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Convergence Theorems on an Iterative Method for Variational Inequality Problems and Fixed Point Problems

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Abstract. In this paper, we propose an explicit viscosity approximation method for finding a common element of the set of fixed points of strict pseudo-contractions and of the set of solutions of variational inequalities with inverse-strongly monotone mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

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1. Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a nonlinear mapping. Recall the following definitions:

(a) A is said to be *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(b) A is said to be *α-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

(c) A is said to be *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

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Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For given $z \in H$ and $u \in C$, we see that the following inequality holds

$$\langle u-z, v-u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that $P_C x$ is characterized by the property: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of the mapping T. Recall the following definitions.

(1) T is said to be *α-contractive* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(2) T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(3) T is said to be *strongly pseudo-contractive* with the coefficient $\lambda \in (0, 1)$ if

$$\langle Tx - Ty, x - y\rangle \leq \lambda\|x - y\|^2, \quad \forall x, y \in C.$$

(4) T is said to be *strictly pseudo-contractive* with the coefficient $k \in (0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

For such a case, T is also said to be a *k-strict pseudo-contraction*.

(5) T is said to be *pseudo-contractive* if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Clearly, the class of strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of strict pseudo-contractions; See, for example [1, 2, 20].

The class of strict pseudo-contractions is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of fixed points for strict pseudo-contractions. Recently, Zhou [21] considered a convex combination method to study strict pseudo-contractions. More precisely, take $t \in (0, 1)$ and define a mapping S_t by

$$S_t x = tx + (1 - t)Tx, \quad \forall x \in C,$$

where T is a strict pseudo-contraction. Under appropriate restrictions on t, it is proved the mapping S_t is nonexpansive. Therefore, the techniques of studying nonexpansive mappings can be applied to study more general strict pseudo-contractions.

Recently, many authors studied the problem of finding a common element of the set of solution of variational for an inverse-strongly monotone mapping and of the set of fixed points of a nonexpansive mapping; see, for example, [5, 6, 9, 11–13, 16, 17, 19] and the references therein. Iiduka and Takahashi [9] proved the following theorem.

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \dots,$ where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Further, Yao and Yao [19] introduced an iterative method for finding a common element of the set of fixed points of a single nonexpansive mapping and the set of solution of variational inequalities for a α-inverse-strongly monotone mapping. To be more precise, they proved the following theorem.

Theorem 1.2. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose that $x_1 = u \in C$ and $\{x_n\}$, $\{y_n\}$ are given by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n A) y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda\}$ is a sequence in $[0, 2a]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda \in [a, b]$ for some a, b with $0 < a < b < 2a$ and

(a) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap\Omega}u$.

In this paper, motivated by Ceng and Yao [6], Iiduka and Takahashi [9], Wang and Guo [17], and Yao and Yao [19], we continue to study the problem of finding a common element of the set of fixed points of strict pseudo-contractions and of the set of solutions to variational inequalities with inverse-strongly monotone mappings by using viscosity approximation methods in the framework of Hilbert spaces.

The results presented in this paper improve and extend the corresponding results announced by many others.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1. [21] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a λ-strict pseudo-contraction with a fixed point. Define $S : C \to C$ by $Sx = \alpha x + (1-\alpha)Tx$ for each $x \in C$. Then, as $\alpha \in [\lambda, 1)$, S is nonexpansive such that $F(S) = F(T)$.*

The following lemma is a corollary of Bruck's result in [4].

Lemma 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1 and T_2 be two nonexpansive mappings from C into itself with a common fixed point. Define a mapping $S : C \to C$ by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Proof. It is obvious that $F(T_1) \cap F(T_2) \subset F(S)$. Fixing $x^* \in F(S)$ and $y \in F(T_1) \cap F(T_2)$, we see that

$$\begin{aligned} \|x^* - y\| &= \|\lambda T_1 x^* + (1-\lambda)T_2 x^* - y\| \\ &\leq \lambda \|T_1 x^* - y\| + (1-\lambda)\|T_2 x^* - y\| \\ &\leq \lambda \|x^* - y\| + (1-\lambda)\|x^* - y\| \\ &= \|x^* - y\|. \end{aligned}$$

Since H is strictly convex, we see that

$$x^* = \lambda T_1 x^* + (1-\lambda)T_2 x^* = T_1 x^* = T_2 x^*.$$

That is, $x^* \in F(T_1) \cap F(T_2)$. This implies that $F(S) = F(T_1) \cap F(T_2)$. On the other hand, it is easy to see that S is also nonexpansive. This completes the proof. ∎

Lemma 1.3. [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4. [18] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $f : C \to C$ be a τ-contraction with the $0 < \tau < 1$ and let $S : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S_k : C \to C$ by $S_k x = kx + (1-k)Sx$, $\forall x \in C$. Assume that $\mathcal{F} := F(S) \cap VI(C,A) \cap VI(C,B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\tag{2.1} \begin{cases} x_1 \in C, \\ z_n = P_C(x_n - \mu_n B x_n), \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n], \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ $\{\delta_{(1,n)}\}$, $\{\delta_{(2,n)}\}$ and $\{\delta_{(3,n)}\}$ are sequences in $[0,1]$. Assume that the control sequences satisfy the following restrictions:

(C1) $\alpha_n + \beta_n + \gamma_n = \delta_{(1,n)} + \delta_{(2,n)} + \delta_{(3,n)} = 1$, $\forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $a \leq \lambda_n \leq 2\alpha$, $b \leq \mu_n \leq 2\beta$, *where a, b are two positive constants;*
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;
(C5) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C6) $\lim_{n\to\infty} \delta_{(i,n)} = \delta_i \in (0,1)$ *for each $i = 1, 2, 3$.*

Then the sequence $\{x_n\}$ defined by the algorithm (2.1) converges strongly to some $\bar{x} \in \mathcal{F}$ which solves uniquely the following variational inequality:

$$\tag{2.2} \langle f(\bar{x}) - \bar{x}, x - \bar{x} \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Proof. The proof is divided into four steps.

Step 1: Show that the sequence $\{x_n\}$ is bounded.

First, we show that the mappings $I - \lambda_n A$ and $I - \mu_n B$ are nonexpansive for each $n \geq 1$. Actually, for any $x, y \in C$, from the condition (C3), we have

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\ &\leq \|x-y\|^2 - 2\lambda_n\langle Ax - Ay, x - y\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - 2\lambda_n\alpha\|Ax - Ay\|^2 + \lambda_n^2\|Ax - Ay\|^2 \\ &= \|x-y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

This implies that $I - \lambda_n A$ is nonexpansive for each $n \geq 1$, so is, $I - \mu_n B$. It follows that

$$\tag{2.3} \|z_n - p\| = \|P_C(x_n - \mu_n B x_n) - p\| \leq \|x_n - p\|$$

and

$$\|y_n - p\| = \|P_C(x_n - \lambda_n A_n) - p\| \le \|x_n - p\| \tag{2.4}$$

for any $p \in F$. On the other hand, from Lemma 1.1, we have S_k is a nonexpansive mapping. Put

$$w_n = \delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n, \quad \forall n \ge 1.$$

From (2.3) and (2.4), we see that

$$\begin{aligned} \|w_n - p\| &= \|\delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n - p\| \\ &\le \delta_{(1,n)} \|S_k x_n - p\| + \delta_{(2,n)} \|y_n - p\| + \delta_{(3,n)} \|z_n - p\| \\ &\le \delta_{(1,n)} \|x_n - p\| + \delta_{(2,n)} \|x_n - p\| + \delta_{(3,n)} \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.5}$$

From the algorithm (2.1), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n w_n - p\| \\ &\le \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\ &\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\ &\le \alpha_n \tau \|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\le [1 - \alpha_n(1 - \tau)] \|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

By mathematical inductions, we obtain that

$$\|x_n - p\| \le \left\{ \|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \tau} \right\}, \quad \forall n \ge 1.$$

This shows that the sequence $\{x_n\}$ is bounded.

Step 2: Show that $x_n - w_n \to 0$ as $n \to \infty$.

First, we estimate $\|y_{n+1} - y_n\|$ and $\|z_{n+1} - z_n\|$. Notice that

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &= \|P_C(I - \lambda_{n+1} A) x_{n+1} - P_C(I - \lambda_n A) x_n\| \\ &\le \|(I - \lambda_{n+1} A) x_{n+1} - (I - \lambda_{n+1} A) x_n + (I - \lambda_{n+1} A) x_n - (I - \lambda_n A) x_n\| \\ &\le \|(I - \lambda_{n+1} A) x_{n+1} - (I - \lambda_{n+1} A) x_n\| + \|(I - \lambda_{n+1} A) x_n - (I - \lambda_n A) x_n\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}| \|A x_n\|. \end{aligned} \tag{2.6}$$

Similarly, we can obtain that

$$\|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\| + |\mu_n - \mu_{n+1}| \|B x_n\|. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} &\|w_{n+1} - w_n\| \\ &= \|\delta_{(1,(n+1))} S_k x_{n+1} + \delta_{(2,(n+1))} y_{n+1} + \delta_{(3,(n+1))} z_{n+1} \\ &\quad - (\delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n)\| \\ &\le \delta_{(1,(n+1))} \|S_k x_{n+1} - S_k x_n\| + \|S_k x_n\| |\delta_{(1,(n+1))} - \delta_{(1,n)}| \\ &\quad + \delta_{(2,(n+1))} \|y_{n+1} - y_n\| + \|y_n\| |\delta_{(2,(n+1))} - \delta_{(2,n)}| \end{aligned}$$

$$+\delta_{(3,(n+1))}\|z_{n+1}-z_n\|+\|z_n\||\delta_{(3,(n+1))}-\delta_{(3,n)}|$$

$$\le \|x_{n+1}-x_n\|+M\left(\sum_{i=1}^{3}|\delta_{(i,(n+1))}-\delta_{(i,n)}|+|\lambda_n-\lambda_{n+1}|+|\mu_n-\mu_{n+1}|\right), \tag{2.8}$$

where M is an appropriate constant such that $M = \max\{\|S_k x_n\|, \|y_n\|, \|z_n\|, \|Ax_n\|, \|Bx_n\| : n \ge 1\}$. Put $l_n = (x_{n+1}-\beta_n x_n)/(1-\beta_n)$ for all $n \ge 1$. That is,

$$x_{n+1}=(1-\beta_n)l_n+\beta_n x_n, \quad \forall n \ge 1. \tag{2.9}$$

Now, we estimate $\|l_{n+1}-l_n\|$. From

$$\begin{aligned} l_{n+1}-l_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}w_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(x_n)+\gamma_n w_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}w_{n+1}-\frac{\alpha_n}{1-\beta_n}f(x_n) \\ &\quad -\frac{1-\beta_n-\alpha_n}{1-\beta_n}w_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-w_{n+1})+\frac{\alpha_n}{1-\beta_n}(w_n-f(x_n))+w_{n+1}-w_n, \end{aligned}$$

we arrive at

$$\begin{aligned} \|l_{n+1}-l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-w_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|w_n-f(x_n)\| \\ &\quad +\|w_{n+1}-w_n\|. \end{aligned} \tag{2.10}$$

Substituting (2.8) into (2.10), we obtain that

$$\begin{aligned} &\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-w_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|w_n-f(x_n)\| \\ &\quad +M\left(\sum_{i=1}^{3}|\delta_{(i,(n+1))}-\delta_{(i,n)}|+|\lambda_n-\lambda_{n+1}|+|\mu_n-\mu_{n+1}|\right). \end{aligned}$$

It follows from the conditions (C2), (C4), (C5) and (C6) that

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_{n+1}\|)<0.$$

It follows from Lemma 1.4 that

$$\lim_{n\to\infty}\|l_n-x_n\|=0. \tag{2.11}$$

Thanks to (2.9), we see that $x_{n+1}-x_n=(1-\beta_n)(l_n-x_n)$. Combining the condition (C5) and (2.11), we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.12}$$

On the other hand, from the iterative algorithm (2.1), we see that

$$x_{n+1}-x_n=\alpha_n(f(x_n)-x_n)+\gamma_n(w_n-x_n)$$

and the conditions (C2) and (C5), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{2.13}$$

Step 3: Show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \le 0$.

To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\rangle. \tag{2.14}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to v. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup v$. Assume also that $\lambda_{n_i} \to \lambda \in [a, 2\alpha]$ and $\mu_{n_i} \to \mu \in [b, 2\beta]$, respectively. Next, we prove that

$$v \in F := F(S) \cap VI(C, A) \cap VI(C, B).$$

In fact, define a mapping $V : C \to C$ by

$$Vx = \delta_1 S_k x + \delta_2 P_C(I - \lambda A)x + \delta_3 P_C(I - \mu B)x, \quad \forall x \in C.$$

From Lemma 1.2, we see that V is a nonexpansive mapping such that

$$F(V) = F(S_k) \cap F(P_C(I - \lambda A)) \cap F(P_C(I - \mu B)) = F(S) \cap VI(C, A) \cap VI(C, B).$$

On the other hand, we have

$$\begin{aligned}
\|Vx_{n_i} - x_{n_i}\| &\le \|Vx_{n_i} - w_{n_i}\| + \|w_{n_i} - x_{n_i}\| \\
&= \|\delta_1 S_k x_{n_i} + \delta_2 P_C(I - \lambda A)x_{n_i} + \delta_3 P_C(I - \mu B)x_{n_i} \\
&\quad - [\delta_{(1,n_i)} S_k x_{n_i} + \delta_{(2,n_i)} y_{n_i} + \delta_{(3,n_i)} z_{n_i}]\| + \|w_{n_i} - x_{n_i}\| \\
&\le |\delta_1 - \delta_{(1,n_i)}|\|S_k x_{n_i}\| + \delta_2\|P_C(I - \lambda A)x_{n_i} - P_C(I - \lambda_{n_i} A)x_{n_i}\| \\
&\quad + \|P_C(I - \lambda_{n_i} A)x_{n_i}\||\delta_2 - \delta_{(2,n)}| + \delta_3\|P_C(I - \mu B)x_{n_i} \\
&\quad - P_C(I - \mu_{n_i} B)x_{n_i}\| + \|P_C(I - \mu_{n_i} B)x_{n_i}\||\delta_3 \\
&\quad - \delta_{(3,n)}| + \|w_{n_i} - x_{n_i}\| \\
&\le |\delta_1 - \delta_{(1,n_i)}|\|S_k x_{n_i}\| + \delta_2|\lambda_{n_i} - \lambda|\|Ax_{n_i}\| \\
&\quad + \|P_C(I - \lambda_{n_i} A)x_{n_i}\||\delta_2 - \delta_{(2,n)}| + \delta_3|\mu_{n_i} - \mu|\|Bx_{n_i}\| \\
&\quad + \|P_C(I - \mu_{n_i} B)x_{n_i}\||\delta_3 - \delta_{(3,n)}| + \|w_{n_i} - x_{n_i}\| \\
&\le M\left(\sum_{i=1}^{3}|\delta_i - \delta_{(i,n)}| + |\mu_{n_i} - \mu| + |\lambda_{n_i} - \lambda|\right) + \|w_{n_i} - x_{n_i}\|.
\end{aligned}$$

From (2.13) and the condition (C6), we arrive at

$$\lim_{i\to\infty}\|Vx_{n_i} - x_{n_i}\| = 0. \tag{2.15}$$

It follows from Lemma 1.3 that

$$v \in F(V) = F(S) \cap VI(C, A) \cap VI(C, B).$$

Thanks to (2.14), we arrive at

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x}\rangle = \langle f(\bar{x}) - \bar{x}, v - \bar{x}\rangle \le 0. \tag{2.16}$$

Step 4: Show that $x_n \to \bar{x}$ as $n \to \infty$.

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n w_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &= \alpha_n \langle f(x_n) - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n \langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad + \gamma_n \langle w_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\le \alpha_n (\langle f(x_n) - f(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle) \\ &\quad + \beta_n \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n \|w_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\le \alpha_n \tau \|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\quad + (1 - \alpha_n)\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\ &\le \frac{[1 - \alpha_n(1 - \tau)]}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) \\ &\quad + \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1} - \bar{x}\|^2 \le [1 - \alpha_n(1 - \tau)]\|x_n - \bar{x}\|^2 + 2\alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle. \tag{2.17}$$

From the condition (C2), (2.16) and applying Lemma 1.5 to (2.17), we obtain that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0.$$

This completes the proof. ∎

Remark 2.1. Theorem 2.1 improve the corresponding result of [19] in the following aspects:

(1) from nonexpansive mappings to strict pseudo-contractions;
(2) from a single inverse-strongly monotone mapping to a pair of inverse-strongly monotone mapping;
(3) the proof line is more concise than that of [19]'s.

If the mapping S is nonexpansive, then $S_k = S_0 = S$. We can obtain the following result from Theorem 2.1 immediately.

Corollary 2.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : C \to H$ be a β-inverse-strongly monotone mapping, respectively. Let $f : C \to C$ be a τ-contraction and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $F := F(S)\cap VI(C, A)\cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 \in C, \\ z_n = P_C(x_n - \mu_n B x_n), \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta_{(1,n)} S x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n], \quad n \ge 1, \end{cases}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ $\{\delta_{(1,n)}\}$, $\{\delta_{(2,n)}\}$ and $\{\delta_{(3,n)}\}$ are sequences in $[0, 1]$. Assume that the control sequences satisfy the following restrictions:

(C1) $\alpha_n + \beta_n + \gamma_n = \delta_{(1,n)} + \delta_{(2,n)} + \delta_{(3,n)} = 1$, $\forall n \ge 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $a \le \lambda_n \le 2\alpha$, $b \le \mu_n \le 2\beta$, *where* a, b *are two positive constants;*
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;
(C5) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(C6) $\lim_{n\to\infty} \delta_{(i,n)} = \delta_i \in (0, 1)$ *for each* $i = 1, 2, 3$.

Then the sequence $\{x_n\}$ *defined by the above algorithm converges strongly to some* $\bar{x} \in \mathcal{F}$.

3. Applications

As some applications of Theorem 2.1, we consider another class of nonlinear mapping: Strict pseudo-contraction.

Theorem 3.1. *Let* H *be a real Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $T_1 : C \to C$ *be an* k_1*-strict pseudo-contraction and let* $T_2 : C \to C$ *be a* k_2*-strict pseudo-contraction. Let* $f : C \to C$ *be a* τ*-contraction with the* $0 < \tau < 1$ *and* $S : C \to C$ *a* k*-strict pseudo-contraction with a fixed point. Define a mapping* $S_k : C \to C$ *by* $S_k x = kx + (1 - k)Sx$, $\forall x \in C$. *Assume that* $\mathcal{F} := F(S) \cap F(T_2) \cap F(T_2) \neq \emptyset$. *Suppose that* $\{x_n\}$ *is generated by the following iterative algorithm:*

$$(3.1) \quad \begin{cases} x_1 \in C, \\ z_n = (1 - \mu_n)x_n + \mu_n B x_n, \\ y_n = (1 - \lambda_n)x_n + \lambda_n A x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n], \quad n \ge 1, \end{cases}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ *are positive sequences and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ $\{\delta_{(1,n)}\}$, $\{\delta_{(2,n)}\}$ *and* $\{\delta_{(3,n)}\}$ *are sequences in* $[0, 1]$. *Assume that the control sequences satisfy the following restrictions:*

(C1) $\alpha_n + \beta_n + \gamma_n = \delta_{(1,n)} + \delta_{(2,n)} + \delta_{(3,n)} = 1, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $a \le \lambda_n \le (1 - k_1)$, $b \le \mu_n \le (1 - k_2)$, *where* a, b *are two positive constants;*
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;
(C5) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(C6) $\lim_{n\to\infty} \delta_{(i,n)} = \delta_i \in (0, 1)$ *for each* $i = 1, 2, 3$.

Then the sequence $\{x_n\}$ *defined by the algorithm* (3.1) *converges strongly to some* $\bar{x} \in F$ *which solves uniquely the following variational inequality:*

$$\langle f(\bar{x}) - \bar{x}, x - \bar{x} \rangle \le 0, \quad \forall x \in \mathcal{F}.$$

Proof. Put $A = I - T_1$ and $B = I - T_2$. We see that A is $(1 - k_1)/2$-inverse-strongly monotone and B is $(1 - k_2)/2$-inverse-strongly monotone. We also have $F(T_1) = VI(C, A)$, $F(T_2) = VI(C, B)$. Notice that

$$P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n A x_n$$

and

$$P_C(x_n - \mu_n B x_n) = (1 - \mu_n)x_n + \mu_n B x_n.$$

From Theorem 2.1, we can obtain the desired conclusion easily. ∎

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse-strongly monotone mapping and let $B : H \to H$ be a β-inverse-strongly monotone mapping. Let $f : H \to H$ be a τ-contraction with the $0 < \tau < 1$ and $S : H \to H$ a nonexpansive mapping with a fixed point. Assume that $F := F(S) \cap A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by the following iterative algorithm:*

$$(3.2) \quad \begin{cases} x_1 \in H, \\ z_n = x_n - \mu_n B x_n, \\ y_n = x_n - \lambda_n A x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta_{(1,n)} S_k x_n + \delta_{(2,n)} y_n + \delta_{(3,n)} z_n], \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\}$, $\{\mu_n\}$ are positive sequences and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ $\{\delta_{(1,n)}\}$, $\{\delta_{(2,n)}\}$ and $\{\delta_{(3,n)}\}$ are sequences in $[0, 1]$. Assume that the control sequences satisfy the following restrictions:

(C1) $\alpha_n + \beta_n + \gamma_n = \delta_{(1,n)} + \delta_{(2,n)} + \delta_{(3,n)} = 1$, $\forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $a \leq \lambda_n \leq 2\alpha$, $b \leq \mu_n \leq 2\beta$, *where a, b are two positive constants;*
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;
(C5) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C6) $\lim_{n\to\infty} \delta_{(i,n)} = \delta_i \in (0, 1)$ *for each $i = 1, 2, 3$.*

Then the sequence $\{x_n\}$ defined by the algorithm (3.2) converges strongly to some $\bar{x} \in F$ which solves the uniquely the following variational inequality:

$$\langle f(\bar{x}) - \bar{x}, x - \bar{x} \rangle \leq 0, \quad \forall x \in F.$$

Proof. Since $A^{-1}(0) = VI(H, A)$, $B^{-1}(0) = VI(H, B)$ and $P_H = I$, we can conclude the desired conclusion from Theorem 2.1 immediately. ∎

Finally, we consider the following convex feasibility problem (CFP):

$$\text{Finding a } x \in \bigcap_{i=1}^{N} C_i,$$

where $N \geq 1$ is an integer and each C_i is assumed to be the solution set of the variational inequality problem (1.1). There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [8, 10], computer tomography [14] and radiation therapy treatment planning [7].

The following result can be concluded from Theorem 2.1 easily. We, therefore, omit the proof here.

Theorem 3.3. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{N} : C \to H$ be a family of η_i-inverse-strongly monotone mappings, for each $i \geq 1$. Let $f : C \to C$ be a τ-contraction with the $0 < \tau < 1$ and let $S : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S_k : C \to C$ by $S_k x = kx + (1 - k)Sx$, $\forall x \in C$. Assume that $\mathcal{F} := \cap_{i=1}^{N} VI(C, A_i) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following* *iterative algorithm:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n[\delta_1 S_k x_n + \sum_{i=1}^{N} \delta_{i+1} P_C(x_n - \lambda_{(i,n)} A_i)x_n], \quad n \geq 1,$$

where $\delta_1, \delta_2, \ldots, \delta_{N+1} \in [0,1]$ *such that* $\sum_{i=1}^{N+1} \delta_i = 1$, $\{\lambda_{(i,n)}\}$ *are positive sequences and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$. *Assume that the control sequences satisfy the following restrictions:*

(C1) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $a_i \leq \lambda_{(i,n)} \leq 2\eta_i$, *where* a_i *is some positive constant for each* $1 \leq i \leq N$;
(C4) $\lim_{n\to\infty}(\lambda_{(i,(n+1))} - \lambda_{(i,n)}) = 0$ *for each* $1 \leq i \leq N$;
(C5) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by the above iterative algorithm converges strongly to some* $\bar{x} \in \mathcal{F}$.

References

[1] F. E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, *Proc. Nat. Acad. Sci. U.S.A.* **53** (1965), 1272–1276.

[2] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967), 197–228.

[3] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, in *Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968)*, 1–308, Amer. Math. Soc., Providence, R. I.

[4] R. E. Bruck, Jr., Properties of fixed-point sets of nonexpansive mappings in Banach spaces, *Trans. Amer. Math. Soc.* **179** (1973), 251–262.

[5] L.-C. Ceng, C. Wang and J.-C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, *Math. Methods Oper. Res.* **67** (2008), no. 3, 375–390.

[6] L.-C. Ceng and J.-C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, *Appl. Math. Comput.* **190** (2007), no. 1, 205–215.

[7] Y. Censor and S. A. Zenios, *Parallel Optimization*, Oxford Univ. Press, New York, 1997.

[8] P. L. Combettes, *The convex feasibility problem in Image Recovery:* In *Advances in Imaging and Electron Physics*, P. Hawkes, Ed., vol. 95, pp. 155–270, Academic Press, Orlando, FL, USA, 1996.

[9] H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* **61** (2005), no. 3, 341–350.

[10] T. Kotzer, N. Cohen and J. Shamir, Images to ration by a novel method of parallel projection onto constraint sets, *Opt. Lett.* **20** (1995), 1172–1174.

[11] X. Qin, M. Shang and H. Zhou, Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces, *Appl. Math. Comput.* **200** (2008), no. 1, 242–253.

[12] X. Qin, M. Shang and Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, *Math. Comput. Modelling* **48** (2008), no. 7-8, 1033–1046.

[13] X. Qin, M. Shang and Y. Su, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, *Nonlinear Anal.* **69** (2008), no. 11, 3897–3909.

[14] M. I. Sezan and H. Stark, Application of convex projection theory to image recovery in tomograph and related areas, in *Image Recovery: Theory and Application*, H. Stark, Ed., pp. 155–270 Academic Press, Orlando, FL, USA, 1987.

[15] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005), no. 1, 227–239.

[16] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* **118** (2003), no. 2, 417–428.

[17] S. Wang and B. Guo, Viscosity approximation methods for nonexpansive mappings and inverse-strongly monotone mappings in Hilbert spaces, *J. Appl. Math. Comput.* **28** (2008), no. 1-2, 351–365.

[18] H.-K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc. (2)* **66** (2002), no. 1, 240–256.

[19] Y. Yao and J.-C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Math. Comput.* **186** (2007), no. 2, 1551–1558.

[20] H. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.* **343** (2008), no. 1, 546–556.

[21] H. Zhou, Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.* **69** (2008), no. 2, 456–462.