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On $(\in, \in\vee \mathrm{q})$-Fuzzy Filters of Pseudo-BL Algebras

[1]Jianming Zhan, [2]Young Bae Jun and [3]Wiesław A. Dudek
[1]Department of Mathematics, Hubei Institute for Nationalities,
Enshi, Hubei Province, 445000, P. R. China
[2]Department of Mathematics Education, Gyeongsang National University,
Chinju 660-701, Korea
[3]Institute of Mathematics and Computer Science, Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
[1]zhanjianming@hotmail.com, [2]skywine@gmail.com, [3]dudek@im.pwr.wroc.pl

Abstract. The aim of this paper is to introduce the notion of $(\in, \in\vee \mathrm{q})$-fuzzy (implicative, MV- and G-) filters of pseudo-BL algebras and to investigate some of their related properties. Some characterization theorems of these generalized fuzzy (implicative) filters are derived. The relationship among these generalized fuzzy filters of pseudo-BL algebras is considered. It is proved that an $(\in, \in\vee \mathrm{q})$-fuzzy filter of a pseudo-BL algebra is an $(\in, \in\vee \mathrm{q})$-fuzzy implicative filter if and only if it is both an $(\in, \in\vee \mathrm{q})$-fuzzy MV-filter and an $(\in, \in\vee \mathrm{q})$-fuzzy G-filter. Finally, we consider the concept of implication-based fuzzy implicative filters of pseudo-BL algebras, in particular, the implication operators in Lukasiewicz system of continuous-valued logic are discussed.

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1. Introduction

BL-algebras were introduced by Hájek as algebraic structures for his Basic Logic, starting from continuous t-norms and their residuals (cf. [14]). MV-algebras [3], product algebras and Gödel algebras are BL-algebras. Filter theory plays an important role in studying these algebras. From logical point of view, various filters correspond to various sets of provable formulae. Hájek [14] introduced the concepts of (prime) filters of BL-algebras. Using prime filters of BL-algebras, he proved the completeness of Basic Logic (BL). BL-algebras are further discussed by Di Nola [5, 6], Iorgulescu [15] and Turunen [19, 20], and so on.

Recent investigations are concerned with non-commutative generalizations for these structures (see [4, 7–13, 17–20, 24, 25]). In [12], Georgescu and Iorgulescu introduced the concept of pseudo-MV algebras as a non-commutative generalization

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of MV-algebras. Several researchers discussed the properties of pseudo MV-algebras (see [7, 8, 17, 18]). Pseudo-BL algebras are a common extension of BL-algebras and pseudo MV-algebras (see [4, 9, 10, 13, 24]). These structures seem to be a very general algebraic concept in order to express the non-commutative reasoning. We remark that a pseudo-BL algebra has two implications and two negations.

The theory of fuzzy sets was first developed by Zadeh [22] and has been applied to many branches in mathematics. A new type of fuzzy subgroup, that is, the $(\in, \in\vee \mathrm{q})$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [16]. With this objective in mind, we also applied this theory to BCI-algebras [23].

The paper is organized as follows. In Section 2, we recall some basic definitions and results of pseudo-BL algebras. In Section 3, we introduce the notion of $(\in, \in\vee \mathrm{q})$-fuzzy filters in pseudo-BL algebras and investigate some of their related properties. Further, the notions of $(\in, \in\vee \mathrm{q})$-fuzzy (implicative, MV- and G-) of pseudo-BL algebras are introduced and the relationship among these generalized fuzzy filters of pseudo-BL algebras is considered in Section 4. Finally, in Section 5 we consider the concept of implication-based fuzzy implicative filters of pseudo-BL algebras, in particular, the implication operators in Lukasiewicz system of continuous-valued logic are discussed.

2. Preliminaries

A pseudo-BL algebra is an algebra $(A; \wedge, \vee, \odot, \to, \hookrightarrow, 0, 1)$ of type (2, 2, 2, 2, 2, 0, 0) such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a monoid and the following axioms hold for all $x, y, z \in A$:

(a_1) $x \odot y \le z \Longleftrightarrow x \le y \to z \Longleftrightarrow y \le x \hookrightarrow z$;
(a_2) $x \wedge y = (x \to y) \odot x = x \odot (x \hookrightarrow y)$;
(a_3) $(x \to y) \vee (y \to x) = (x \hookrightarrow y) \vee (y \hookrightarrow x) = 1$.

In the sequel, we shall agree that the operation $\vee, \wedge, \odot$ have priority towards the operation $\to, \hookrightarrow$.

Example 2.1. [4] Let $(G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and let θ be the symbol distinct from the elements of G. If $G^- = \{x' \in G | x' \le 0\}$, then we define on $G^* = \{\theta\} \cup G^-$ the following operations:

$$x' \odot y' = \begin{cases} x' + y' & \text{if} \quad x', y' \in G^-, \\ \theta & \text{otherwise,} \end{cases}$$

$$x' \to y' = \begin{cases} (y' - x') \wedge 0 & \text{if} \quad x', y' \in G^-, \\ \theta & \text{if} \quad x' \in G^-,\ y' = \perp, \\ 0 & \text{if} \quad x' = \perp, \end{cases}$$

$$x' \hookrightarrow y' = \begin{cases} (-x' + y') \wedge 0 & \text{if} \quad x', y' \in G^-, \\ \theta & \text{if} \quad x' \in G^-,\ y' = \perp, \\ 0 & \text{if} \quad x' = \perp. \end{cases}$$

If we put $\theta \leq x'$, for any $x' \in G$, then $(G^*, \leq)$ becomes a lattice with first element θ, and the last element 0. The structure $G^* = (G^*, \vee, \wedge, \odot, \rightarrow, \hookrightarrow, 0 = \theta, 1 = 0)$ is a pseudo-BL algebra.

Let A be a pseudo-BL algebra and $x, y, z \in A$. The following statements are true (for details see [4, 11, 23]):

(1) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
(2) $(y \odot x) \hookrightarrow z = x \hookrightarrow (y \hookrightarrow z)$,
(3) $x \leq y \Longleftrightarrow x \rightarrow y = 1 \Longleftrightarrow x \hookrightarrow y = 1$,
(4) $(x \hookrightarrow y) \hookrightarrow x \leq (x \hookrightarrow y) \rightarrow ((x \hookrightarrow y) \hookrightarrow y)$,
(5) $x \leq y \Rightarrow x \odot z \leq y \odot z$,
(6) $x \leq y \Rightarrow z \odot x \leq z \odot y$,
(7) $x \odot y \leq x, \ \ x \odot y \leq y$,
(8) $x \odot 0 = 0 \odot x = 0$,
(9) $1 \rightarrow x = 1 \hookrightarrow x = x$,
(10) $y \leq x \rightarrow y$.

A non-empty subset I of a pseudo-BL algebra A is called a *filter* of A if it satisfies the following two conditions:

(i) $x \odot y \in I$ for all $x, y \in I$;
(ii) $x \leq y \Rightarrow y \in I$ for all $x \in I$ and $y \in A$.

Remind that a filter I is called

implicative if $\begin{cases} (x \rightarrow y) \hookrightarrow x \in I \text{ implies } x \in I, \\ (x \hookrightarrow y) \rightarrow x \in I \text{ implies } x \in I, \end{cases}$

MV-filter if $\begin{cases} x \rightarrow y \in I \text{ implies } ((y \rightarrow x) \hookrightarrow x) \rightarrow y \in I, \\ x \hookrightarrow y \in I \text{ implies } ((y \hookrightarrow x) \rightarrow x) \hookrightarrow y \in I, \end{cases}$

G-filter if $\begin{cases} x \rightarrow (x \rightarrow y) \in I \text{ implies } x \rightarrow y \in I, \\ x \hookrightarrow (x \hookrightarrow y) \in I \text{ implies } x \hookrightarrow y \in I. \end{cases}$

Now, we introduce the concept of fuzzy (implicative) filters of pseudo-BL algebras as follows:

Definition 2.1. *A fuzzy set μ of a pseudo-BL algebra A is called a* fuzzy filter *of A if*

(i) $\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $x \leq y \Longrightarrow \mu(x) \leq \mu(y)$,

is satisfied for all $x, y \in A$.

Definition 2.2. *A fuzzy filter μ of a pseudo-BL algebra A is called* fuzzy implicative *if*

(iii) $\mu(x) \geq \max\{\mu((x \rightarrow y) \hookrightarrow x), \mu((x \hookrightarrow y) \rightarrow x)\}$,

holds for all $x, y, z \in A$.

For any fuzzy set μ of A and $t \in (0, 1]$, the set $\mu_t = \{x \in A \mid \mu(x) \geq t\}$ is called a *level subset* of μ.

It is not difficult to verify that the following theorem is true.

Theorem 2.1. *A fuzzy set μ of a pseudo-BL algebra A is a fuzzy filter of A if and only if each its nonempty level subset μ_t is a filter of A.*

Let $0 < t \leq 1$. A fuzzy set μ of the form

$$\mu(y) = \begin{cases} t & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support* x *and value* t and is denoted by $U(x;t)$. A fuzzy point $U(x;t)$ is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set μ, written as $U(x;t) \in \mu$ (resp. $U(x;t)\mathrm{q}\mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $U(x;t) \in \mu$ or (resp. and) $U(x;t)\mathrm{q}\mu$, then we write $U(x;t) \in\vee\,\mathrm{q}\mu$ (resp. $\in\wedge\,\mathrm{q}\mu$). The symbol $\overline{\in\vee\,\mathrm{q}}$ means that $\in\vee\,\mathrm{q}$ does not hold. Using the notion of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, the concept of (α,β)-fuzzy subsemigroup, where α and β are any two of $\{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$ with $\alpha \neq \in\wedge\,\mathrm{q}$, was introduced in [2]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in\vee\,\mathrm{q})$-fuzzy subgroup. For more information of $(\in, \in\vee\,\mathrm{q})$-fuzzy subgroups, the reader is referred to [1] if necessary.

3. $(\in, \in\vee\,\mathrm{q})$-fuzzy filters

In what follows, A is a pseudo-BL algebra unless otherwise specified.

Definition 3.1. *A fuzzy set* μ *of* A *is said to be an* $(\in, \in\vee\,\mathrm{q})$-fuzzy filter *of* A *if for all* $t, r \in (0,1]$ *and* $x, y \in A$,

(F_1) $U(x;t) \in \mu$ *and* $U(y;r) \in \mu$ *imply* $U(x \odot y; \min\{t,r\}) \in\vee\,\mathrm{q}\mu$,
(F_2) $U(x;r) \in \mu$ *implies* $U(y;r) \in\vee\,\mathrm{q}\mu$ *if* $x \leq y$.

Example 3.1. Let I be a filter of a pseudo-BL algebra A and let μ_I be a fuzzy set in A defined by

$$\mu_I(x) = \begin{cases} 0.7 & \text{if} \quad x \in I, \\ 0.3 & \text{otherwise.} \end{cases}$$

It is easily to verify that μ_I is an $(\in, \in\vee\,\mathrm{q})$-fuzzy filter of A.

Theorem 3.1. *A fuzzy set* μ *of* A *is an* $(\in, \in\vee\,\mathrm{q})$*-fuzzy filter if and only if for all* $x, y \in A$, *the following two conditions are satisfied:*

(F_3) $\mu(x \odot y) \geq \min\{\mu(x), \mu(y), 0.5\}$,
(F_4) $x \leq y \Longrightarrow \mu(y) \geq \min\{\mu(x), 0.5\}$.

Proof. At first we prove that the conditions (F_1) and (F_3) are equivalent. Suppose that (F_1) do not implies (F_3), i.e., (F_1) holds but (F_3) is not satisfied. In this case there are $x, y \in A$ such that $\mu(x \odot y) < \min\{\mu(x), \mu(y), 0.5\}$. If $\min\{\mu(x), \mu(y)\} < 0.5$, then $\mu(x \odot y) < \min\{\mu(x), \mu(y)\}$. This means that for every $t \in R$ satisfying the condition $\mu(x \odot y) < t < \min\{\mu(x), \mu(y)\}$ we have $U(x;t) \in \mu$ and $U(y;t) \in \mu$, but $U(x \odot y; t)\overline{\in\vee\,\mathrm{q}}\mu$, which contradicts to ($F_1$). So, this case is impossible. Therefore $\min\{\mu(x), \mu(y)\} \geq 0.5$. In this case $\mu(x \odot y) < 0.5$, $U(x; 0.5) \in \mu$, $U(y; 0.5) \in \mu$ and $U(x \odot y; 0.5)\overline{\in\vee\,\mathrm{q}}\mu$, which also is impossible. So, (F_1) implies (F_3).

Conversely, if (F_3) holds and $U(x;t) \in \mu$, $U(y;r) \in \mu$, then $\mu(x) \geq t$, $\mu(y) \geq r$ and $\mu(x \odot y) \geq \min\{t, r, 0.5\}$. If $\min\{t,r\} > 0.5$, then $\mu(x \odot y) \geq 0.5$, which implies $\mu(x \odot y) + \min\{t,r\} > 1$, i.e., $U(x \odot y; \min\{t,r\})\mathrm{q}\mu$. If $\min\{t,r\} \leq 0.5$, then $\mu(x \odot y) \geq \min\{t,r\}$. Thus $U(x \odot y; \min\{t,r\}) \in \mu$. Therefore, $U(x \odot y; \min\{t,r\}) \in \vee\,\mathrm{q}\mu$. Summarizing, ($F_3$) implies ($F_1$). So ($F_1$) and ($F_3$) are equivalent.

Now we prove that (F_2) and (F_4) are equivalent. Suppose that (F_4) is not satisfied, i.e., $\mu(y_0) < \min\{\mu(x_0), 0.5\}$ for some $x_0 \leq y_0$. If $\mu(x_0) < 0.5$, then $\mu(y_0) < \mu(x_0)$, which means that there exists $s \in R$ such that $\mu(y_0) < s < \mu(x_0)$ and $\mu(y_0)+\mu(x_0) < 1$. Thus $U(y_0; s) \in \mu$ and $U(x_0; s)\overline{\in\vee q}\mu$, which contradicts to (F_2). So, $\mu(x_0) \geq 0.5$. But in this case $\mu(y_0) < \min\{\mu(x_0), 0.5\}$ gives $U(x_0; 0.5) \in \mu$ and $U(x_0; 0.5)\overline{\in\vee q}\mu$, which contradicts to (F_2). Hence $\mu(y) \geq \min\{\mu(x), 0.5\}$ for all $x \leq y$, i.e., (F_2) implies (F_4).

Conversely, if (F_4) holds then $x \leq y$ and $U(x; t) \in \mu$ imply $\mu(x) \geq t$, and so $\mu(y) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\}$. Thus $\mu(y) \geq t$ or $\mu(y) \geq 0.5$, according to $t \leq 0.5$ or $t > 0.5$. Therefore, $U(y; t) \in\vee q\mu$. Hence (F_4) implies (F_2). ∎

Corollary 3.1. *In a pseudo-BL algebra the conditions* (F_1) *and* (F_3) (*similarly* (F_2)*and* (F_4)) *are equivalent.*

In a similar way we can prove the following proposition.

Proposition 3.1. *A fuzzy set* μ *of* A *is an* $(\in, \in\vee q)$*-fuzzy filter if and only if*

(F_5) $\mu(1) \geq \min\{\mu(x), 0.5\}$ *holds for all* $x \in A$
and one of the following conditions:
(F_6) $\mu(y) \geq \min\{\mu(x), \mu(x \to y), 0.5\}$,
(F_6') $\mu(y) \geq \min\{\mu(x), \mu(x \hookrightarrow y), 0.5\}$
is satisfied for all $x, y \in A$.

Also, using Theorem 3.1, it is not difficult to verify that the following theorem is true.

Theorem 3.2. *A fuzzy set* μ *of* A *is its* $(\in, \in\vee q)$*-fuzzy filter if and only if for* $0 < t \leq 0.5$ *each nonempty level subset* μ_t *is a filter of* A.

Naturally, we can establish a corresponding result when μ_t is a filter of A for $0.5 < t \leq 1$.

Theorem 3.3. *For* $0.5 < t \leq 1$, *each nonempty level subset* μ_t *of a fuzzy set* μ *of* A *is a filter if and only if for all* $x, y \in A$ *the following two conditions are satisfied:*

(F_7) $\max\{\mu(x \odot y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$,
(F_8) $\max\{\mu(y), 0.5\} \geq \mu(x)$ *if* $x \leq y$.

Proof. Let μ_t be a nonempty level subset of μ. Assume that μ_t is a filter of A. If $\max\{\mu(x \odot y), 0.5\} < \min\{\mu(x), \mu(y)\} = t$ for some $x, y \in A$, then $0.5 < t \leq 1$, $\mu(x \odot y) < t$ and $x, y \in \mu_t$. Thus $x \odot y \in \mu_t$. Whence $\mu(x \odot y) \geq t$, which contradicts to $\mu(x \odot y) < t$. So, (F_7) is satisfied.

If there exist $x, y \in A$ such that $\max\{\mu(y), 0.5\} < \mu(x) = t$, then $0.5 < t \leq 1$, $\mu(y) < t$ and $x \in \mu_t$. Since $x \leq y$ we also have $y \in \mu_t$. Thus $\mu(y) \geq t$, which is impossible. Therefore $\max\{\mu(y), 0.5\} \geq \mu(x)$ for $x \leq y$.

Conversely, suppose that the conditions (F_7) and (F_8) are satisfied. In order to prove that for $0.5 < t \leq 1$ each nonempty level subset μ_t is a filter of A assume that $x, y \in \mu_t$. In this case $0.5 < t \leq \min\{\mu(x), \mu(y)\} \leq \max\{\mu(x \odot y), 0.5\} = \mu(x \odot y)$ which proves $x \odot y \in \mu_t$. If $x \leq y$ and $x \in \mu_t$, then $0.5 < t \leq \mu(x) \leq \max\{\mu(y), 0.5\} = \mu(y)$, and so $y \in \mu_t$. This completes the proof. ∎

Let $J = \{t \in (0,1] \mid \mu_t \neq \emptyset\}$, where μ is a fuzzy set of A. For $J = (0,1]$ μ is an ordinary fuzzy filter of A (Theorem 2.1), for $J = (0, 0.5]$ it is an $(\in, \in\vee \mathrm{q})$-fuzzy filter of A (Theorem 3.2).

In [21], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and also Bhakat and Das's fuzzy subgroup. Based on the results of [21], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy filter with thresholds in the following way:

Definition 3.2. *Let* $0 \leq \alpha < \beta \leq 1$. *A fuzzy set* μ *of* A *is called a* fuzzy filter with thresholds (α, β) *if for all* $x, y \in A$ *the following two conditions are satisfied:*

(F_9) $\max\{\mu(x \odot y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$,

(F_{10}) $\max\{\mu(y), \alpha\} \geq \min\{\mu(x), \beta\}$ *if* $x \leq y$.

Theorem 3.4. *A fuzzy set* μ *of* A *is a fuzzy filter with thresholds* (α, β) *if and only if each nonempty* μ_t*, where* $\alpha < t \leq \beta$ *is a filter of* A.

Proof. The proof is similar to the proof of Theorems 3.2 and 3.3. ∎

4. $(\in, \in\vee \mathrm{q})$-fuzzy implicative filters

Definition 4.1. *An* $(\in, \in\vee \mathrm{q})$*-fuzzy filter* μ *of* A *is called an* $(\in, \in\vee \mathrm{q})$-fuzzy implicative filter *if for all* $x, y \in A$ *it satisfies the condition:*

$$(F_{11}) \quad \begin{cases} \mu(x) \geq \min\{\mu((x \to y) \hookrightarrow x), 0.5\}, \\ \mu(x) \geq \min\{\mu((x \hookrightarrow y) \to x), 0.5\}. \end{cases}$$

The following proposition is obvious.

Proposition 4.1. *If* μ *is an* $(\in, \in\vee \mathrm{q})$*-fuzzy implicative filter of* A*, then*

$$(1) \quad \begin{cases} \mu(x) \geq \min\{\mu((x \to y) \to x), 0.5\}, \\ \mu(x) \geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x), 0.5\}, \end{cases}$$

$$(2) \quad \begin{cases} \mu(((y \to x) \hookrightarrow x) \to y) \geq \min\{\mu(x \to y), 0.5\}, \\ \mu(((y \hookrightarrow x) \to x) \hookrightarrow y) \geq \min\{\mu(x \hookrightarrow y), 0.5\}, \end{cases}$$

$$(3) \quad \begin{cases} \mu((y \hookrightarrow x) \to x) \geq \min\{\mu((x \to y) \hookrightarrow y), 0.5\}, \\ \mu((y \to x) \hookrightarrow x) \geq \min\{\mu((x \hookrightarrow y) \to y), 0.5\}, \end{cases}$$

$$(4) \quad \begin{cases} \mu((y \to x) \hookrightarrow x) \geq \min\{\mu((x \to y) \hookrightarrow y), 0.5\}, \\ \mu((y \hookrightarrow x) \to x) \geq \min\{\mu((x \hookrightarrow y) \hookrightarrow y), 0.5\} \end{cases}$$

hold for all $x, y \in A$.

Theorem 4.1. *A fuzzy set* μ *of* A *is an* $(\in, \in\vee \mathrm{q})$*-fuzzy implicative filter if and only if for* $0 < t \leq 0.5$ *each nonempty level subset* μ_t *is an implicative filter of* A.

Proof. Let μ be an $(\in, \in\vee \mathrm{q})$-fuzzy implicative filter of A and $0 < t \leq 0.5$. Then, by Theorem 3.2, each nonempty μ_t is a filter of A. For all $x, y \in A$ from $(x \to y) \hookrightarrow x \in \mu_t$ it follows $\mu((x \to y) \hookrightarrow x) \geq t$. This, according to (F_{11}), gives $\mu(x) \geq \min\{\mu((x \to y) \hookrightarrow x), 0.5\} \geq \min\{t, 0.5\} = t$. So, $x \in \mu_t$. Analogously $(x \hookrightarrow y) \to x \in \mu_t$ implies $x \in \mu_t$. Therefore μ_t is an implicative filter of A.

Conversely, if μ is a fuzzy set of A such that for $0 < t \leq 0.5$ each nonempty level set μ_t is an implicative filter of A, then, by Theorem 3.2, μ is an $(\in, \in\vee \mathrm{q})$-fuzzy filter of A. Putting $\mu((x \to y) \hookrightarrow x) \geq s_0 = \min\{\mu((x \to y) \hookrightarrow x), 0.5\}$, we obtain

$(x \to y) \hookrightarrow x \in \mu_{s_0}$. Consequently, $x \in \mu_{s_0}$, i.e., $\mu(x) \geq s_0 = \min\{\mu((x \to y) \hookrightarrow x), 0.5\}$. This proves the firs condition of (F_{11}). Similarly, we prove the second condition. ∎

Basing on our Theorem 3.3 the above result can be extended to the case $0.5 < t \leq 1$ in the following way:

Theorem 4.2. *For* $0.5 < t \leq 1$ *each nonempty level subset* μ_t *of a fuzzy set* μ *of* A *is an implicative filter of* A *if and only if the conditions* (F_7), (F_8) *and*

$$(F_{12}) \begin{cases} \max\{\mu(x), 0.5\} \geq \mu((x \to y) \hookrightarrow x), \\ \max\{\mu(x), 0.5\} \geq \mu((x \hookrightarrow y) \to x) \end{cases}$$

are satisfied for all $x, y \in A$.

Proof. According to Theorem 3.3, for $0.5 < t \leq 1$ each nonempty level subset μ_t of μ is a filter of A if and only if μ satisfies (F_7) and (F_8). So, we shall prove only that a filter μ_t is implicative if and only if μ satisfies (F_{12}).

To prove (F_{12}) suppose the existence of $x, y \in A$ such that $\max\{\mu(x), 0.5\} < t = \mu((x \to y) \hookrightarrow x)$. In this case $0.5 < t \leq 1$, $\mu(x) < t$ and $(x \to y) \hookrightarrow x \in \mu_t$. Since μ_t is an implicative filter of A, we have $x \in \mu_t$, and so $\mu(x) \geq t$, which is a contradiction. Similarly, we can prove the second inequality of (F_{12}).

Conversely, suppose that a fuzzy set μ satisfies (F_{12}) and each nonempty μ_t is a filter of A. If $0.5 < t \leq 1$ and $(x \to y) \hookrightarrow x \in \mu_t$, then $0.5 < t \leq \min\{\mu((x \to y) \hookrightarrow x) \leq \max\{\mu(x), 0.5\} = \mu(x)$, which implies $x \in \mu_t$. Similarly, from $(x \hookrightarrow y) \to x \in \mu_t$ it follows $x \in \mu_t$. Thus, μ_t is an implicative filter of A. ∎

Basing on the method presented in [21], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy implicative filter with thresholds.

Definition 4.2. *Let* $0 \leq \alpha < \beta \leq 1$. *A fuzzy set* μ *of* A *is called a* fuzzy implicative filter with thresholds (α, β) *of* A *if for all* $x, y \in A$ *it satisfies* (F_9), (F_{10}) *and*

$$(F_{13}) \begin{cases} \max\{\mu(x), \alpha\} \geq \min\{\mu((x \to y) \hookrightarrow x), \beta\}, \\ \max\{\mu(x), \alpha\} \geq \min\{\mu((x \hookrightarrow y) \to x), \beta\}. \end{cases}$$

Theorem 4.3. *A fuzzy set* μ *of* A *is a fuzzy implicative filter with thresholds* (α, β) *if and only if each nonempty* μ_t, *where* $\alpha < t \leq \beta$ *is an implicative filter of* A.

Proof. The proof is similar to the proof of Theorems 4.1 and 4.2. ∎

Definition 4.3. *An* $(\in, \in\vee q)$*-fuzzy filter of* A *is called an* $(\in, \in\vee q)$-fuzzy MV-filter *of* A *if the following two inequalities:*

$$(F_{14}) \begin{cases} \mu(((y \to x) \hookrightarrow x) \to y) \geq \min\{\mu(x \to y), 0.5\}, \\ \mu(((y \hookrightarrow x) \to x) \hookrightarrow y) \geq \min\{\mu(x \hookrightarrow y), 0.5\} \end{cases}$$

are satisfied for all $x, y \in A$.

It follows from Proposition 4.1(2) that every $(\in, \in\vee q)$-fuzzy implicative filter is an $(\in, \in\vee q)$-fuzzy MV-filter.

Definition 4.4. *An* $(\in, \in\vee q)$*-fuzzy filter of* A *is called an* $(\in, \in\vee q)$-fuzzy G-filter *of* A *if the condition:*

$$(F_{15}) \begin{cases} \mu(x \to y) \geq \min\{\mu(x \to (x \to y)), 0.5\}, \\ \mu(x \hookrightarrow y) \geq \min\{\mu(x \hookrightarrow (x \hookrightarrow y)), 0.5\} \end{cases}$$

is satisfied for all $x, y \in A$.

Lemma 4.1. *Every* $(\in, \in\vee\mathrm{q})$*-fuzzy implicative filter is an* $(\in, \in\vee\mathrm{q})$*-fuzzy* G*-filter.*

Proof. Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of A. As it is well-known (see [4]) for any $x, y \in A$ we have $x \hookrightarrow (x \hookrightarrow y) \leq ((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)$. Whence, according to (F_4) we obtain

$$\mu(((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)) \geq \min\{\mu(x \hookrightarrow (x \hookrightarrow y)),\, 0.5\}.$$

From this, applying (F_{11}), we get

$$\mu(x \hookrightarrow y) \geq \min\{\mu(((x \hookrightarrow y) \hookrightarrow y) \to (x \hookrightarrow y)),\, 0.5\} \geq \min\{\mu(x \hookrightarrow (x \hookrightarrow y)),\, 0.5\}.$$

This proves the second inequality of (F_{15}). The proof of the first inequality is similar. Hence μ is an $(\in, \in\vee\mathrm{q})$-fuzzy G-filter of A. ∎

Theorem 4.4. *For any* $(\in, \in\vee\mathrm{q})$*-fuzzy filter* μ *of* A *satisfying the identity* $\mu(x \to y) = \mu(x \hookrightarrow y)$ *the following conditions are equivalent:*

(1) μ *is an* $(\in, \in\vee\mathrm{q})$*-fuzzy implicative filter;*
(2) μ *is an* $(\in, \in\vee\mathrm{q})$*-fuzzy* MV*-filter and an* $(\in, \in\vee\mathrm{q})$*-fuzzy* G*-filter.*

Proof.
(1) $\Longrightarrow$ (2). By Proposition 4.1 (2) and Lemma 4.1.
(2) $\Longrightarrow$ (1). Let μ be an $(\in, \in\vee\mathrm{q})$-fuzzy MV-filter and an $(\in, \in\vee\mathrm{q})$-fuzzy G-filter of A. From $(x \hookrightarrow y) \hookrightarrow x \leq (x \hookrightarrow y) \to ((x \hookrightarrow y) \hookrightarrow y)$ (see [4]), we have

$$\begin{aligned}\mu((x \hookrightarrow y) \hookrightarrow ((x \hookrightarrow y) \hookrightarrow y)) &= \mu((x \hookrightarrow y) \to ((x \hookrightarrow y) \hookrightarrow y)) \\ &\geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x),\, 0.5\},\end{aligned}$$

which together with the fact that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy G-filter of A gives

$$\begin{aligned}\mu((x \hookrightarrow y) \hookrightarrow y) &\geq \min\{\mu((x \hookrightarrow y) \hookrightarrow ((x \to y) \hookrightarrow y)),\, 0.5\} \\ &\geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x),\, 0.5\}.\end{aligned}$$

Moreover, from $y \leq x \to y$ we get $(x \hookrightarrow y) \hookrightarrow x \leq y \hookrightarrow x$, and consequently

$$\mu(y \hookrightarrow x) \geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x),\, 0.5\}.$$

The fact that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy MV-filter of A implies

$$\mu(((x \hookrightarrow y) \to y) \hookrightarrow x) \geq \min\{\mu(y \hookrightarrow x),\, 0.5\} \geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x),\, 0.5\}.$$

Since μ is an $(\in, \in\vee\mathrm{q})$-fuzzy filter of A, we also have

$$\mu(x) \geq \min\{\mu(((x \hookrightarrow y) \to y) \hookrightarrow x),\, \mu((x \hookrightarrow y) \to y),\, 0.5\}.$$

Summarizing the above, we obtain $\mu(x) \geq \min\{\mu((x \hookrightarrow y) \hookrightarrow x),\, 0.5\}$. Hence, μ is an $(\in, \in\vee\mathrm{q})$-fuzzy implicative filter of A. ∎

Problem. Prove or disprove that in Theorem 4.4 the assumption $\mu(x \to y) = \mu(x \hookrightarrow y)$ is essential.

5. Implication-based fuzzy implicative filters

Fuzzy logic is an extension of set theoretic variables to terms of the linguistic variable truth. Some operators, like $\wedge, \vee, \neg, \rightarrow$ in fuzzy logic also can be defined by using the tables of valuations. Also, the extension principle can be used to derive definitions of the operators.

In the fuzzy logic, the truth value of fuzzy proposition P is denoted by $[P]$. The correspondence between fuzzy logical and set-theoretical notations is presented below:
$[x \in \mu] = \mu(x)$,
$[x \notin \mu] = 1 - \mu(x)$,
$[P \wedge Q] = \min\{[P], [Q]\}$,
$[P \vee Q] = \max\{[P], [Q]\}$,
$[P \rightarrow Q] = \min\{1, 1 - [P] + [Q]\}$,
$[\forall x P(x)] = \inf[P(x)]$,
$\models P \Longleftrightarrow [P] = 1$ for all valuations.

Of course, various implication operators can be defined similarly. In the table presented below we give the example of such definitions. In this table α denotes the degree of truth (or degree of membership) of the premise, β is the values for the consequence, and I is the result for the corresponding implication:

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\}$,
Lukasiewicz	$I_a(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\}$,
Standard Star (Gödel)	$I_g(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ \beta & \text{if} \quad \alpha > \beta, \end{cases}$
Contraposition of Gödel	$I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ 1 - \alpha & \text{if} \quad \alpha > \beta, \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha \leq \beta, \\ 0 & \text{if} \quad \alpha > \beta, \end{cases}$
Kleene-Dienes	$I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}$.

The "quality" of these implication operators could be evaluated either by empirical or by axiomatic methods.

Below we consider the implication operator defined in the Lukasiewicz system of continuous-valued logic.

Definition 5.1. *A fuzzy set μ of A is called a* fuzzifying implicative filter *of A if for any $x, y, z \in A$ it satisfies the following four conditions:*

(F_{16}) $\models ((x \in \mu \wedge y \in \mu) \rightarrow x \odot y \in \mu)$,
(F_{17}) $\models (x \in \mu \rightarrow y \in \mu)$ *for any* $x \leq y$,
(F_{18}) $\models ((x \rightarrow y) \hookrightarrow x \rightarrow x \in \mu)$,
(F_{19}) $\models ((x \hookrightarrow y) \rightarrow x \rightarrow x \in \mu)$.

Clearly, the above definition is equivalent to Definition 2.2.

The concept of the "standard" tautology can be generalized to the *t-tautology*, where $0 < t \leq 1$, in the following way:

$$\models_t P \Longleftrightarrow [P] \geq t$$

for all valuations.

This definition and results obtained in [21] gives for us the possibility to introduce such definition:

Definition 5.2. *Let $0 < t \leq 1$ be fixed. A fuzzy set μ of A is called a t-*implication-based fuzzy implicative filter *of A if for all $x, y, z \in A$ the following conditions hold:*

(F_{20}) $\models_t ((x \in \mu \wedge y \in \mu) \to x \odot y \in \mu)$,
(F_{21}) $\models_t (x \in \mu \to y \in \mu)$ *for any* $x \leq y$,
(F_{22}) $\models_t ((x \to y) \hookrightarrow x \to x \in \mu)$,
(F_{23}) $\models_t ((x \hookrightarrow y) \to x \to x \in \mu)$.

In a special case when an implication operator is defined as I we obtain:

Corollary 5.1. *A fuzzy set μ of A is a t-implication-based fuzzy implicative filter if and only if for all $x, y, z \in A$ it satisfies:*

(F_{23}) $I(\mu(x) \wedge \mu(y), \mu(x \odot y)) \geq t$,
(F_{24}) $I(\mu(x) \wedge \mu(y)) \geq t$ *for all* $x \leq y$,
(F_{25}) $I(\mu((x \to y) \hookrightarrow x), \mu(x)) \geq t$,
(F_{26}) $I(\mu((x \hookrightarrow y) \to x), \mu(x)) \geq t$.

This gives a very good base for future study of filters in various algebraic systems with implication operators. As an example we present one theorem. In a similar way we can obtain other typical results.

Theorem 5.1. *Let μ be a fuzzy set of A.*

(i) *If $I = I_{gr}$, then μ is an 0.5-implication-based fuzzy implicative filter of A if and only if μ is a fuzzy implicative filter with thresholds $(r = 0, s = 1)$.*
(ii) *If $I = I_g$, then μ is an 0.5-implication-based fuzzy implicative filter of A if and only if μ is a fuzzy implicative filter with thresholds $(r = 0, s = 0.5)$.*
(iii) *If $I = I_{cg}$, then μ is an 0.5-implication-based fuzzy implicative filter of A if and only if μ is a fuzzy implicative filter with thresholds $(r = 0.5, s = 1)$.*

Proof. The proofs are straightforward and hence are omitted. ∎

6. Conclusions

In this paper, we considered different type of $(\in, \in \vee\, q)$-fuzzy filters of pseudo-BL algebras and investigated the relationship between these filters. Finally, we proposed the concept of implication-based fuzzy implicative filters of pseudo-BL algebras, which seems to be a good support for future study. The other direction of future study is an investigation of (α, β)-fuzzy (implicative) filters, where α, β are one of $\in, q, \in \vee\, q$ or $\in \wedge\, q$.

References

[1] S. K. Bhakat, $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups, *Fuzzy Sets and Systems* **112** (2000), no. 2, 299–312.

[2] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroup, *Fuzzy Sets and Systems* **80** (1996), no. 3, 359–368.

[3] C. C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[4] A. Di Nola, G. Georgescu and A. Iorgulescu, Pseudo-BL algebras. I, *Mult.-Valued Log.* **8** (2002), no. 5-6, 673–714.

[5] A. Di Nola, G. Georgescu and L. Leuştean, Boolean products of BL-algebras, *J. Math. Anal. Appl.* **251** (2000), no. 1, 106–131.

[6] A. Di Nola and L. Leuştean, Compact representations of BL-algebras, *Arch. Math. Logic* **42** (2003), no. 8, 737–761.

[7] A. Dvurečenskij, States on pseudo MV-algebras, *Studia Logica* **68** (2001), no. 3, 301–327.

[8] A. Dvurečenskij, On pseudo MV-algebras, *Soft Computing* **5** (2001), 347–354.

[9] A. Dvurečenskij, Every linear pseudo-BL algebra admits a state, *Soft Computing* **11** (2007), 495–501.

[10] P. Flonder, G. Georgescu, A. Iorgulescu, Pseudo-t-norms and pseudo-BL algebras, *Soft Computing* **5** (2001), 355–371.

[11] G. Georgescu, Bosbach states on fuzzy structures, *Soft Computing* **8** (2004), 217–230.

[12] G. Georgescu and A. Iorgulescu, Pseudo-MV algebras, *Mult.-Valued Log.* **6** (2001), no. 1-2, 95–135.

[13] G. Georgescu and L. Leuştean, Some classes of pseudo-BL algebras, *J. Aust. Math. Soc.* **73** (2002), no. 1, 127–153.

[14] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht, 1998.

[15] A. Iorgulescu, Iséki algebras. Connection with BL-algebras, *Soft Computing* **8** (2004), 449–463.

[16] P. M. Pu and Y. M. Liu, Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.* **76** (1980), no. 2, 571–599.

[17] J. Rachunek, Prime spectra of non-commutative generalizations of MV-algebras, *Algebra Universalis* **48** (2002), no. 2, 151–169.

[18] J. Rachunek, A non-commutative generalization of MV-algebras, *Czechoslovak Math. J.* **52(127)** (2002), no. 2, 255–273.

[19] E. Turunen, Boolean deductive systems of BL-algebras, *Arch. Math. Logic* **40** (2001), no. 6, 467–473.

[20] E. Turunen and S. Sessa, Local BL-algebras, *Mult.-Valued Log.* **6** (2001), no. 1-2, 229–249.

[21] X. Yuan, C. Zhang and Y. Ren, Generalized fuzzy groups and many-valued implications, *Fuzzy Sets and Systems* **138** (2003), no. 1, 205–211.

[22] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.

[23] J. Zhan and Y. B. Jun, Generalized fuzzy ideals of BCI-algebras, *Bull. Malays. Math. Sci. Soc.* (2) **32** (2009), no. 2, 119–130.

[24] X. H. Zhang, W. H. Li, On pseudo-BL algebras and BCC-algebras, *Soft Computing* **10** (2006), 941–952.

[25] X. Zhang, K. Qin and W. A. Dudek, Ultra LI-ideals in lattice implication algebras and MTL-algebras, *Czechoslovak Math. J.* **57(132)** (2007), no. 2, 591–605.