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Completion of Rectangular Matrices and Power-Free Modules

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Abstract. We prove, in this article, that every finitely generated stably free R-module is power-free if and only if for any right invertible rectangular matrix (a_{ij}) over R, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix. Furthermore, we prove that every right invertible rectangular matrix over generalized stable, right repetitive rings can be completed in this way.

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1. Introduction

Let R be an associative ring with identity, and Let $s \in \mathbb{N}$. We say that a right R-module P is finitely generated stably free of rank s if there exists $m \in \mathbb{N}$ such that $P \oplus R^m \cong R^{m+s}$. We say that a finitely generated stably free right R-module P of rank s is power-free if there exists some $n \in \mathbb{N}$ such that $P^n \cong R^{ns}$. As is well known, every finitely generated stably free module of positive rank over a commutative ring is power-free (cf. [13, Theorem IV: 44]). In this article, we establish the equivalent conditions under which every finitely generated stably free module of positive rank over an associative ring is power-free.

We say that R is a generalized stable ring provided that $aR+bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in K(R)$, where $K(R) = \{r \in R \mid \exists s, t \in R$ such that $srt = 1\}$ (cf. [4]). The class of generalized stable ring is very large. It includes exchange rings having stable range one, exchange rings satisfying general comparability, exchange rings satisfying related comparability, purely infinite simple ring, etc. So far, all known exchange rings belong to such class of rings. Following Goodearl [8], an element $u \in R$ is right repetitive provided that for each finitely generated right ideal I of R, the right ideal $\sum_{i=0}^{\infty} u^i I$ is finitely generated,

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or, equivalently, $\sum_{i=0}^{\infty} u^i I = \sum_{i=0}^{k} u^i I$ for some $k \in \mathbb{N}$. A ring R is right repetitive if all elements of R are right repetitive. Clearly, all commutative rings and all right noetherian rings are right repetitive. Also we know that every ring integral over its center is right repetitive. It is well known that all surjective endomorphisms of finitely generated right R-modules are automorphism if and only if all matrix rings $M_n(R)$ are right repetitive (cf. [8, Theorem 7]). Let R be a generalized stable, right repetitive ring. We prove that for any right invertible rectangular matrix (a_{ij}) over R, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.

Throughout, all rings are associative with identity and all modules are right unitary modules. We use $\mathbb{N}$ to denote the set of all natural numbers. $P^n (n \in \mathbb{N})$ denotes the n copies of right R-module P. A $n \times m$ matrix (a_{ij}) is called a rectangular matrix in case $n < m$. $R^{1\times n}$ and $R^{n\times 1}$ stand for the R-$M_n(R)$-bimodule and $M_n(R)$-R-bimodule of all tuples of row vectors and column vectors, respectively.

2. Completion of matrices

Let $\sigma : R^m \to R^n$ be a R-morphism. Then we have a matrix $A \in M_{n\times m}(R)$ corresponding to σ. Let $A = (a_{ij}) \in M_{n\times m}(R)$, and let $I_s = \mathrm{diag}(1, \cdots, 1) \in M_s(R)$. We use the Kronecker product $A \otimes I_s$ to stand for the matrix $(a_{ij}I_s) \in M_{ns\times ms}(R)$.

Theorem 2.1. *Let R be a ring. Then the following are equivalent:*

(1) *Every finitely generated stably free right R-module of positive rank is power-free.*

(2) *For any right invertible rectangular matrix (a_{ij}), there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof.
(2) $\Longrightarrow$(1) Let P be a finitely generated stably free right R-module of positive rank. Then there exist $m, n \in \mathbb{N}$ such that $0 \longrightarrow P \longrightarrow R^m \xrightarrow{\alpha} R^n \longrightarrow 0$ $(n < m)$ is an exact sequence. Since R^n is a projective right R-module, there exists $\beta : R^n \to R^m$ such that $\alpha\beta = 1$. Let $\{\eta_1, \cdots, \eta_m\}$ and $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be bases of R^m and R^n, respectively. Then $\alpha(\eta_1, \cdots, \eta_m) = (\varepsilon_1, \cdots, \varepsilon_n)A$, where $A \in M_{n\times m}(R)$. Clearly, we have some $m \times n$ matrix B such that $AB = I_n$. By the hypothesis, there exists $s \in \mathbb{N}$ and an invertible $ms \times ms$ matrix

$$\begin{pmatrix} A \otimes I_s \\ * \end{pmatrix}_{ms\times ms};$$

hence, we have $C \in M_{ms}(R)$ such that

$$\begin{pmatrix} A \otimes I_s \\ * \end{pmatrix}_{ms\times ms} C = \mathrm{diag}(I_s, \cdots, I_s)_{ms\times ms}.$$

Clearly, there exists an exact sequence $0 \longrightarrow P^s \longrightarrow R^{ms} \xrightarrow{\varphi} R^{ns} \longrightarrow 0$, where the corresponding matrix of φ is $\mathrm{diag}(A, A, \cdots, A)_{ns\times ms}$. By elementary row and column transformations, we have

$$\mathrm{diag}(A, A, \cdots, A)_{ns\times ms} \to \begin{pmatrix} A \otimes I_s \end{pmatrix}_{ns\times ms}.$$

Hence there exists an exact sequence $0 \longrightarrow P^s \longrightarrow R^{ms} \xrightarrow{\psi} R^{ns} \longrightarrow 0$, where the corresponding matrix of ψ is

$$\begin{pmatrix} A \otimes I_s \end{pmatrix}_{ns\times ms}.$$

Let $\{\delta_1, \cdots, \delta_{ms}\}$ and $\{\zeta_1, \cdots, \zeta_{ns}\}$ be bases of R^{ms} and R^{ns}, respectively. Then

$$\psi(\delta_1, \cdots, \delta_{ms}) = (\zeta_1, \cdots, \zeta_{ns}) \begin{pmatrix} A \otimes I_s \end{pmatrix}_{ns\times ms}.$$

Hence,

$$\begin{aligned} \psi(\delta_1, \cdots, \delta_{ms})C &= (\zeta_1, \cdots, \zeta_{ns}) \begin{pmatrix} A \otimes I_s \end{pmatrix}_{ns\times ms} C \\ &= (\zeta_1, \cdots, \zeta_{ns}) \begin{pmatrix} I_s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & I_s & 0 & \cdots & 0 \end{pmatrix}_{ns\times ms}. \end{aligned}$$

Let $(\xi_1, \cdots, \xi_{ms}) = (\delta_1, \cdots, \delta_{ms})C$. Since C is an invertible matrix, $(\xi_1, \cdots, \xi_{ms})$ is a basis of R^{ms} as well. In addition,

$$\psi(\xi_1, \cdots, \xi_{ms}) = (\zeta_1, \cdots, \zeta_{ns}) \begin{pmatrix} I_s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & I_s & 0 & \cdots & 0 \end{pmatrix}_{ns\times ms}.$$

As a result, we get

$$\begin{aligned} P^s &\cong \text{Ker } \psi \\ &= \left\{ \textstyle\sum_{i=1}^{ms} \xi_i r_i \mid \psi(\xi_1, \cdots, \xi_{ms}) \begin{pmatrix} r_1 \\ \vdots \\ r_{ms} \end{pmatrix} = 0, r_1, \cdots, r_{ms} \in R \right\} \\ &= \left\{ \textstyle\sum_{i=1}^{ms} \xi_i r_i \mid (\zeta_1, \cdots, \zeta_{ns}) \begin{pmatrix} I_s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & I_s & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_{ms} \end{pmatrix} = 0 \right\} \\ &= \{ \textstyle\sum_{i=1}^{ms} \xi_i r_i \mid r_1 = \cdots = r_{ns} = 0, r_{ns+1}, \cdots, r_{ms} \in R \} \\ &= \{ \xi_{ns+1} r_{ns+1} + \cdots + \xi_{ms} r_{ms} \mid r_{ns+1}, \cdots, r_{ms} \in R \} \\ &= R^{(m-n)s}, \end{aligned}$$

as required.

(1)$\Longrightarrow$(2) Let $A \in M_{n\times m}(R)$ be a right invertible rectangular matrix. Then we have $B \in M_{m\times n}(R)$ such that $AB = I_n$. Let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ and $\{\eta_1, \cdots, \eta_m\}$ be bases of R^n and R^m, respectively. Construct two maps $\varphi : R^n \to R^m, \varphi(\varepsilon_1, \cdots, \varepsilon_n) = (\eta_1, \cdots, \eta_m)B; \phi : R^m \to R^n, \phi(\eta_1, \cdots, \eta_m) = (\varepsilon_1, \cdots, \varepsilon_n)A$. Clearly, $\phi\varphi(\varepsilon_1, \cdots, \varepsilon_n) = \phi(\eta_1, \cdots, \eta_m)B = (\varepsilon_1, \cdots, \varepsilon_n)AB = (\varepsilon_1, \cdots, \varepsilon_n)$. Hence $\phi\varphi = 1$, and we get a split exact sequence $0 \longrightarrow \text{Ker } \phi \longrightarrow R^m \xrightarrow{\phi} R^n \longrightarrow 0$. So Ker ϕ is a finitely generated stably free module of rank $n - m$. By the hypothesis, we can find $s \in \mathbb{N}$ such that $(\text{Ker } \phi)^s \cong R^{(m-n)s}$. Clearly, we have an exact sequence $0 \longrightarrow (\text{Ker } \phi)^s \longrightarrow R^{ms} \xrightarrow{\psi} R^{ns} \longrightarrow 0$, where the matrix corresponding to ψ is $\text{diag}(A, \cdots, A)_{ns\times ms}$. Thus, we have some $E \cong R^{ns}$ such that $R^{ms} = \text{Ker } \psi \oplus E$. Let $\{\delta_1, \cdots, \delta_{(m-n)s}\}$ be a basis of Ker ψ, and let $\{\delta_{(m-n)s+1},\cdots,\delta_{ms}\}$ be a basis of E. Then $\{\delta_1,\cdots,\delta_{ms}\}$ is a basis of R^{ms}.

Let $\{\zeta_1,\cdots,\zeta_{ms}\}$ and $\{\mu_1,\cdots,\mu_{ns}\}$ be bases of R^{ms} and R^{ns}, respectively. In addition, $\psi(\zeta_1,\cdots,\zeta_{ms})=(\mu_1,\cdots,\mu_{ns})\mathrm{diag}(A,\cdots,A)_{ns\times ms}$. Obviously, we have a $C\in GL_{ms}(R)$ such that $(\zeta_1,\cdots,\zeta_{ms})=(\delta_1,\cdots,\delta_{ms})C$. So for $r_1,\cdots,r_{ms}\in R$,

$$\begin{aligned}\text{Ker }\psi &= \left\{(\zeta_1,\cdots,\zeta_{ms})\begin{pmatrix} r_1\\ \vdots\\ r_{ms}\end{pmatrix} \mid (\mu_1,\cdots,\mu_{ns})\mathrm{diag}(A,\cdots,A)\begin{pmatrix} r_1\\ \vdots\\ r_{ms}\end{pmatrix}=0,\right\}\\ &= \left\{(\delta_1,\cdots,\delta_{ms})C\begin{pmatrix} r_1\\ \vdots\\ r_{ms}\end{pmatrix} \mid (\mu_1,\cdots,\mu_{ns})\mathrm{diag}(A,\cdots,A)\begin{pmatrix} r_1\\ \vdots\\ r_{ms}\end{pmatrix}=0.\right\}\end{aligned}$$

Let

$$C\begin{pmatrix} r_1\\ \vdots\\ r_{ms}\end{pmatrix}=\begin{pmatrix} k_1\\ \vdots\\ k_{ms}\end{pmatrix}.$$

Then for $k_1,\cdots,k_{ms}\in R$,

$$\text{Ker }\psi = \left\{(\delta_1,\cdots,\delta_{ms})\begin{pmatrix} k_1\\ \vdots\\ k_{ms}\end{pmatrix} \mid (\mu_1,\cdots,\mu_{ns})\mathrm{diag}(A,\cdots,A)C^{-1}\begin{pmatrix} k_1\\ \vdots\\ k_{ms}\end{pmatrix}=0.\right\}$$

Let $\mathrm{diag}(A,\cdots,A)_{ns\times ms}C^{-1}=(T_1,\cdots,T_{ms})$. It follows that

$$\begin{aligned}\text{Ker }\psi &= \left\{(\delta_1,\cdots,\delta_{ms})\begin{pmatrix} k_1\\ \vdots\\ k_{ms}\end{pmatrix} \mid T_1k_1+\cdots+T_{ms}k_{ms}=0_{ns\times 1}\right\}\\ &= \delta_1R\oplus\cdots\oplus\delta_{(m-n)s}R.\end{aligned}$$

As

$$\delta_1=(\delta_1,\cdots,\delta_{ms})\begin{pmatrix} 1\\ 0\\ \vdots\\ 0\end{pmatrix}_{ms\times 1}\in\text{Ker }\psi,$$

we get $T_1=0_{ns\times 1}$. Likewise, $T_2=\cdots=T_{(m-n)s}=0_{ns\times 1}$. Let

$$C=(c_{ij})\in M_{ms\times ms}(R)\quad\text{and}\quad D=(c_{ij})_{i\geq (m-n)s+1}.$$

Then

$$\begin{aligned}I_{ns} &= \mathrm{diag}(A,\cdots,A)\mathrm{diag}(B,\cdots,B)_{ms\times ns}\\ &= (0_{ns\times 1},\cdots,0_{ns\times 1},T_{(m-n)s+1},\cdots,T_{ms})C\mathrm{diag}(B,\cdots,B)_{ms\times ns}\\ &= (T_{(m-n)s+1},\cdots,T_{ms})D\mathrm{diag}(B,\cdots,B)_{ms\times ns}.\end{aligned}$$

Set $(\lambda_1,\cdots,\lambda_{ns})=I_{ns}-D\mathrm{diag}(B,\cdots,B)(T_{(m-n)s+1},\cdots,T_{ms})$. Since

$$(T_{(m-n)s+1},\cdots,T_{ms})(I_{ns}-D\mathrm{diag}(B,\cdots,B)_{ms\times ns}(T_{(m-n)s+1},\cdots,T_{ms}))=0_{ns},$$

we derive that $(\delta_{(m-n)s+1},\cdots,\delta_{ms})\lambda_1 \in \text{Ker } \psi$; hence, $\lambda_1 = 0$. Analogously, we deduce that $\lambda_2 = \cdots = \lambda_{ns} = 0$. So

$$D\text{diag}(B,\cdots,B)_{ms\times ns}(T_{(m-n)s+1},\cdots,T_{ms}) = I_{ns}.$$

It follows that $(T_{(m-n)s+1},\cdots,T_{ms}) \in GL_{ns}(R)$. Clearly, $\text{diag}(A,\cdots,A)_{ns\times ms} = (T_{(m-n)s+1},\cdots,T_{ms})D$, and that there exist $U \in GL_{ns}(R)$ and $V \in GL_{ms}(R)$ such that $U\text{diag}(A,\cdots,A)_{ns\times ms}V = (a_{ij}I_s)_{ns\times ms}$. Hence

$$U^{-1}(a_{ij}I_s)_{ns\times ms}V^{-1} = (T_{(m-n)s+1},\cdots,T_{ms})(0_{ns\times(m-n)s}, I_{ns})C.$$

Thus

$$(a_{ij}I_s) = U(T_{(m-n)s+1},\cdots,T_{ms})(0_{ns\times(m-n)s}, I_{ns})CV,$$

so

$$\begin{pmatrix} A\otimes I_s \\ * \end{pmatrix}_{ms\times ms} = \text{diag}(I_{(m-n)s}, U(T_{(m-n)s+1},\cdots,T_{ms}))CV\begin{pmatrix} 0 & I_{ns} \\ I_{(m-n)s} & 0 \end{pmatrix},$$

as desired. ∎

Corollary 2.1. *Let (a_{ij}) be a right invertible rectangular matrix over a commutative ring R. Then there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Let P be a finitely generated stably free right R-module of positive rank. Then we have $P \oplus R^n \cong R^m (n < m)$. In view of [13, Theorem IV 44], there exists $s \in \mathbb{N}$ such hat $P^s \cong R^t$. Hence $R^{t+ns} \cong R^{ms}$. As R is commutative, it has the invariant basis property. We infer that $t + ns = ms$; hence, $t = (m-n)s$. So P is power-free. In view of Theorem 2.1, we get the result. ∎

Corollary 2.2. *Let (a_{ij}) be a left invertible $n \times m (m < n)$ matrix over a commutative ring R. Then there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Since $A \in M_{n\times m}(R)$ is left invertible, one checks that $(A^T)^{op} \in M_{m\times n}(R^{op})$ is right invertible, where R^{op} is the opposite ring of R. By virtue of Corollary 2.1, we have $s \in \mathbb{N}$ such that $(A^T)^{op} \otimes (I_s^T)^{op}$ can be completed to an invertible $ns \times ns$ matrix over R^{op} by the addition of $(n-m)s$ further rows. Therefore $A \otimes I_s$ can be completed to an invertible $ns \times ns$ matrix by the addition of $(n-m)s$ further columns, as asserted. ∎

We note that a right invertible rectangular matrix over a ring R may not be invertible. Let $\mathbb{F}$ be a field, and let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $A \in M_{2\times 3}(\mathbb{F})$ is right invertible, while it is not left invertible. Let $\mathbb{R}$ be the field of real numbers, let $R = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ and let x, y, z denote the images in R of X, Y, Z. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{3\times 1}(R)$$

is left invertible. By [12, Proposition 11.2.3],

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

cannot be completed to an invertible 3×3 matrix by the addition of 2 further columns. In view of Corollary 2.2, we have some $s \in \mathbb{N}$ such that

$$\begin{pmatrix} xI_s \\ yI_s \\ zI_s \end{pmatrix}$$

can be completed to an invertible $3s \times 3s$ matrix by the addition of $2s$ further columns. More explicitly, we may choose any $s \geq 2$.

Let R be a simple ring. If $M_n(R)$ is directly finite for all $n \in \mathbb{N}$, we claim that every stably free right R-module of positive rank is power free. It follows from Theorem 2.1 that for any right invertible rectangular matrix (a_{ij}), there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.

3. Generalized stable, right repetitive rings

The main purpose of this section is to investigate completion of rectangular matrices over a generalized stable ring. Recall that n is in the general linear range of a ring R provided that $P \oplus R \cong R^{n+1}$ implies that $P \cong R^n$. If $m-n$ is in the general linear range of R, then $0 \to P \to R^m \stackrel{\sigma}{\to} R^n \to 0 (n < m)$ implies that $P \cong R^{m-n}$. Now we extend this fact and generalize [13, Theorem IV 36] to non-commutative rings by a similar route.

Lemma 3.1. *Let $0 \to P \to R^m \stackrel{\sigma}{\to} R^n \to 0 (n < m)$ be an exact sequence of right R-modules. If there exists a basis $\{\varepsilon_1, \cdots, \varepsilon_m\}$ of R^m such that R^n can be generated by $\{\sigma(\varepsilon_1), \cdots, \sigma(\varepsilon_t)\} (t \leq m-n)$, then $P \cong R^{m-n}$.*

Proof. Let $\{\eta_1, \cdots, \eta_n\}$ be a basis of R^n. Then there is $n \times m$ matrix A such that $\sigma(\varepsilon_1, \cdots, \varepsilon_m) = (\eta_1, \cdots, \eta_n)A$. Since R^n can be generated by $\{\sigma(\varepsilon_1), \cdots, \sigma(\varepsilon_t)\} (t \leq m-n)$, we have $n \times t$ matrix C such that $(\eta_1, \cdots, \eta_n) = \sigma(\varepsilon_1, \cdots, \varepsilon_t)C$. Let $A = [B, D]$ with $B \in M_{n \times t}(R)$. Then

$$\sigma(\varepsilon_1, \cdots, \varepsilon_m) = \sigma(\varepsilon_1, \cdots, \varepsilon_t)CA = \sigma(\varepsilon_1, \cdots, \varepsilon_t)C[B, D],$$

so $\sigma(\varepsilon_1, \cdots, \varepsilon_t) = \sigma(\varepsilon_1, \cdots, \varepsilon_t)CB$. Hence

$$(\eta_1, \cdots, \eta_n) = \sigma(\varepsilon_1, \cdots, \varepsilon_t)C = \sigma(\varepsilon_1, \cdots, \varepsilon_t)CBC = (\eta_1, \cdots, \eta_n)BC,$$

and then $BC = I_n$. As $m - t \geq n$, we have $D = [E, F]$ with $E \in M_n(R)$. By elementary transformations, we get $A = [B, E, F] \to [B, E - BC(E - I_n), F] \to [B, I_n, F] \to [I_n, 0]$. So we have $U \in GL_n(R)$ and $V \in GL_m(R)$ such that $UAV = [I_n, 0]$. As a result,

$$P \cong \text{Ker } \sigma = \left\{ \sum_{i=1}^{m} \varepsilon_i r_i \mid (\eta_1, \cdots, \eta_n)A \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = 0, r_1, \cdots, r_m \in R \right\}$$

$$= \left\{ \sum_{i=1}^{m} \varepsilon_i r_i | A \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = 0, r_1, \cdots, r_m \in R \right\}.$$

Let

$$\begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} = V^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$$

and $(\varepsilon_1', \cdots, \varepsilon_m') = (\varepsilon_1, \cdots, \varepsilon_m)V$. Then

$$P \cong \left\{ (\varepsilon_1, \cdots, \varepsilon_m)V \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} \mid (I_n, 0) \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} = 0, s_1, \cdots, s_m \in R \right\}$$

$$= \left\{ (\varepsilon_1', \cdots, \varepsilon_m') \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} | s_1 = \cdots = s_n = 0, s_{n+1}, \cdots, s_m \in R \right\}.$$

Clearly, $\{\varepsilon_1', \cdots, \varepsilon_m'\}$ is a basis of R^m. Therefore $P \cong \varepsilon_{n+1}' R \oplus \cdots \oplus \varepsilon_m R \cong R^{m-n}$, as asserted. ∎

Lemma 3.2. *Let $0 \to P \to R^m \xrightarrow{\sigma} R^n \to 0$ $(n < m)$ be an exact sequence of right R-modules, and let A be the matrix representing σ with respect to the standard bases of R^m and R^n. If there exist $U \in GL_n(R)$ and $V \in GL_m(R)$ such that $UAV = [B_{n\times t}, 0] (t \leq m - n)$, then $P \cong R^{m-n}$.*

Proof. Let $\{\varepsilon_1, \cdots, \varepsilon_m\}$ and $\{\eta_1, \cdots, \eta_n\}$ be bases of R^m and R^n, respectively. Suppose that $(\varepsilon_1, \cdots, \varepsilon_m) = (\eta_1, \cdots, \eta_n)A$. Then

$$R^n = \mathrm{Im}\ \sigma = \left\{ \sigma(\varepsilon_1, \cdots, \varepsilon_m) \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \mid r_1, \cdots, r_m \in R \right\}$$

$$= \left\{ (\eta_1, \cdots, \eta_n)U^{-1}(UAV)V^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \mid r_1, \cdots, r_m \in R \right\}.$$

Set

$$\begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} = V^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}.$$

Then we get

$$R^n = \left\{ (\eta_1, \cdots, \eta_n)U^{-1}(\beta_1, \cdots, \beta_t, 0, \cdots, 0) \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} | s_1, \cdots, s_m \in R \right\}$$

$$= \left\{ (\eta_1, \cdots, \eta_n)U^{-1}(\beta_1, \cdots, \beta_t, 0, \cdots, 0) \begin{pmatrix} s_1 \\ \vdots \\ s_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} | s_1, \cdots, s_t \in R \right\}$$

$$= \left\{ (\eta_1, \cdots, \eta_n)AV \begin{pmatrix} s_1 \\ \vdots \\ s_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} | s_1, \cdots, s_t \in R \right\}$$

$$= \left\{ \sigma((\varepsilon_1, \cdots, \varepsilon_m)V) \begin{pmatrix} s_1 \\ \vdots \\ s_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} | s_1, \cdots, s_t \in R \right\}.$$

Let $(\varepsilon_1', \cdots, \varepsilon_m') = (\varepsilon_1, \cdots, \varepsilon_m)V$. Then $\{\varepsilon_1', \cdots, \varepsilon_m'\}$ is a basis of R^m as well. Further,

$$R^n = \left\{ \sigma(\varepsilon_1', \cdots, \varepsilon_m') \begin{pmatrix} s_1 \\ \vdots \\ s_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid s_1, \cdots, s_t \in R \right\}$$

$$= \left\{ \sigma(\varepsilon_1', \cdots, \varepsilon_t') \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} | s_1, \cdots, s_t \in R \right\}.$$

In other words, R^n is generated by $\{\sigma(\varepsilon_1'), \cdots, \sigma(\varepsilon_t')\}$. It follows from Lemma 3.1 that $P \cong R^{m-n}$. ∎

Lemma 3.3. *Let $A \in M_{n\times m}(R)$ be a right invertible rectangular matrix over a generalized stable ring R. Then there exist $U \in GL_n(R)$ and $V \in GL_m(R)$ such that*

$$UAV = \begin{pmatrix} I_{n-1} & \mathbf{0}_{(n-1)\times(m-n+1)} \\ \mathbf{0}_{1\times(n-1)} & b_n \quad b_{n+1} \quad \cdots \quad b_m \end{pmatrix},$$

where $b_n, \cdots, b_m \in R$.

Proof. The result is trivial if $n = 1$. Assume now that $n \geq 2$. Let $A = (a_{ij})_{n\times m}$. Since A is a right invertible matrix, we have $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 1$ for some $x_1, x_2, \cdots, x_m \in R$. As R is a generalized stable ring, there exists some $z \in R$ such that $a_{11} + a_{12}x_2z + \cdots + a_{1m}x_mz = w \in K(R)$. Assume that $swt = 1$ for $s, t \in R$. Then

$$\begin{pmatrix} \begin{matrix} s & 0 \\ 1-wts & wt \end{matrix} & \mathbf{0}_{2\times(n-2)} \\ \mathbf{0}_{(n-2)\times 2} & I_{n-2} \end{pmatrix} A \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & x_2z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_mz & \cdots & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \begin{matrix} sw & 0 \\ 1-tsw & t \end{matrix} & \mathbf{0}_{2\times(m-2)} \\ \mathbf{0}_{(m-2)\times 2} & I_{m-2} \end{pmatrix}$$
$$= \begin{pmatrix} * & 1 & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix},$$

so we prove that

$$\begin{pmatrix} 1 & \mathbf{0} \\ * & I_{n-1} \end{pmatrix} \begin{pmatrix} \begin{matrix} s & 0 \\ 1-wts & wt \end{matrix} & \mathbf{0}_{2\times(n-2)} \\ \mathbf{0}_{(n-2)\times 2} & I_{n-2} \end{pmatrix} A \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & x_2z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_mz & \cdots & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \begin{matrix} sw & 0 \\ 1-tsw & t \end{matrix} & \mathbf{0}_{2\times(m-2)} \\ \mathbf{0}_{(m-2)\times 2} & I_{m-2} \end{pmatrix} \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ & I_{m-2} \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & I_{m-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \mathbf{0}_{1\times(m-1)} \\ \mathbf{0}_{(n-1)\times 1} & * \end{pmatrix}.$$

Clearly,

$$\begin{pmatrix} s & 0 \\ 1-wts & wt \end{pmatrix}^{-1} = \begin{pmatrix} wt & 1-wts \\ 0 & s \end{pmatrix}, \begin{pmatrix} sw & 0 \\ 1-tsw & t \end{pmatrix}^{-1} = \begin{pmatrix} t & 1-tsw \\ 0 & sw \end{pmatrix}.$$

Thus we can reduce A to the form

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times(m-1)} \\ \mathbf{0}_{(n-1)\times 1} & * \end{pmatrix}_{n\times m}$$

by elementary transformations. By induction, we complete the proof. ∎

Theorem 3.1. *Let R be a generalized stable, right repetitive ring. Then for any right invertible rectangular matrix (a_{ij}) over R, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Let P be a finitely generated stably free right R-module of positive rank. Then there is an exact sequence $0 \to P \to R^m \stackrel{\sigma}{\to} R^n \to 0 (n < m)$. Let $\{\varepsilon_1, \cdots, \varepsilon_m\}$ and $\{\eta_1, \cdots, \eta_n\}$ be bases of R^m and R^n, respectively. Then we have a $n \times m$ matrix

A such that $\sigma(\varepsilon_1, \cdots, \varepsilon_m) = (\eta_1, \cdots, \eta_n)A$. Since σ is a split epimorphism, we can find a $m \times n$ matrix B such that $AB = I_n$. As R is a generalized stable ring, by virtue of Lemma 3.3, there exist $U \in GL_n(R)$ and $V \in GL_m(R)$ such that

$$UAV = \begin{pmatrix} I_{n-1} & \multicolumn{4}{c}{\mathbf{0}_{(n-1)\times(m-n+1)}} \\ \mathbf{0}_{1\times(n-1)} & b & b_{n+1} & \cdots & b_m \end{pmatrix}.$$

Observing that

$$P \cong \operatorname{Ker}\sigma \cong \left\{ (\varepsilon_1, \cdots, \varepsilon_m) \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} | (\eta_1, \cdots, \eta_n)A \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = 0, r_1, \cdots, r_m \in R \right\},$$

for $r_1, \cdots, r_m \in R$ we have

$$P \cong \left\{ \sum_{i=1}^{m} \varepsilon_i r_i | (\eta_1, \cdots, \eta_n) U^{-1} \begin{pmatrix} I_{n-1} & \multicolumn{4}{c}{\mathbf{0}_{(n-1)\times(m-n+1)}} \\ \mathbf{0}_{1\times(n-1)} & b & b_{n+1} & \cdots & b_m \end{pmatrix} \right.$$
$$\left. \times V^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} = 0. \right\}$$

Set $(\varepsilon'_1, \cdots, \varepsilon'_m) = (\varepsilon_1, \cdots, \varepsilon_m)V$ and

$$\begin{pmatrix} r'_1 \\ \vdots \\ r'_m \end{pmatrix} = V^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}.$$

So

$$P \cong \left\{ (\varepsilon'_1, \cdots, \varepsilon'_m) \begin{pmatrix} r'_1 \\ \vdots \\ r'_m \end{pmatrix} \mid \begin{pmatrix} I_{n-1} & \multicolumn{4}{c}{\mathbf{0}_{(n-1)\times(m-n+1)}} \\ \mathbf{0}_{1\times(n-1)} & b & b_{n+1} & \cdots & b_m \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} r'_1 \\ \vdots \\ r'_m \end{pmatrix} = 0, r'_1, \cdots, r'_m \in R \right\}.$$

Clearly, $(\varepsilon'_1, \cdots, \varepsilon'_m)$ is a basis of R^m. Thus,

$$P \cong \left\{ (\varepsilon'_n, \cdots, \varepsilon'_m) \begin{pmatrix} r'_n \\ \vdots \\ r'_m \end{pmatrix} | \begin{pmatrix} b, & b_{n+1}, & \cdots, & b_m \end{pmatrix} \begin{pmatrix} r'_n \\ \vdots \\ r'_m \end{pmatrix} = 0, \right\}$$

for $r'_n, \cdots, r'_m \in R$. Obviously, $\{\varepsilon'_n, \cdots, \varepsilon'_m\}$ is a basis of R^{m-n+1}. Let δ be a basis of R. Construct a map $\varphi : R^{m-n+1} \to R$ given by $\varphi(\varepsilon'_n, \cdots, \varepsilon'_m) = \delta(b, b_{n+1}, \cdots, b_m)$. It suffices to prove that Ker $\varphi \cong P$ is stably free of positive rank. Let $B = (b, b_{n+1}, \cdots, b_m)$. Then we have $(m - n + 1) \times 1$ matrix C such that $BC = 1$. Let $\alpha = [b_{n+1}, \cdots, b_m]$. By elementary transformations, one easily checks that

$$\begin{bmatrix} B & 0 & 0 \\ 0 & b & \alpha \end{bmatrix} \to \begin{bmatrix} B & -BC & 0 \\ 0 & b & \alpha \end{bmatrix} \to \begin{bmatrix} B & -1 & 0 \\ 0 & b & \alpha \end{bmatrix}$$
$$\to \begin{bmatrix} 0 & -1 & 0 \\ bB & 0 & \alpha \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b^2 & b\alpha & \alpha \end{bmatrix}.$$

Furthermore, we get

$$\begin{bmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & b & \alpha \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & b^2 & b\alpha & \alpha \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & -1 & 0 & 0 \\ 0 & 0 & b^2 & b\alpha & \alpha \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & b^2B & b^2 & b\alpha & \alpha \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^3 & b^2\alpha & b\alpha & \alpha \end{bmatrix}$$

By induction, we prove that

$$\begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & b^t & b^{t-1}\alpha & \cdots & b\alpha & \alpha \end{bmatrix}$$

using elementary row and column transformations. Let $\{\zeta_1, \cdots, \zeta_{(1+t(m-n))}\}$ be a basis of $R^{1+t(m-n)}$. We construct a map $\psi : R^{1+t(m-n)} \to R$ given by

$$\psi(\zeta_1, \cdots, \zeta_{(1+t(m-n))}) = \delta(b^t, b^{t-1}\alpha, \cdots, b\alpha, \alpha).$$

It is easy to see that $P^t \cong \text{Ker } \psi$.

Since R is right repetitive, there exists some $i_{(n+1)} \in \mathbb{N}$ such that

$$b^{i_{(n+1)}} b_{n+1} = b^{i_{(n+1)}-1} b_{(n+1)} r_{11} + \cdots + bb_{(n+1)} r_{1i_{(n+1)}} + b_{(n+1)} r_{1(i_{(n+1)}+1)}.$$

Analogously, we have $i_{(n+2)}, \cdots, i_m \in \mathbb{N}$ such that

$$b^{i_{(n+2)}} b_{n+2} = b^{i_{(n+2)}-1} b_{(n+2)} r_{21} + \cdots + bb_{(n+2)} r_{2i_{(n+2)}} + b_{(n+2)} r_{2(i_{(n+2)}+1)}, \cdots, b^{i_m} b_m$$
$$= b^{i_m - 1} b_m r_{m1} + \cdots + bb_m r_{mi_m} + b_m r_{m(i_m+1)}.$$

Choose $p = \max(i_{(n+1)}, i_{(n+2)}, \cdots, i_m)$. If $i \geq p$, then

$$b^i b_j = b^{p-1} b_j s_{(p-1)j} + b^{p-2} b_j s_{(p-2)j} + \cdots + bb_j s_{1j} + b_j s_{0j}$$

for $n+1 \leq j \leq m$. Assume that $t \geq p+1$. Since $b^i\alpha = (b^i b_{(n+1)}, \cdots, b^i b_m)$, we can find invertible matrices U, V such that

$$U(b^t, b^{t-1}\alpha, \cdots, b\alpha, \alpha)V = (b^t, b^{p-1}\alpha, \cdots, b\alpha, \alpha, 0 \cdots, 0).$$

As $n < m$, it follows from $t \geq p+1$ that $1 + p(m-n) \leq (1 + t(m-n)) - 1$. By virtue of Lemma 3.2, $P^t \cong R^{(m-n)t}$. Therefore we complete the proof from Theorem 2.1. ∎

Corollary 3.1. *If R is a generalized stable ring which is integral over its center, then for any right invertible rectangular matrix (a_{ij}) over R, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Let $a, x \in R$. Since R is integral over its center, we have $n \in \mathbb{N}$ and central elements $r_0, \cdots, r_{n-1}$ such that $a^n = a^{n-1}r_{n-1} + \cdots + ar_1 + r_0$. Hence $a^n x = a^{n-1}xr_{n-1} + \cdots + axr_1 + xr_0$. This means that R is right repetitive. By virtue of Theorem 3.1, the proof is true. ∎

Corollary 3.2. *If R is a right noetherian generalized stable ring, then for any right invertible rectangular matrix (a_{ij}) over R, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Since R is a right noetherian ring, it is right repetitive. According to Theorem 3.1, we complete the proof. ∎

Lemma 3.4. *If every finitely generated stably free right R-module of positive rank is power-free, then every finitely generated stably free right $M_n(R)$-module of positive rank is power-free.*

Proof. Let P be a finitely generated stably free right $M_n(R)$-module of positive rank. Then we have

$$P \oplus M_n(R)^s \cong M_n(R)^t (s < t).$$

Hence

$$P \bigotimes_{M_n(R)} R^{n\times 1} \oplus M_n(R)^s \bigotimes_{M_n(R)} R^{n\times 1} \cong M_n(R)^t \bigotimes_{M_n(R)} R^{n\times 1},$$

and so

$$P \bigotimes_{M_n(R)} R^{n\times 1} \oplus R^{ns} \cong R^{nt}.$$

That is, $P \bigotimes_{M_n(R)} R^{n\times 1}$ is a stably free right R-module of positive rank. Thus we can find some $p \in \mathbb{N}$ such that $(P \bigotimes_{M_n(R)} R^{n\times 1})^p \cong R^{pn(t-s)}$. As a result, $P^p \bigotimes_{M_n(R)} R^{n\times 1} \cong R^{pn(t-s)}$, and so

$$\left(P^p \bigotimes_{M_n(R)} R^{n\times 1}\right) \bigotimes_R R^{1\times n} \cong R^{pn(t-s)} \bigotimes_R R^{1\times n}.$$

It follows that

$$P^p \bigotimes_{M_n(R)} \left(R^{n\times 1} \bigotimes_R R^{1\times n}\right) \cong \left(R^{n\times 1} \bigotimes_R R^{1\times n}\right)^{p(t-s)}.$$

As

$$R^{n\times 1} \bigotimes_R R^{1\times n} \cong M_n(R),$$

we deduce that $P^p \cong M_n(R)^{p(t-s)}$, as required. ∎

Proposition 3.1. *Let R be a generalized stable, right repetitive ring. Then for any right invertible rectangular matrix (a_{ij}) over $M_n(R)$, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. It is obvious by Theorem 3.1, Theorem 2.1 and Lemma 3.4. ∎

Corollary 3.3. *Let R be a right noetherian generalized stable ring. Then for any right invertible rectangular matrix (a_{ij}) over $M_n(R)$, there exists $s \in \mathbb{N}$ such that $(a_{ij}I_s)$ can be completed to an invertible matrix.*

Proof. Since R is a right noetherian ring, it is right repetitive ring. Therefore the result follows from Proposition 3.1. ∎

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