

On Uniqueness Theorems of Meromorphic Functions Concerning Weighted Sharing of Three Values

¹XIAO-MIN LI AND ²HONG-XUN YI

¹Department of Mathematics, Ocean University of China,
Qingdao, Shandong 266071, P. R. China

²Department of Mathematics, Shandong University,
Jinan, Shandong 250100, P. R. China

¹xmli01267@gmail.com, ²hxyi@sdu.edu.cn

Abstract. In this paper, we deal with the problem of meromorphic functions that have three weighted sharing values, and obtain a uniqueness theorem which improves those given by Ozawa, H. X. Yi, I. Lahiri, Q. C. Zhang, and others. Some examples are provided to show that the results in this paper are the best possible.

2000 Mathematics Subject Classification: 30D30, 30D35

Key words and phrases: Meromorphic functions, weighted sharing values, uniqueness theorems.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [4]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$ ($r \rightarrow \infty, r \notin E$).

Let f and g be two nonconstant meromorphic functions, and let $a \in C \cup \{\infty\}$, where $C \cup \{\infty\}$ denotes the extended complex plane. We say that f and g share the value a CM, provided that f and g have the same a -points with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that f and g have the same a -points ignoring multiplicities (see [13]). In addition, we need the following three definitions.

Definition 1.1. [1, Definition 1] Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_p(r, 1/(f - a))$ we denote the counting function of those a -points of f (counted with

Communicated by Lee See Keong.

Received: September 8, 2007; Revised: March 30, 2009.

proper multiplicities) whose multiplicities are not greater than p , by $\overline{N}_p(r, 1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p)}(r, 1/(f-a))$ we denote the counting function of those a -points of f (counted with proper multiplicities) whose multiplicities are not less than p , by $\overline{N}_{(p)}(r, 1/(f-a))$ we denote the corresponding reduced counting function (ignoring multiplicities).

Definition 1.2. [5, Definition 4] For $a \in C \cup \{\infty\}$, we put

$$\delta_p(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_p(r, 1/(f-a))}{T(r, f)},$$

where p is a positive integer. Clearly

$$0 \leq \delta_p(a, f) \leq \delta_{p-1}(a, f) \leq \cdots \leq \delta_1(a, f) \leq 1.$$

Definition 1.3. [7, Definition 3] For $a \in C \cup \{\infty\}$, we put

$$\Theta_p(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}_p(r, 1/(f-a))}{T(r, f)},$$

where p is a positive integer. Clearly

$$0 \leq \Theta_p(a, f) \leq \Theta_{p-1}(a, f) \leq \cdots \leq \Theta_1(a, f) \leq 1.$$

In 1976, M. Ozawa proved the following result.

Theorem 1.1. [11] Let f and g be two nonconstant entire functions of finite order, such that f and g share 0 and 1 CM. If $2\delta(0, f) > 1$, then either $f \equiv g$ or $fg \equiv 1$.

In 1983, H. Ueda proved the following result, which improved Theorem 1.1.

Theorem 1.2. [12] Let f and g be two nonconstant meromorphic functions such that f and g share 0, 1 and ∞ CM. If $\limsup_{r \rightarrow \infty} (N(r, 1/f) + N(r, f))/T(r, f) < 1/2$, then either $f \equiv g$ or $fg \equiv 1$.

In 1990, H. X. Yi proved the following result, which improved Theorem 1.2.

Theorem 1.3. [14] Let f and g be two nonconstant meromorphic functions such that f and g share 0, 1 and ∞ CM. If $N_1(r, 1/f) + N_1(r, f) < (\lambda + o(1))T(r, f)$ ($r \in I$), where λ satisfying $0 < \lambda < 1/2$ is a positive integer, and $I \subseteq (0, +\infty)$ is a set of infinite linear measure, then either $f \equiv g$ or $fg \equiv 1$.

Regarding Theorem 1.1–1.3, it is natural to ask the following question.

Question 1.1. [6] Is it really possible to relax in any way the nature of sharing any one of 0, 1 and ∞ in Theorem 1.1, Theorem 1.2 and Theorem 1.3?

In this paper, we will study Question 1.1. For this purpose, we will explain the notion of weighted sharing by the following definition.

Definition 1.4. [5] Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$, and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k .

Remark 1.1. Definition 1.4 implies that if f, g share a value a with weight k , then z_0 is a zero of $f - a$ with multiplicity $m (\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m (\leq k)$, and z_0 is a zero of $f - a$ with multiplicity $m (> k)$, if and only if it is a zero of $g - a$ with multiplicity $n (> k)$, where m is not necessarily equal to n . Throughout this paper, we write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly, if f, g share (a, k) , then f, g share (a, p) for all integer p , $0 \leq p < k$. Also we note that f, g share a value a *IM* or *CM* if and only if f, g share $(a, 0)$ or (a, ∞) , respectively.

Using the idea of weighted sharing, I. Lahiri proved the following result which improved Theorem 1.3.

Theorem 1.4. [7, Theorem 1] *Let f and g be two nonconstant meromorphic functions such that f and g share $(0, 1)$, $(1, m)$ and (∞, k) , where m and k are positive integers satisfying*

$$(1.1) \quad (m-1)(km-1) > (1+m)^2.$$

If

$$(1.2) \quad A_0 = 2\delta_1(0, f) + 2\delta_1(\infty, f) + \min \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} > 3,$$

then either $f \equiv g$ or $fg \equiv 1$.

Regarding Theorem 1.4, it is natural to ask the following question.

Question 1.2. What can be said if we relax in any way the condition (1.2) of Theorem 1.4?

In this paper, we will prove the following two theorems, of which Theorem 1.5 improves Theorem 1.1, Theorem 1.6 improves Theorem 1.4. Moreover, Theorem 1.5 and Theorem 1.6 deal with Questions 1.1–1.2.

Theorem 1.5. *Let f and g be two nonconstant entire functions such that f and g share $(a_1, 1)$, $(a_2, 2)$, where $\{a_1, a_2\} = \{0, 1\}$. If*

$$(1.3) \quad \begin{aligned} A_1 = 2\delta_1(0, f) + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} \\ + \max \{ \delta_1(1, f), \delta_1(1, g) \} \\ > 1, \end{aligned}$$

then either $f \equiv g$ or $fg \equiv 1$.

Theorem 1.6. *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2, k_3 are three positive integers satisfying*

$$(1.4) \quad k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2.$$

If

$$(1.5) \quad A_2 = 2\delta_1(0, f) + 2\delta_1(\infty, f) + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} > 3,$$

then either $f \equiv g$ or $fg \equiv 1$.

From Theorem 1.6 we get the following corollary.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.4). If*

$$(1.6) \quad \delta_1(0, f) + \delta_1(\infty, f) > 3/2,$$

then either $f \equiv g$ or $fg \equiv 1$.

Recently, Q. C. Zhang proved the following two results, which also improved Theorem 1.4 and dealt with Question 1.2.

Theorem 1.7. [17, Theorem 2] *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2, k_3 are three positive integers satisfying (1.4). If both*

$$(1.7) \quad 2\delta_1(0, f) + 2\delta_1(\infty, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \max\{\delta_1(1, f), \delta_1(1, g)\} > 3$$

and

$$(1.8) \quad 2\delta_1(0, g) + 2\delta_1(\infty, g) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \max\{\delta_1(1, f), \delta_1(1, g)\} > 3$$

hold, then either $f \equiv g$ or $fg \equiv 1$.

Theorem 1.8. [17, Theorem 3] *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2, k_3 are three positive integers satisfying (1.4). If*

$$(1.9) \quad 2\delta_1(0, f) + 2\delta_1(\infty, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \max\{\delta_1(1, f), \delta_1(1, g)\} > 3,$$

and

$$(1.10) \quad 2\delta_1(0, g) + 2\delta_1(\infty, g) + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \max\{\delta_1(1, f), \delta_1(1, g)\} > 3$$

hold, then one of the following four equalities holds:

- (i) $f \equiv g$;
- (ii) $f - 1 \equiv A(g - 1)$;
- (iii) $1/f - 1 \equiv A(1/g - 1)$;
- (iv) $fg \equiv 1$;

where $A(\neq 0, 1)$ is a finite complex number.

In this paper, we will prove the following two theorems, of which Theorem 1.9 supplements Theorem 1.7 by replacing (1.7) and (1.8) with (1.11), Theorem 1.10 supplements Theorem 1.8 by replacing (1.9) with (1.12) and by replacing (1.10) with (1.13).

Theorem 1.9. *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2, k_3 are three positive integers satisfying (1.4). If*

$$(1.11) \quad \begin{aligned} & 2\delta_1(0, f) + 2\delta_1(\infty, f) + 2\delta_1(0, g) + 2\delta_1(\infty, g) \\ & + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \delta_1(1, f) + \delta_1(1, g) > 6, \end{aligned}$$

then either $f \equiv g$ or $fg \equiv 1$.

Theorem 1.10. *Let f and g be two transcendental meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2, k_3 are three positive integers satisfying (1.4). If*

$$(1.12) \quad \begin{aligned} & 2\delta_1(0, f) + 2\delta_1(\infty, f) \\ & + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} + \delta_1(1, f) + \delta_1(1, g) > 3 \end{aligned}$$

and

$$(1.13) \quad \begin{aligned} & 2\delta_1(0, g) + 2\delta_1(\infty, g) \\ & + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} + \delta_1(1, f) + \delta_1(1, g) > 3, \end{aligned}$$

then f is a Möbius transformation of g , moreover, f and g assume one of the following four relations:

- (i) $f \equiv g$;
- (ii) $fg \equiv 1$;
- (iii) $(f - 1)(g - 1) \equiv 1$;
- (iv) $f + g \equiv 1$.

We give the following five examples.

Example 1.1. Let $f = 1 + e^z$ and $g = 1 + e^{-z}$, and let $k_1 = 1$, $k_2 = 2$ and $k_3 = 6$. Then it follows that k_1, k_2 and k_3 satisfy (1.4), and that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) . Moreover, we can verify that $\delta_1(0, f) = 0$,

$$\sum_{a \neq 0, 1, \infty} \delta_2(a, f) = \sum_{a \neq 0, 1, \infty} \delta_2(a, g) = 0$$

and $\delta_1(1, f) = \delta_1(1, g) = 1$, and so $A_1 = 1$, where A_1 is defined as (1.3). However, neither $f \equiv g$ nor $fg \equiv 1$. This example shows that the condition (1.3) in Theorem 1.5 is the best possible.

Example 1.2. [8] Let $f = e^z - 1$ and $g = 2 - 2e^{-z}$, and let $k_1 = 1$, $k_2 = 2$ and $k_3 = 6$, and so k_1, k_2, k_3 satisfy (1.4). We can verify that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) . Moreover, we can verify that $\delta_1(0, f) = 0$, $\delta_1(\infty, f) = 1$ and

$$\sum_{a \neq 0, 1, \infty} \delta_2(a, f) = \sum_{a \neq 0, 1, \infty} \delta_2(a, g) = 1,$$

and so $A_2 = 3$, where A_2 is defined as (1.5). However, neither $f \equiv g$ nor $fg \equiv 1$. This example shows that the condition (1.5) in Theorem 1.6 is the best possible.

Example 1.3. [6] Let $f = (e^z - 1)^2$ and $g = e^z - 1$. Then f and g share $(0, 0)$, $(1, 2)$ and $(\infty, 6)$. Moreover, we can verify that $\delta_1(0, f) = \delta_1(1, f) = 1$, and so the condition (1.2) holds. However, neither $f \equiv g$ nor $fg \equiv 1$. The example shows that the condition that f and g share $(0, k_1)$ ($k_1 \geq 1$) can not be relaxed to the condition that f and g share $(0, 0)$ in Theorem 1.5 and Theorem 1.6.

Let $k_1 = 1$, $k_2 = m$ and $k_3 = k$, then (1.4) can be rewritten as $(m-1)(km-1) > (1+m)^2$.

Example 1.4. [7] Let $f(z) = 4e^z/(1+e^z)^2$, $g(z) = 2e^z/(1+e^z)$ and $m = k = 0$. Then f and g share $(0, \infty)$, $(1, m)$, (∞, k) , and $\delta_1(0, f) = \delta_1(\infty, f) = 1$. Moreover, $(m-1)(km-1) = (1+m)^2$. But neither $f \equiv g$ nor $fg \equiv 1$. This example shows that the conclusion of Theorem 1.6 does not hold, if $(m-1)(km-1) = (1+m)^2$.

The following example shows that the quantity A_2 in the condition (1.5) of Theorem 1.6 can not be replaced with any one of the larger quantities B_2 and C_2 , where B_2 and C_2 are respectively defined as

$$B_2 = \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_1(a, f), \sum_{a \neq 0, 1, \infty} \delta_1(a, g) \right\} + \max\{\delta_1(1, f), \delta_1(1, g)\} \\ + 2\delta_1(0, f) + 2\delta_1(\infty, f)$$

and

$$C_2 = \max \left\{ \sum_{a \neq 0, 1, \infty} \Theta_2(a, f), \sum_{a \neq 0, 1, \infty} \Theta_2(a, g) \right\} + \max\{\delta_1(1, f), \delta_1(1, g)\} \\ + 2\delta_1(0, f) + 2\delta_1(\infty, f).$$

Example 1.5. [5] Let $f = e^z(1 - e^z)$ and $g = e^{-z}(1 - e^{-z})$, and let $k_1 = 1$, $k_2 = 2$ and $k_3 = 6$, and so the condition (1.4) holds. We can verify that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) . Since $f - 1/4 = -(e^z - 1/2)^2$, we can deduce that $A_2 = 3$, $B_2 > 3$ and $C_2 > 3$. However, neither $f \equiv g$ nor $fg \equiv 1$.

From Theorem 1.6 we get the following corollary.

Corollary 1.2. *Theorem 1.6 still holds, if k_1, k_2 and k_3 assume any one of the following three cases.*

- (i) $k_1 = 1, k_2 = 2$ and $k_3 = 6$, or $k_2 = 3$ and $k_3 = 4$, or $k_2 = 4$ and $k_3 = 3$, or $k_2 = 6$ and $k_3 = 2$;
- (ii) $k_2 = 1, k_1 = 2$ and $k_3 = 6$, or $k_1 = 3$ and $k_3 = 4$, or $k_1 = 4$ and $k_3 = 3$, or $k_1 = 6$ and $k_3 = 2$;

- (iii) $k_3 = 1$, $k_1 = 6$ and $k_2 = 2$, or $k_1 = 4$ and $k_2 = 3$, or $k_1 = 3$ and $k_2 = 4$, or $k_1 = 2$ and $k_2 = 6$.

2. Some lemmas

Let f and g share $0, 1, \infty$ IM, next we denote by $N_0(r)$ ($\overline{N}_0(r)$) the counting function of the zeros of $f - g$ that are not zeros of f , $f - 1$ and $1/f$ (ignoring multiplicities) (see [13]).

Lemma 2.1. [9, Lemma 6] *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1$ and ∞ IM, If f is a fractional linear transformation (Möbius transformation) of g , then f and g satisfy one of the following relations:*

- (i) $f \cdot g \equiv 1$;
- (ii) $(f - 1)(g - 1) \equiv 1$;
- (iii) $f + g \equiv 1$;
- (iv) $f \equiv cg$;
- (v) $f - 1 \equiv c(g - 1)$,
- (vi) $[(c - 1)f + 1] \cdot [(c - 1)g - c] \equiv -c$;

where $c (\neq 0, 1)$ is a finite complex number.

Lemma 2.2. [16, Lemma 2.6] *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are three positive integers satisfying (1.4). Then*

- (i) $\overline{N}_{(2)}(r, 1/f) + \overline{N}_{(2)}(r, 1/(f - 1)) + \overline{N}_{(2)}(r, f) = S(r, f)$,
- (ii) $\overline{N}_{(2)}(r, 1/g) + \overline{N}_{(2)}(r, 1/(g - 1)) + \overline{N}_{(2)}(r, g) = S(r, f)$.

Lemma 2.3. [18, Lemma 6] *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying $\overline{N}(r, f_j) + \overline{N}(r, 1/f_j) = S(r)$ ($j = 1, 2$). Then either $\overline{N}_0(r, 1; f_1, f_2) = S(r)$ or there exist two integers s, t ($|s| + |t| > 0$) such that $f_1^s f_1^t \equiv 1$, where $\overline{N}_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points, and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o\{T(r)\}$ ($r \rightarrow \infty, r \notin E$) depending only on f_1 and f_2 .*

Lemma 2.4. [18, Proof of Theorem 1 and Theorem 2] *Let f and g be two distinct nonconstant meromorphic functions sharing $0, 1$ and ∞ CM, and let $N_0(r) \neq S(r, f)$. If f is a fractional linear transformation of g , then $N_0(r) = T(r, f) + S(r, f)$. If f is not a fractional linear transformation of g , then*

$$(2.1) \quad N_0(r) \leq \frac{1}{2} T(r, f) + S(r, f),$$

and f and g assume one of the following three relations:

(i)

$$f \equiv \frac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}, \quad g \equiv \frac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1};$$

(ii)

$$f \equiv \frac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}, \quad g \equiv \frac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1};$$

(iii)

$$f \equiv \frac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}, \quad g \equiv \frac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1};$$

where γ is a nonconstant entire function, s and k (≥ 2) are positive integers such that s and $k+1$ are relatively prime and $1 \leq s \leq k$.

Lemma 2.5. [18] *Let s (> 0) and t are relatively prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.*

Lemma 2.6. [13, Proof of Theorem 1.12 and Theorem 1.13] *Let f be a nonconstant meromorphic function, and let*

$$F = \frac{\sum_{k=0}^p a_k f^k}{\sum_{j=0}^q b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_p \neq 0$ and $b_q \neq 0$. Then $T(r, F) = dT(r, f) + O(1)$, where $d = \max\{p, q\}$.

Lemma 2.7. [3, Lemma 2 and Lemma 3] or [10, Theorem 1.1] *Let f and g be two distinct nonconstant meromorphic functions sharing $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , where k_1, k_2 and k_3 are positive integers satisfying (1.4). If f is not a Möbius transformation of g , then*

- (i) $T(r, f) + T(r, g) = \bar{N}(r, 1/f) + \bar{N}(r, 1/(f-1)) + \bar{N}(r, f) + N_0(r) + S(r, f)$;
- (ii) $T(r, f) = N(r, 1/(f-a)) + S(r, f)$, $T(r, g) = N(r, 1/(g-a)) + S(r, f)$;
- (iii) $N_{(3)}(r, 1/(f-a)) + N_{(3)}(r, 1/(g-a)) = S(r, f)$;

where a ($\neq 0, 1$) is an arbitrary finite complex number.

3. Proof of theorems

Proof of Theorem 1.5. Suppose that $f \not\equiv g$. We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g . From Lemma 2.1 and the condition that f and g are two nonconstant entire functions sharing 0 and 1 IM, we deduce that f and g assume one of the relations (i), (ii) and (vi). If f and g assume (i) of Lemma 2.1, then 0 is a Picard exceptional value of f and g , and so $f = e^{\alpha_1}$ and $g = e^{-\alpha_1}$, where α_1 is a nonconstant entire function. From this we deduce $\delta_{(1)}(0, f) = 1$ and

$$\max \left\{ \sum_{a \neq 0, 1, \infty} \delta_{(2)}(a, f), \sum_{a \neq 0, 1, \infty} \delta_{(2)}(a, g) \right\} + \max\{\delta_{(1)}(1, f), \delta_{(1)}(1, g)\} = 0.$$

Thus (1.3) is valid and we get the conclusion $fg \equiv 1$. If f and g assume (ii) of Lemma 2.1, then 1 is a Picard exceptional value of f and g , then $f = 1 + e^{\alpha_2}$ and $g = 1 + e^{-\alpha_2}$, where α_2 is a nonconstant entire function. From this we deduce $\max\{\delta_{(1)}(1, f), \delta_{(1)}(1, g)\} = 1$ and

$$2\delta_{(1)}(0, f) + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_{(2)}(a, f), \sum_{a \neq 0, 1, \infty} \delta_{(2)}(a, g) \right\} = 0,$$

from this and (1.3) we get a contradiction. If f and g assume (vi) of Lemma 2.1, then

$$(3.1) \quad f = (e^{\alpha_3} - 1)/(c - 1), \quad g = (e^{-\alpha_3} - 1)/(c^{-1} - 1),$$

where α_3 is a nonconstant entire function. From (3.1) and the second fundamental theorem, we deduce

$$(3.2) \quad \sum_{a \neq 0, 1, \infty} \delta_2(a, f) \leq 1 \quad \text{and} \quad \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \leq 1,$$

and so we have

$$(3.3) \quad \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} \leq 1.$$

On the other hand, from (3.1) we also deduce $\delta_1(0, f) = \delta_1(1, f) = 0$ and $\delta_1(0, g) = \delta_1(1, g) = 0$. Combining (1.3) and (3.3) we get a contradiction.

Case 2. Suppose that f is not a Möbius transformation of g . Let

$$(3.4) \quad \frac{f-1}{g-1} = h_1, \quad \frac{f}{g} = h_2$$

and

$$(3.5) \quad h_0 = \frac{h_1}{h_2},$$

where none of h_1 , h_2 and h_0 is a constant such that

$$(3.6) \quad T(r, g) + T(r, h_1) + T(r, h_2) = O\{T(r, f)\} \quad (r \notin E).$$

From Lemma 2.2 and the condition that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3) , we deduce

$$(3.7) \quad \overline{N}(r, h_j) + \overline{N}(r, 1/h_j) = S(r, f) \quad (j = 0, 1, 2).$$

From (3.6), (3.7) and the second fundamental theorem we get

$$\begin{aligned} T(r, h_j) &\leq \overline{N}(r, h_j) + \overline{N}(r, 1/h_j) + \overline{N}(r, 1/(h_j - 1)) + S(r, h_j) \\ &\leq \overline{N}(r, 1/(h_j - 1)) + S(r, h_j) + S(r, f) \\ &\leq N(r, 1/(h_j - 1)) + S(r, f) \leq T(r, h_j) + S(r, f) \quad (j = 0, 1, 2), \end{aligned}$$

which implies

$$(3.8) \quad N_{(2)}(r, 1/(h_j - 1)) = S(r, f) \quad (j = 0, 1, 2).$$

From (3.4) and (3.5) we have

$$(3.9) \quad f = \frac{h_1 - 1}{h_0 - 1}, \quad g = \frac{h_1^{-1} - 1}{h_0^{-1} - 1},$$

$$(3.10) \quad h_1 - 1 = \frac{f - g}{g - 1}$$

and

$$(3.11) \quad h_2 - 1 = \frac{f - g}{g}.$$

From (3.8), (3.10) and (3.11) we get

$$(3.12) \quad N_0(r) - \overline{N}_0(r) \leq N_{(2)}(r, 1/(h_1 - 1)) + N_{(2)}(r, 1/(h_2 - 1)) = S(r, f).$$

and

$$(3.13) \quad \overline{N}_0(r) \leq \overline{N}_0(r, 1; h_1, h_2) + S(r, f).$$

Let z_0 be a common simple zero of $h_1 - 1$ and $h_2 - 1$, then at least one of $g(z_0) \neq 0$ and $g(z_0) \neq 1$ holds. If $g(z_0) \neq 1$, then it follows from (3.10) that z_0 is a simple zero of $f - g$, and so it follows from (3.10) that $g(z_0) \neq 0$. Suppose that z_0 is a zero of $h_0 - 1$ with multiplicity ≥ 2 , then it follows from (3.8) that

$$(3.14) \quad \overline{N}(r, S_1) \leq N_{(2)}(r, 1/(h_0 - 1)) = S(r, f),$$

where S_1 denotes a subset of the complex plane, in which every element is a simple zero of $h_1 - 1$ and a zero of $h_0 - 1$ with multiplicity ≥ 2 , $\overline{N}(r, S_1)$ denotes the reduced counting function of the elements in $S_1 \cap \{z : |z| < r\}$. Suppose that z_0 is a simple zero of $h_0 - 1$, then from (3.9) we get $f(z_0) \neq \infty$. From (3.8), (3.12), (3.14), the above analysis and the condition that f and g share $0, 1, \infty$ IM, we have

$$\begin{aligned} \overline{N}_0(r, 1; h_1, h_2) &\leq N_0(r) + N_{(2)}(r, 1/(h_1 - 1)) + N_{(2)}(r, 1/(h_2 - 1)) + \overline{N}(r, S_1) + S(r, f) \\ &\leq N_0(r) + S(r, f) \leq \overline{N}_0(r) + S(r, f), \end{aligned}$$

namely

$$(3.15) \quad \overline{N}_0(r, 1; h_1, h_2) \leq N_0(r) + S(r, f) \leq \overline{N}_0(r) + S(r, f).$$

From (3.8), (3.13) and (3.15) we get

$$(3.16) \quad N_0(r) = \overline{N}_0(r, 1; h_1, h_2) + S(r, f) = \overline{N}_0(r, 1; h_1, h_0) + S(r, f).$$

If $g(z_0) \neq 0$, in the same manner as above, we also get (3.16). Next we prove

$$(3.17) \quad T(r, f) = T(r, g) + S(r, f).$$

Suppose that $N_0(r) \neq S(r, f)$. From this and (3.16) we have

$$(3.18) \quad \overline{N}_0(r, 1; h_1, h_2) \neq S(r, f).$$

Thus from (3.7), (3.18) and Lemma 2.3 we see that there exist two integers s and t ($|s| + |t| > 0$) such that

$$(3.19) \quad h_1^s h_2^t \equiv 1.$$

Substituting (3.4) into (3.19) we get

$$(3.20) \quad f^t (f - 1)^s \equiv g^t (g - 1)^s.$$

From (3.20) and the condition that f is not a Möbius transformation of g , we deduce that $s \neq 0$, and $t \neq 0$ and $|s| \neq |t|$, and so f and g share $0, 1$ CM. Combining Lemma 2.4 and the condition $N_0(r) \neq S(r, f)$, we have (2.1) and that f, g assume one of the relations (i) and (iii) of Lemma 2.4. If f and g assume (i) of Lemma 2.4, then from Lemma 2.5 we have $s = 1$ and

$$(3.21) \quad f = 1 + e^\gamma + e^{2\gamma} + \cdots + e^{k\gamma}, \quad g = 1 + e^{-\gamma} + e^{-2\gamma} + \cdots + e^{-k\gamma}.$$

From (3.21) and Lemma 2.6 we deduce (3.17). If f and g assume (iii) of Lemma 2.4, then $s = k$ and

$$(3.22) \quad f = -e^{k\gamma} - e^{(k-1)\gamma} - \dots - e^{2\gamma} - e^\gamma, \quad g = -e^{-k\gamma} - e^{-(k-1)\gamma} - \dots - e^{-2\gamma} - e^{-\gamma}.$$

From (3.22) and Lemma 2.6 we get (3.17). Suppose that $N_0(r) = S(r, f)$. Then it follows from (3.16) that $\overline{N}_0(r, 1; h_1, h_0) = S(r, f)$. Combining (3.9) and the condition that f and g are nonconstant entire functions, we deduce $\overline{N}(r, 1/(h_0 - 1)) = S(r, f)$. Combining (3.6), (3.7) and the second fundamental theorem, we get

$$(3.23) \quad T(r, h_0) \leq \overline{N}(r, 1/h_0) + \overline{N}(r, 1/(h_0 - 1)) + \overline{N}(r, h_0) + S(r, h_0) = S(r, f).$$

Again from (3.9) and (3.23) we get (3.17). On the other hand, from (i) of Lemma 2.7 and the supposition that f is not a Möbius transformation of g , we get

$$(3.24) \quad T(r, f) + T(r, g) = \overline{N}(r, 1/f) + \overline{N}(r, 1/(f - 1)) + N_0(r) + S(r, f).$$

From (3.17), (3.24) and Lemma 2.2 we get

$$(3.25) \quad 2T(r, f) = N_1(r, 1/f) + N_1(r, 1/(f - 1)) + N_0(r) + S(r, f).$$

We discuss the following two subcases.

Subcase 2.1. Suppose that $N_0(r) \neq S(r, f)$. Then from the above supposition and Lemma 2.4 we have (2.1), moreover, f and g assume one of the three expressions (i)–(iii) of Lemma 2.4. From (2.1) and (3.25) we get

$$(3.26) \quad \frac{3}{2}T(r, f) \leq N_1\left(r, \frac{1}{f}\right) + N_1\left(r, \frac{1}{f-1}\right) + S(r, f),$$

From (3.26) we deduce

$$(3.27) \quad \delta_1(0, f) + \delta_1(1, f) \leq 1/2.$$

From the condition that f and g share 0, 1 IM, we deduce $S(r, f) = S(r, g)$. From this, in the same manner as above, we get

$$(3.28) \quad \delta_1(0, g) + \delta_1(1, g) \leq 1/2.$$

On the other hand, from Lemmas 2.5–2.6 and (i)–(iii) of Lemma 2.4 we get

$$T(r, f) = T(r, g) + O(1).$$

From the condition that f and g share $(a_1, 1)$ and $(a_2, 2)$, where $\{a_1, a_2\} = \{0, 1\}$, we have

$$N_1(r, 1/f) = N_1(r, 1/g)$$

and

$$N_1(r, 1/(f - 1)) = N_1(r, 1/(g - 1)).$$

Thus $\delta_1(0, f) = \delta_1(0, g)$ and $\delta_1(1, f) = \delta_1(1, g)$, and so it follows from (3.27) and (3.28) that

$$(3.29) \quad \delta_1(0, f) + \max\{\delta_1(1, f), \delta_1(1, g)\} \leq 1/2.$$

On the other hand, from (ii)–(iii) of Lemma 2.7 we deduce

$$(3.30) \quad T(r, f) = N_2(r, 1/(f - a)) + S(r, f), \quad T(r, g) = N_2(r, 1/(g - a)) + S(r, f),$$

where $a (\neq 0, 1)$ is an arbitrary finite complex number. From (3.30) we deduce

$$(3.31) \quad \sum_{a \neq 0, 1, \infty} \delta_2(a, f) = \sum_{a \neq 0, 1, \infty} \delta_2(a, g) = 0.$$

From (1.3) and (3.31) we see that (1.3) can be rewritten as

$$(3.32) \quad 2\delta_1(0, f) + \max\{\delta_1(1, f), \delta_1(1, g)\} > 1.$$

From (3.29) and (3.32) we deduce $\max\{\delta_1(1, f), \delta_1(1, g)\} < 0$, which is impossible.

Subcase 2.2. Suppose that $N_0(r) = S(r, f)$. Then from (3.12) and (3.25) we get

$$\begin{aligned} 2T(r, f) &= N_1\left(r, \frac{1}{f}\right) + N_1\left(r, \frac{1}{f-1}\right) + S(r, f) \\ &\leq T\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{f-1}\right) + S(r, f) \leq 2T(r, f) + S(r, f), \end{aligned}$$

which implies that

$$(3.33) \quad T(r, f) = N_1\left(r, \frac{1}{f}\right) + S(r, f) = N_1\left(r, \frac{1}{f-1}\right) + S(r, f).$$

From (3.33) we deduce

$$(3.34) \quad \delta_1(0, f) = \delta_1(1, f) = 0.$$

Similarly

$$(3.35) \quad \delta_1(0, g) = \delta_1(1, g) = 0.$$

On the other hand, from (ii)–(iii) of Lemma 2.7 and the supposition that f is not a Möbius transformation of g , we deduce (3.30) and (3.31). From (1.3), (3.31), (3.34) and (3.35) we see that (1.3) can be rewritten by $2\delta_1(0, f) > 1$, which contradicts (3.34). Theorem 1.5 is thus completely proved. \blacksquare

Proof of Theorem 1.6. Suppose that $f \not\equiv g$. We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g . Then f and g assume one of the six relations (i)–(vi) of Lemma 2.1. Moreover, from (i)–(vi) of Lemma 2.1 and Lemma 2.6 we get $T(r, f) = T(r, g) + O(1)$. Suppose that f and g assume one of the three relations (i)–(iii) of Lemma 2.1, then two of the three values 0, 1, ∞ are Picard exceptional values of f and g . If f and g assume (i) of Lemma 2.1, then 0 and ∞ are Picard exceptional values of f and g . From this and the second fundamental theorem and the condition $a \neq 0, 1, \infty$, we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, 1/f) + \overline{N}(r, 1/(f-a)) + S(r, f) \\ &= \overline{N}(r, 1/(f-a)) + S(r, f) \leq N(r, 1/(f-a)) + S(r, f) \\ &\leq T(r, f) + S(r, f), \end{aligned}$$

which implies $T(r, f) = N_1(r, 1/(f-a)) + S(r, f)$, $N_{(2)}(r, 1/(f-a)) = S(r, f)$ and $N_2(r, 1/(f-a)) = N_1(r, 1/(f-a)) + S(r, f)$. From this we get the left equality of (3.30). Similarly, we get the right equality of (3.30). From (3.30) we get (3.31).

Suppose that f and g assume one of the two relations (ii)–(iii) of Lemma 2.1, in the same manner as above, we get (3.30) and (3.31), and so (1.5) can be rewritten as

$$(3.36) \quad 2\delta_1(0, f) + 2\delta_1(\infty, f) > 3.$$

From (3.36) and (i)–(iii) of Lemma 2.1 we get $fg \equiv 1$. If f and g assume one of the three relations (iv)–(vi) of Lemma 2.1, then only one of $0, 1, \infty$ is a Picard exceptional value of f and g , and there exists some nonconstant entire function γ and some finite complex number $c (\neq 0, 1)$, such that f and g are given as one of the following three expressions respectively:

- (a) $f = (e^\gamma - 1)/(e^\gamma/c - 1), g = (e^\gamma - 1)/(e^\gamma - c)$;
- (b) $f = (c - 1)/(e^\gamma - 1), g = (c^{-1} - 1)/(e^{-\gamma} - 1)$;
- (c) $f = (e^\gamma - 1)/(c - 1), g = (e^{-\gamma} - 1)/(c^{-1} - 1)$.

If f and g assume (a), then

$$(3.37) \quad \begin{aligned} \delta_1(1, f) = \delta_1(1, g) = 1, \quad \delta_1(0, f) = \delta_1(0, g) = 0 \\ \text{and} \quad \delta_1(\infty, f) = \delta_1(\infty, g) = 0. \end{aligned}$$

On the other hand, in the same manner as in the proof of Theorem 1.5 we get (3.2). From (3.2) and (3.37) we get

$$2\delta_1(0, f) + 2\delta_1(\infty, f) + \max \left\{ \sum_{a \neq 0, 1, \infty} \delta_2(a, f), \sum_{a \neq 0, 1, \infty} \delta_2(a, g) \right\} \leq 1,$$

which contradicts (1.5). Similarly, if f and g assume one of the relations (b) and (c) we get contradictions.

Case 2. Suppose that f is not a Möbius transformation of g . First we will prove

$$(3.38) \quad \delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f) \leq \frac{3}{2}$$

and

$$(3.39) \quad \delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g) \leq \frac{3}{2}.$$

In fact, if $N_0(r) \neq S(r, f)$, in the same manner as Case 2 of the proof of Theorem 1.5 we get (3.4)–(3.20). From (3.20) and the condition that f is not a Möbius transformation of g , we deduce that f and g share $0, 1, \infty$ CM such that (2.1) holds and such that one of the three cases (i)–(iii) of Lemma 2.4 occurs. From (i)–(iii) of Lemma 2.4, Lemma 2.5 and Lemma 2.6 we deduce (3.17). From (2.1), (3.17), Lemma 2.2 and (i) of Lemma 2.7 we deduce

$$(3.40) \quad \frac{3}{2}T(r, f) \leq N_1(r, 1/f) + N_1(r, 1/(f - 1)) + N_1(r, f) + S(r, f).$$

From (3.40) we deduce (3.38). Similarly, we get (3.39). On the other hand, from (ii)–(iii) of Lemma 2.7 we deduce (3.31), and so (1.5) can be rewritten as (3.36). From (3.36) and (3.38) we get a contradiction. If $N_0(r) = S(r, f)$, from Lemma 2.2, (i) of Lemma 2.7 and the condition that f and g share $0, 1, \infty$ IM, we get

$$(3.41) \quad \begin{aligned} T(r, f) + T(r, g) = N_1(r, 1/f) + N_1(r, 1/(f - 1)) + N_1(r, f) + S(r, f) \\ \leq 3T(r, g) + S(r, f), \end{aligned}$$

which implies that

$$(3.42) \quad T(r, f) \leq 2T(r, g) + S(r, f).$$

From (3.41) and (3.42) we deduce (3.38). Similarly, we get (3.39). Finally, from (3.36) and (3.38) we get a contradiction. Theorem 1.6 is thus completely proved. \blacksquare

Proof of Theorem 1.9. Suppose that $f \not\equiv g$. We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g . Then f and g assume one of the six relations (i)–(vi) of Lemma 2.1. If f and g assume one of the three relations (i)–(iii) of Lemma 2.1, then two of the three values $0, 1, \infty$ are Picard exceptional values of f and g . Combining the second fundamental theorem, we deduce (3.30) and (3.31). From (3.31) we see that (1.11) can be rewritten as

$$(3.43) \quad 2\delta_1(0, f) + 2\delta_1(\infty, f) + 2\delta_1(0, g) + 2\delta_1(\infty, g) + \delta_1(1, f) + \delta_1(1, g) > 6.$$

From (3.43) and (i)–(iii) of Lemma 2.1 we get $fg \equiv 1$. If f and g assume one of the three relations (iv)–(vi) of Lemma 2.1, then only one of $0, 1, \infty$ is a Picard exceptional value of f and g , and there exists some nonconstant entire function γ and some finite complex number $c (\neq 0, 1)$, such that f and g are given as one of the three expressions (a), (b) and (c) in Case 1 of the proof of Theorem 1.6. If f and g assume (a), then (3.2) and (3.37) hold. From (3.2) and (3.37) we deduce

$$(3.44) \quad \begin{aligned} & 2\delta_1(0, f) + 2\delta_1(\infty, f) + 2\delta_1(0, g) + 2\delta_1(\infty, g) \\ & + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \delta_1(1, f) + \delta_1(1, g) \leq 4. \end{aligned}$$

From (1.11) and (3.44) we get a contradiction. If f and g assume (b), then we have (3.2) and

$$(3.45) \quad \begin{aligned} & \delta_1(0, f) = \delta_1(0, g) = 1, \quad \delta_1(1, f) = \delta_1(1, g) = 0 \\ & \text{and} \quad \delta_1(\infty, f) = \delta_1(\infty, g) = 0. \end{aligned}$$

From (3.2) and (3.45) we get

$$(3.46) \quad \begin{aligned} & 2\delta_1(0, f) + 2\delta_1(\infty, f) + 2\delta_1(0, g) + 2\delta_1(\infty, g) \\ & + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \delta_1(1, f) + \delta_1(1, g) \leq 6. \end{aligned}$$

From (1.11) and (3.46) we get a contradiction. If f and g assume (c), then we have (3.2) and

$$(3.47) \quad \begin{aligned} & \delta_1(0, f) = \delta_1(0, g) = 0, \quad \delta_1(1, f) = \delta_1(1, g) = 0 \\ & \text{and} \quad \delta_1(\infty, f) = \delta_1(\infty, g) = 1. \end{aligned}$$

From (3.2) and (3.46) we get (3.47), which contradicts (1.11).

Case 2. Suppose that f is not a Möbius transformation of g . Proceeding as in Case 2 of the proof of Theorem 1.6 we get (3.31), (3.38) and (3.39). From (3.31), (3.38) and (3.39) we get

$$2\delta_1(0, f) + 2\delta_1(\infty, f) + 2\delta_1(0, g) + 2\delta_1(\infty, g)$$

$$\begin{aligned}
& + \sum_{a \neq 0, 1, \infty} \delta_2(a, f) + \sum_{a \neq 0, 1, \infty} \delta_2(a, g) + \delta_1(1, f) + \delta_1(1, g) \\
& \leq 2\{\delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f)\} + 2\{\delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g)\} \\
(3.48) \quad & \leq 6,
\end{aligned}$$

which contradicts (1.11). Theorem 1.9 is thus completely proved. \blacksquare

Proof of Theorem 1.10. Suppose that $f \not\equiv g$. We discuss the following two cases.

Case 1. Suppose that f is a Möbius transformation of g . Then f and g assume one of the six relations (i)–(vi) of Lemma 2.1. If f and g assume one of the three relations (i)–(iii) of Lemma 2.1, then two of the three values $0, 1, \infty$ are Picard exceptional values of f and g . Combining the second fundamental theorem, we deduce (3.31). From (3.31), (1.12) and (1.13) we get

$$(3.49) \quad \delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f) + \delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g) > 3.$$

From (3.49) and (i)–(iii) of Lemma 2.1 we get the conclusion of Theorem 1.10. If f and g assume one of the three relations (iv)–(vi) of Lemma 2.1, then only one of $0, 1, \infty$ is a Picard exceptional value of f and g , and there exists some nonconstant entire function γ and some finite complex number $c (\neq 0, 1)$, such that f and g are given as one of the three expressions (a), (b) and (c) in Case 1 of the proof of Theorem 1.6, and $\delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f) + \delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g) = 2$, which contradicts (3.49).

Case 2. Suppose that f is not a Möbius transformation of g . Proceeding as in Case 2 of the proof of Theorem 1.6 we get (3.31), (3.38) and (3.39). From (3.31), (3.38) and (3.39) we get

$$(3.50) \quad \delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f) + \delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g) \leq 3.$$

However, from (3.31), (1.12) and (1.13) we get

$$\delta_1(0, f) + \delta_1(1, f) + \delta_1(\infty, f) + \delta_1(0, g) + \delta_1(1, g) + \delta_1(\infty, g) > 3,$$

which contradicts (3.50). Theorem 1.10 is thus completely proved. \blacksquare

Acknowledgement. The authors wish to express their thanks to the referee for his valuable suggestions and comments. Project supported by the NSFC (No. 10771121), the NSFC & RFBR (Joint Project) (No. 10911120056), the NSF of Shandong Province, China (No. Z2008A01) and the NSF of Shandong Province, China (No. 2009ZRB02536).

References

- [1] T. C. Alzahary and H.-X. Yi, Weighted sharing three values and uniqueness of meromorphic functions, *J. Math. Anal. Appl.* **295** (2004), no. 1, 247–257.
- [2] G. G. Gundersen, Meromorphic functions that share three or four values, *J. London Math. Soc. (2)* **20** (1979), no. 3, 457–466.
- [3] Q. Han and H.-X. Yi, On meromorphic functions that share three values of finite weights, *J. Math. Anal. Appl.* **334** (2007), no. 1, 314–332.
- [4] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

- [5] I. Lahiri, Weighted sharing of three values and uniqueness of meromorphic functions, *Kodai Math. J.* **24** (2001), no. 3, 421–435.
- [6] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.* **161** (2001), 193–206.
- [7] I. Lahiri, On a result of Ozawa concerning uniqueness of meromorphic functions. II, *J. Math. Anal. Appl.* **283** (2003), no. 1, 66–76.
- [8] I. Lahiri, Weighted sharing and a result of Ozawa, *Hokkaido Math. J.* **30** (2001), no. 3, 679–688.
- [9] I. Lahiri and A. Sarkar, On a uniqueness theorem of Tohge, *Arch. Math. (Basel)* **84** (2005), no. 5, 461–469.
- [10] X.-M. Li and H.-X. Yi, Results on weighted sharing of values for meromorphic functions, *Ann. Polon. Math.* **92** (2007), no. 1, 55–67.
- [11] M. Ozawa, Unicity theorems for entire functions, *J. Analyse Math.* **30** (1976), 411–420.
- [12] H. Ueda, Unicity theorems for meromorphic or entire functions. II, *Kodai Math. J.* **6** (1983), no. 1, 26–36.
- [13] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Acad. Publ., Dordrecht, 2003.
- [14] H.-X. Yi, Meromorphic functions that share two or three values, *Kodai Math. J.* **13** (1990), no. 3, 363–372.
- [15] H.-X. Yi, Meromorphic functions that share three values, *Bull. Hong Kong Math. Soc.* **2** (1998), no. 1, 55–64.
- [16] H.-X. Yi, Meromorphic functions with weighted sharing of three values, *Complex Var. Theory Appl.* **50** (2005), no. 12, 923–934.
- [17] Q. Zhang, On the results of Lahiri concerning uniqueness of meromorphic functions, *J. Math. Anal. Appl.* **318** (2006), no. 2, 707–725.
- [18] Q. C. Zhang, Meromorphic functions sharing three values, *Indian J. Pure Appl. Math.* **30** (1999), no. 7, 667–682.