}+1)/2)$ and $\pi((3^{p'-1}+1)/2) = \pi((3^p+1)/4)$. If $x \in \pi((3^{p-1}+1)/2))$, then $3^{p-1} \equiv -1 \pmod{x}$ and so, $ord_x(3) \mid 2(p-1)$. On the other hand, $x \in \pi((3^{p'}+1)/4)$, which implies that $ord_x(3) = 2p'$. Therefore $2p' \mid 2(p-1)$ so $(2^{n'}+1) \mid 2^n$, which is a contradiction. Therefore $\pi((3^{p'}+1)/4) = \pi((3^p+1)/4)$ and $\pi((3^{p'-1}+1)/2) = \pi((3^{p-1}+1)/2)$. Hence without loss of generality we can assume that $p' \leq p$. If x is a primitive prime of $3^{2p'}-1$, since $\pi((3^{p'}+1)/4) = \pi((3^p+1)/4)$, then x is a primitive prime of $3^{2p}-1$, which implies that $p = p'$ by Lemma 2.6, and so $K/H \cong {}^2D_p(3)$. Now the proof of this theorem is completed. ∎

4. Some related results

We note that $\Gamma(\mathbb{Z}_6)$ is a graph with two vertices, i.e. $V = \{2,3\}$ and there exists an edge between 2 and 3. But $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_{2^k}) = \Gamma(\mathbb{Z}_6)$ for every $k > 0$. Also $S_3 \times \mathbb{Z}_{2^k}$, where $k > 0$, is a nonabelian group and $\Gamma(S_3 \times \mathbb{Z}_{2^k}) = \Gamma(\mathbb{Z}_6)$. Therefore there exist infinitely many non isomorphic groups G such that $\Gamma(G) = \Gamma(\mathbb{Z}_6)$. Also note that even if $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, then we can not conclude that $G \cong M$.

As a consequence of the main theorem we can prove the following corollaries.

Corollary 4.1. *Let G be a finite group satisfying $|G| = |{}^2D_p(3)|$, where $p = 2^n+1$ ($n \geq 2$) is a prime. If $\Gamma(G) = \Gamma({}^2D_p(3))$, then $G \cong {}^2D_p(3)$.*

Proof. By assumption, $\Gamma(G) = \Gamma({}^2D_p(3))$. Now by using the main theorem it follows that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong {}^2D_p(3)$. Also $|G| = |{}^2D_p(3)|$ and so $H = 1$ and $K = G$. Therefore, $G \cong {}^2D_p(3)$. ∎

Remark 4.1. Shi and Bi in [17] put forward the following conjecture.

Conjecture 4.1. *Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if*

(i) $|G| = |M|$;
(ii) $\omega(G) = \omega(M)$.

This conjecture is valid for sporadic simple groups, alternating groups and some simple groups of Lie type. As a consequence of the main theorem, we can give a new proof for this conjecture for the groups under discussion.

Corollary 4.2. *Let G be a finite group satisfying $|G| = |{}^2D_p(3)|$, where $p = 2^n+1$ ($n \geq 2$) is a prime. If $\omega(G) = \omega({}^2D_p(3))$, then $G \cong {}^2D_p(3)$.*

Obviously Corollary 4.1 is a generalization of Shi-Bi conjecture and so Corollary 4.2 is an immediate consequence of Corollary 4.1.

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References

[1] O. A. Alekseeva and A. S. Kondrat′ev, *Sibirsk. Mat. Zh.* **44** (2003), no. 2, 241–255; translation in *Siberian Math. J.* **44** (2003), no. 2, 195–207.
[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford Univ. Press, Eynsham, 1985.
[3] P. P. Crescenzo, A Diophantine equation which arises in the theory of finite groups, *Advances in Math.* **17** (1975), no. 1, 25–29.
[4] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.
[5] K. W. Gruenberg and K. W. Roggenkamp, Decomposition of the augmentation ideal and of the relation modules of a finite group, *Proc. London Math. Soc. (3)* **31** (1975), no. 2, 149–166.
[6] M. Hagie, The prime graph of a sporadic simple group, *Comm. Algebra* **31** (2003), no. 9, 4405–4424.
[7] B. Huppert, *Endliche Gruppen. I*, Springer, Berlin, 1967.
[8] A. Khosravi and B. Khosravi, *Sibirsk. Mat. Zh.* **48** (2007), no. 3, 707–716; translation in *Siberian Math. J.* **48** (2007), no. 3, 570–577.
[9] A. Khosravi and B. Khosravi, A new characterization of some alternating and symmetric groups. II, *Houston J. Math.* **30** (2004), no. 4, 953–967 (electronic).
[10] A. Khosravi and B. Khosravi, *Sibirsk. Mat. Zh.* **49** (2008), no. 4, 934–944; translation in *Siberian Math. J.* **49** (2008), no. 4, 749–757.
[11] B. Khosravi, B. Khosravi and B. Khosravi, Groups with the same prime graph as a CIT simple group, *Houston J. Math.* **33** (2007), no. 4, 967–977 (electronic).
[12] B. Khosravi, B. Khosravi and B. Khosravi, On the prime graph of $\mathrm{PSL}(2,p)$ where $p>3$ is a prime number, *Acta Math. Hungar.* **116** (2007), no. 4, 295–307.
[13] B. Khosravi, n-recognition by prime graph of the simple group $\mathrm{PSL}(2,q)$, *J. Algebra Appl.* **7** (2008), no. 6, 735–748.
[14] B. Khosravi and S. S. S. Amiri, On the prime graph of $L_2(q)$ where $q=p^\alpha<100$, *Quasigroups Related Systems* **14** (2006), no. 2, 179–190.
[15] V. D. Mazurov, Characterizations of finite groups by the set of orders of their elements, *Algebra i Logika* **36** (1997), no. 1, 37–53, 117; translation in *Algebra and Logic* **36** (1997), no. 1, 23–32.
[16] W. Sierpiński, *Elementary Theory of Numbers*, Translated from Polish by A. Hulanicki. Monografie Matematyczne, Państwowe Wydawnictwo Naukowe, Warsaw, 1964.
[17] W. J. Shi and J. X. Bi, A characteristic property for each finite projective special linear group, in *Groups—Canberra 1989*, 171–180, Lecture Notes in Math., 1456, Springer, Berlin.
[18] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logika* **44** (2005), no. 6, 682–725, 764; translation in *Algebra Logic* **44** (2005), no. 6, 381–406.
[19] A. V. Vasil'ev and M. A. Grechkoseeva, On recognition of the finite simple orthogonal groups of dimension 2^m, 2^m+1 and 2^m+2 over a field of characteristic 2, *Sibirsk. Mat. Zh.* **45** (2004), no. 3, 512–526; translation in *Siberian Math. J.* **45** (2004), no. 3, 420–431.
[20] J. S. Williams, Prime graph components of finite groups, *J. Algebra* **69** (1981), no. 2, 487–513.
[21] A. V. Zavarnitsin, Recognition of finite groups by the prime graph, *Algebra Logika* **45** (2006), no. 4, 390–408, 502; translation in *Algebra Logic* **45** (2006), no. 4, 220–231.
[22] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), no. 1, 265–284.