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A Fixed Point Approach to the Stability of Differential Equations $y' = F(x, y)$

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Abstract. Using a fixed point method, the Hyers-Ulam-Rassias stability will be proved for the differential equations of the form $y'(x) = F(x, y(x))$.

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1. Introduction

Let Y be a normed space and let I be an open interval. Assume that for any function $f : I \to Y$ satisfying the differential inequality

$$\|a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x)\| \le \varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0$$

such that $\|f(x) - f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only. Then, we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where φ, $\Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability, refer to [5, 6].

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [14, 15]). Here, we will introduce a result of

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Alsina and Ger (see [1]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \le 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura and Miyajima: They proved in [17] that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [11]).

Recently, Miura, Miyajima and Takahasi also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(x) + g(x)y(x) = 0$, where $g(x)$ is a continuous function, while the author proved the Hyers-Ulam stability of linear differential equations of other type (see [7, 8, 9, 12, 13]).

In this paper, for a bounded and continuous function $F(x, y)$, we will adopt the idea of Cădariu and Radu [2, 3] and prove the Hyers-Ulam-Rassias stability as well as the Hyers-Ulam stability of the differential equations of the form

$$y'(x) = F(x, y(x)). \tag{1.1}$$

2. Preliminaries

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;
(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M_3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [4]. This theorem will play an important rôle in proving our main theorems.

Theorem 2.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:*

(a) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;
(b) x^* is the unique fixed point of Λ in

$$X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\};$$

(c) If $y \in X^*$, then

$$d(y, x^*) \le \frac{1}{1 - L} d(\Lambda y, y).$$

3. Hyers-Ulam-Rassias stability

Recently, Cădariu and Radu [2] applied the fixed point method to the investigation of the Jensen's functional equation. Using such an idea, they could present a proof for the Hyers-Ulam stability of that equation (ref. [3, 10, 16]).

In this section, by using the idea of Cădariu and Radu, we will prove the Hyers-Ulam-Rassias stability of the differential equation (1.1).

Theorem 3.1. *For given real numbers a and b with $a < b$, let $I = [a, b]$ be a closed interval and choose a $c \in I$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition*

$$|F(x, y) - F(x, z)| \leq L|y - z| \tag{3.1}$$

for any $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - F(x, y(x))| \leq \varphi(x) \tag{3.2}$$

for all $x \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left| \int_c^x \varphi(\tau)\, d\tau \right| \leq K\varphi(x) \tag{3.3}$$

for each $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))\, d\tau \tag{3.4}$$

(*consequently, y_0 is a solution to* (1.1)) *and*

$$|y(x) - y_0(x)| \leq \frac{K}{1 - KL}\, \varphi(x) \tag{3.5}$$

for all $x \in I$.

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\} \tag{3.6}$$

and introduce a generalized metric on X as follows:

$$d(f, g) = \inf\{C \in [0, \infty] \mid |f(x) - g(x)| \leq C\varphi(x)\ \forall\, x \in I\}. \tag{3.7}$$

(We will here give a proof for the triangle inequality only. Assume that $d(f, g) > d(f, h) + d(h, g)$ would hold for some $f, g, h \in X$. Then, by (3.7), there would exist an $x_0 \in I$ with

$$\begin{aligned} |f(x_0) - g(x_0)| &> \{d(f, h) + d(h, g)\}\varphi(x_0) \\ &= d(f, h)\varphi(x_0) + d(h, g)\varphi(x_0) \\ &\geq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)|, \end{aligned}$$

a contradiction.)

We assert that (X, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (X, d). Then, for any $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) \leq \varepsilon$ for all $m, n \geq N_\varepsilon$. It further follows from (3.7) that

$$\forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N}\ \forall\, m, n \geq N_\varepsilon\ \forall\, x \in I:\ |h_m(x) - h_n(x)| \leq \varepsilon\varphi(x). \tag{3.8}$$

If x is fixed, (3.8) implies that $\{h_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{h_n(x)\}$ converges for each $x \in I$. Thus, we can define a function $h : I \to \mathbb{R}$ by

$$h(x) = \lim_{n\to\infty} h_n(x).$$

If we let m increase to infinity, it then follows from (3.8) that

$$\forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N}\ \forall\, n \geq N_\varepsilon\ \forall\, x \in I:\ |h(x) - h_n(x)| \leq \varepsilon\varphi(x), \tag{3.9}$$

that is, since φ is bounded on I, $\{h_n\}$ converges uniformly to h. Hence, h is continuous and $h \in X$.

Further if we consider (3.7) and (3.9), then we may conclude that

$$\forall\, \varepsilon > 0\ \exists\, N_\varepsilon \in \mathbb{N}\ \forall\, n \geq N_\varepsilon:\ d(h, h_n) \leq \varepsilon,$$

that is, the Cauchy sequence $\{h_n\}$ converges to h in (X, d). Hence, (X, d) is complete.

Now, an operator $\Lambda : X \to X$ is defined by

$$(\Lambda f)(x) = y(c) + \int_c^x F(\tau, f(\tau))\, d\tau \quad (x \in I) \tag{3.10}$$

for all $f \in X$. (Indeed, according to the Fundamental Theorem of Calculus, Λf is continuously differentiable on I, since F and f are continuous functions. Hence, we may conclude that $\Lambda f \in X$.)

We prove that Λ is strictly contractive on X. For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$, that is, by (3.7), we have

$$|f(x) - g(x)| \leq C_{fg}\varphi(x) \tag{3.11}$$

for any $x \in I$. It then follows from (3.1), (3.3), (3.7), (3.10) and (3.11) that

$$\begin{aligned}
|(\Lambda f)(x) - (\Lambda g)(x)| &= \left| \int_c^x \{F(\tau, f(\tau)) - F(\tau, g(\tau))\}\, d\tau \right| \\
&\leq \left| \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))|\, d\tau \right| \\
&\leq L \left| \int_c^x |f(\tau) - g(\tau)|\, d\tau \right| \\
&\leq LC_{fg} \left| \int_c^x \varphi(\tau)\, d\tau \right| \\
&\leq KLC_{fg}\varphi(x)
\end{aligned}$$

for all $x \in I$, that is, $d(\Lambda f, \Lambda g) \leq KLC_{fg}$. Hence, we can conclude that $d(\Lambda f, \Lambda g) \leq KLd(f, g)$ for any $f, g \in X$, where we note that $0 < KL < 1$.

It follows from (3.6) and (3.10) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(x) - g_0(x)| = \left| y(c) + \int_c^x F(\tau, g_0(\tau))\, d\tau - g_0(x) \right| \leq C\varphi(x)$$

for all $x \in I$, since $F(x, g_0(x))$ and $g_0(x)$ are bounded on I and $\min_{x\in I} \varphi(x) > 0$. Thus, (3.7) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

Therefore, according to Theorem 2.1 (a), there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 satisfies equation (3.4) for every $x \in I$.

We will now verify that $\{g \in X \mid d(g_0, g) < \infty\} = X$. For any $g \in X$, since g and g_0 are bounded on I and $\min_{x\in I} \varphi(x) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(x) - g(x)| \leq C_g \varphi(x)$$

for any $x \in I$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is, $\{g \in X \mid d(g_0, g) < \infty\} = X$. Hence, in view of Theorem 2.1 (b), we conclude that y_0 is the unique continuous function with the property (3.4).

On the other hand, it follows from (3.2) that

$$-\varphi(x) \leq y'(x) - F(x, y(x)) \leq \varphi(x)$$

for all $x \in I$. If we integrate each term in the above inequality from c to x, then we obtain

$$\left| y(x) - y(c) - \int_c^x F(\tau, y(\tau))\, d\tau \right| \leq \left| \int_c^x \varphi(\tau)\, d\tau \right|$$

for any $x \in I$. Thus, by (3.3) and (3.10), we get

$$|y(x) - (\Lambda y)(x)| \leq \left| \int_c^x \varphi(\tau)\, d\tau \right| \leq K\varphi(x)$$

for each $x \in I$, which implies that

$$d(y, \Lambda y) \leq K. \tag{3.12}$$

Finally, Theorem 2.1 (c) together with (3.12) implies that

$$d(y, y_0) \leq \frac{1}{1 - KL}\, d(\Lambda y, y) \leq \frac{K}{1 - KL},$$

which means that the inequality (3.5) holds true for all $x \in I$. ∎

In the last theorem, we have investigated the Hyers-Ulam-Rassias stability of the differential equation (1.1) defined on a bounded and closed interval. We will now prove the theorem for the case of unbounded intervals. More precisely, Theorem 3.1 is also true if I is replaced by an unbounded interval such as $(-\infty, b]$, $\mathbb{R}$, or $[a, \infty)$, as we see in the following theorem.

Theorem 3.2. *For given real numbers a and b, let I denote either $(-\infty, b]$ or $\mathbb{R}$ or $[a, \infty)$. Set either $c = a$ for $I = [a, \infty)$ or $c = b$ for $I = (-\infty, b]$ or c is a fixed real number if $I = \mathbb{R}$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition* (3.1) *for all $x \in I$ and all $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality* (3.2) *for all $x \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function satisfying the condition* (3.3) *for any $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ which satisfies* (3.4) *and* (3.5) *for all $x \in I$.*

Proof. We will give the proof for the case $I = \mathbb{R}$ only. The other cases can similarly be proved.

For any $n \in \mathbb{N}$, we define $I_n = [c-n, c+n]$. (We set $I_n = [b-n, b]$ for $I = (-\infty, b]$ and $I_n = [a, a+n]$ for $I = [a, \infty)$.) According to Theorem 3.1, there exists a unique continuous function $y_n : I_n \to \mathbb{R}$ such that

$$y_n(x) = y(c) + \int_c^x F(\tau, y_n(\tau))\, d\tau \tag{3.13}$$

and

$$|y(x) - y_n(x)| \le \frac{K}{1-KL}\, \varphi(x) \tag{3.14}$$

for all $x \in I_n$. The uniqueness of y_n implies that if $x \in I_n$, then

$$y_n(x) = y_{n+1}(x) = y_{n+2}(x) = \cdots . \tag{3.15}$$

For any $x \in \mathbb{R}$, let us define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}.$$

Moreover, we define a function $y_0 : \mathbb{R} \to \mathbb{R}$ by

$$y_0(x) = y_{n(x)}(x), \tag{3.16}$$

and we assert that y_0 is continuous. For an arbitrary $x_1 \in \mathbb{R}$, we choose the integer $n_1 = n(x_1)$. Then, x_1 belongs to the interior of I_{n_1+1} and there exists an $\varepsilon > 0$ such that $y_0(x) = y_{n_1+1}(x)$ for all x with $x_1 - \varepsilon < x < x_1 + \varepsilon$. Since y_{n_1+1} is continuous at x_1, so is y_0. That is, y_0 is continuous at x_1 for any $x_1 \in \mathbb{R}$.

We will now show that y_0 satisfies (3.4) and (3.5) for all $x \in \mathbb{R}$. For an arbitrary $x \in \mathbb{R}$, we choose the integer $n(x)$. Then, it holds that $x \in I_{n(x)}$ and it follows from (3.13) and (3.16) that

$$y_0(x) = y_{n(x)}(x) = y(c) + \int_c^x F(\tau, y_{n(x)}(\tau))\, d\tau = y(c) + \int_c^x F(\tau, y_0(\tau))\, d\tau,$$

where the last equality holds true because $n(\tau) \le n(x)$ for any $\tau \in I_{n(x)}$ and it follows from (3.15) and (3.16) that

$$y_{n(x)}(\tau) = y_{n(\tau)}(\tau) = y_0(\tau).$$

Since $x \in I_{n(x)}$ for every $x \in \mathbb{R}$, by (3.14) and (3.16), we have

$$|y(x) - y_0(x)| = |y(x) - y_{n(x)}(x)| \le \frac{K}{1-KL}\, \varphi(x)$$

for any $x \in \mathbb{R}$.

Finally, we show that y_0 is unique. Let $z_0 : \mathbb{R} \to \mathbb{R}$ be another continuous function which satisfies (3.4) and (3.5), with z_0 in place of y_0, for all $x \in \mathbb{R}$. Suppose x is an arbitrary real number. Since the restrictions $y_0|_{I_{n(x)}}$ $(= y_{n(x)})$ and $z_0|_{I_{n(x)}}$ both satisfy (3.4) and (3.5) for all $x \in I_{n(x)}$, the uniqueness of $y_{n(x)} = y_0|_{I_{n(x)}}$ implies that

$$y_0(x) = y_0|_{I_{n(x)}}(x) = z_0|_{I_{n(x)}}(x) = z_0(x),$$

as required. ∎

4. Hyers-Ulam stability

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.1) defined on a finite and closed interval.

Theorem 4.1. *Given $c \in \mathbb{R}$ and $r > 0$, let I denote a closed ball of radius r and centered at c, that is, $I = \{x \in \mathbb{R} \mid c - r \le x \le c + r\}$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipschitz condition* (3.1) *for all $x \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < Lr < 1$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality*

$$|y'(x) - F(x, y(x))| \le \varepsilon \tag{4.1}$$

for all $x \in I$ and for some $\varepsilon \ge 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying equation (3.4) *(y_0 is a solution to* (1.1)*) and*

$$|y(x) - y_0(x)| \le \frac{r}{1 - Lr}\,\varepsilon \tag{4.2}$$

for any $x \in I$.

Proof. First, we define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$

and introduce a generalized metric on X as follows:

$$d(f, g) = \inf\{C \in [0, \infty] \mid |f(x) - g(x)| \le C \ \forall\ x \in I\}.$$

It is then obvious that (X, d) is a generalized complete metric space (see the proof of Theorem 3.1).

Let us define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(x) = y(c) + \int_c^x F(\tau, f(\tau))\, d\tau \quad (x \in I) \tag{4.3}$$

for any $f \in X$. (It is true that $\Lambda f \in X$ because Λf is continuously differentiable in view of the Fundamental Theorem of Calculus.)

We now assert that Λ is strictly contractive on X. For $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \le C_{fg}$, that is, let us assume that

$$|f(x) - g(x)| \le C_{fg} \tag{4.4}$$

for all $x \in I$. Moreover, it follows from (3.1), (4.3) and (4.4) that

$$\begin{aligned}
|(\Lambda f)(x) - (\Lambda g)(x)| &= \left| \int_c^x \{F(\tau, f(\tau)) - F(\tau, g(\tau))\}\, d\tau \right| \\
&\le \left| \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))|\, d\tau \right| \\
&\le L \left| \int_c^x |f(\tau) - g(\tau)|\, d\tau \right| \\
&\le LrC_{fg}
\end{aligned}$$

for each $x \in I$, that is, $d(\Lambda f, \Lambda g) \le LrC_{fg}$. Thus, it follows that $d(\Lambda f, \Lambda g) \le Lrd(f, g)$ for all $f, g \in X$ and we note that $0 < Lr < 1$.

Analogously to the proof of Theorem 3.1, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$. Therefore, Theorem 2.1 (a) implies that there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$, and such that $y_0 = \Lambda y_0$, that is, y_0 satisfies equation (3.4) for any $x \in I$.

If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with

$$|g_0(x) - g(x)| \le C$$

for all $x \in I$. This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently, $\{g \in X \mid d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 2.1 (b), y_0 is a unique continuous function with the property (3.4). Furthermore, it follows from (4.1) that

$$-\varepsilon \le y'(x) - F(x, y(x)) \le \varepsilon$$

for all $x \in I$. If we integrate each term of the above inequality from c to x, then we have

$$|(\Lambda y)(x) - y(x)| \le \varepsilon r$$

for any $x \in I$, that is, it holds that $d(\Lambda y, y) \le \varepsilon r$.

It now follows from Theorem 2.1 (c) that

$$d(y, y_0) \le \frac{1}{1 - Lr}\, d(\Lambda y, y) \le \frac{r}{1 - Lr}\, \varepsilon,$$

which implies the validity of (4.2) for each $x \in I$. ∎

It is an open problem, whether the differential equation (1.1) has the Hyers-Ulam stability when the relevant domain is an infinite interval.

5. Examples

In this section, we show that there certainly exist functions $y(x)$ which satisfy all the conditions given in Theorems 3.1, 3.2 and 4.1.

Example 5.1. We choose positive constants K and L with $KL < 1$. For a positive number $\varepsilon < 2K$, let $I = [0, 2K - \varepsilon]$ be a closed interval. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - Ly(x) - p(x)| \le x + \varepsilon$$

for all $x \in I$. If we set $F(x, y) = Ly + p(x)$ and $\varphi(x) = x + \varepsilon$, then the above inequality has the identical form with (3.2). Moreover, we obtain

$$\left| \int_0^x \varphi(\tau)\, d\tau \right| = \frac{1}{2}\, x^2 + \varepsilon x \le K\varphi(x)$$

for each $x \in I$, since $K\varphi(x) - x^2/2 - \varepsilon x \ge 0$ for all $x \in I$. According to Theorem 3.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(0) + \int_0^x \{Ly_0(\tau) + p(\tau)\}\, d\tau$$

and

$$|y(x) - y_0(x)| \le \frac{K}{1 - KL}\, (x + \varepsilon)$$

for any $x \in I$.

Example 5.2. Let a be a constant greater than 1 and choose a constant L with $0 < L < \ln a$. Given an interval $I = [0, \infty)$ and a polynomial $p(x)$, suppose $y : I \to \mathbb{R}$ is a continuously differentiable function satisfying

$$|y'(x) - Ly(x) - p(x)| \le a^x$$

for all $x \in I$. If we set $\varphi(x) = a^x$, then we have

$$\left| \int_0^x \varphi(\tau)\, d\tau \right| \le \frac{1}{\ln a}\, \varphi(x)$$

for any $x \in I$. In view of Theorem 3.2, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ with

$$y_0(x) = y(0) + \int_0^x \{Ly_0(\tau) + p(\tau)\}\, d\tau$$

and

$$|y(x) - y_0(x)| \le \frac{1}{\ln a - L}\, a^x$$

for each $x \in I$.

Example 5.3. Let r and L be positive constants with $0 < Lr < 1$ and define a closed interval $I = \{x \in \mathbb{R} \mid c - r \le x \le c + r\}$ for some real number c. Assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - Ly(x) - p(x)| \le \varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$, where $p(x)$ is a polynomial. Then, by Theorem 4.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x \{Ly_0(\tau) + p(\tau)\}\, d\tau$$

and

$$|y(x) - y_0(x)| \le \frac{r}{1 - Lr}\, \varepsilon$$

for all $x \in I$.

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References

[1] C. Alsina and R. Ger, On some inequalities and stability results related to the exponential function, *J. Inequal. Appl.* **2** (1998), no. 4, 373–380.

[2] L. Cădariu and V. Radu, Fixed points and the stability of Jensen's functional equation, *JIPAM. J. Inequal. Pure Appl. Math.* **4** (2003), no. 1, Article 4, 7 pp. (electronic).

[3] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: a fixed point approach, in *Iteration Theory (ECIT '02)*, 43–52, Karl-Franzens-Univ. Graz, Graz, Austria.

[4] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative, for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.* **74** (1968), 305–309.

[5] D. H. Hyers, G. Isac and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser Boston, Boston, MA, 1998.

[6] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, FL, 2001.

[7] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.* **17** (2004), no. 10, 1135–1140.

[8] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order. II, *Appl. Math. Lett.* **19** (2006), no. 9, 854–858.

[9] S.-M. Jung, Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients, *J. Math. Anal. Appl.* **320** (2006), no. 2, 549–561.

[10] S.-M. Jung, A fixed point approach to the stability of isometries, *J. Math. Anal. Appl.* **329** (2007), no. 2, 879–890.

[11] T. Miura, S.-M. Jung and S.-E. Takahasi, Hyers-Ulam-Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$, *J. Korean Math. Soc.* **41** (2004), no. 6, 995–1005.

[12] T. Miura, S. Miyajima and S.-E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators, *J. Math. Anal. Appl.* **286** (2003), no. 1, 136–146.

[13] T. Miura, S. Miyajima and S.-E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, *Math. Nachr.* **258** (2003), 90–96.

[14] M. Obłoza, Hyers stability of the linear differential equation, *Rocznik Nauk.-Dydakt. Prace Mat. No. 13* (1993), 259–270.

[15] M. Obłoza, Connections between Hyers and Lyapunov stability of the ordinary differential equations, *Rocznik Nauk.-Dydakt. Prace Mat. No. 14* (1997), 141–146.

[16] V. Radu, The fixed point alternative and the stability of functional equations, *Fixed Point Theory* **4** (2003), no. 1, 91–96.

[17] S.-E. Takahasi, T. Miura and S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, *Bull. Korean Math. Soc.* **39** (2002), no. 2, 309–315.