

A Dynamic System Approach to Quadratic Programming Problems with Penalty Method

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Abstract. In this work, we propose a dynamical system (state space) model approach to find a unique minimum of quadratic programming (QP) problems with equality constrained. The unique minimum of the optimization problem is also proved to be asymptotically stable equilibrium point of the state space model. To obtain the optimal solution of QP optimization problem, we seek the limit point of the solution of the state space model by using the transfer function rather than discretization scheme. The numerical results are shown that the applicability and efficiency of the approach by compared with sequential quadratic programming (SQP) method in three examples.

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1. Introduction

Optimization is an area that finds optimal solution for problems which are defined mathematically in science, economics, engineering and other related areas. Quadratic programming (QP) problem is a subclass of optimization problem with a quadratic objective function and linear constraints.

Many efficient methods have been developed for solving these problems in the last three decades. One of them is called penalty method. In this class of methods we replace the original constrained problem with a sequence of unconstrained sub-problems that minimizes the penalty function. Thereby, it is an important approach to solve constrained optimization problems. Further information can be found in references, [3, 4, 6, 16].

The further approaches are Ordinary Differential Equation (ODE) methods and gradient flow method. ODE methods, which is for solving equality constrained optimization problem is proposed by Arrow and Hurwicz [2], and developed by Pan [15]. Recently, Jin and Zhang [10] have prepared a differential equation approach for

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solving nonlinear programming problems. Therewithal, gradient flow method was first introduced by Evtushenko and Zhadan in 1994 [7, 8], and then used for solving nonlinear optimization problems by Andrei [1] and Wang *et al.* [17].

In the same manner, state space approach can be shown as a gradient flow algorithm for solving quadratic programming problems. This approach is based on the concept of state. An important advantage of the state variable representation is that it allows the systems to operate with several inputs and outputs without changing the notational framework. Furthermore, state variables give us the internal behavior of the system. It is represented as the transfer function $G(s)$ which indicates the input-output behavior of the system. For more details, see Kwakernaak [11] and Ogata [14].

To obtain the solution of QP problem, we seek the limit point of the solution of the state space model by using the transfer function of the corresponding dynamical system rather than the forward Euler's time-stepping scheme proposed in [7, 8] which is known to be unstable for solving ODEs, and two level implicit time discretization scheme in [17].

In this work, we apply the technique of state space (dynamic system) modeling to the equality constrained QP optimization problem to find the unique optimum. The state space approach is constructed from the gradient vector of the unconstrained optimization problem, which is obtained from QP problem by using the quadratic penalty function method. The paper also shows that the unique optimal solution of the original optimization is also the asymptotically stable equilibrium point of the state space model, vice versa.

The structure of this paper is given as follows. In Section 2, we present essential background material. In Section 3, we prove the main result, which states a new approach for solving QP problem with equality constrained for strictly convex functions. In Section 4, the approach presented in this paper is illustrated by three numerical examples. And finally, conclusions will be presented.

2. Problems and preliminaries

Let us consider the following equality constrained quadratic programming problem,

$$(2.1) \quad \text{minimize } f(x) = \frac{1}{2}x^T Gx + g^T x \quad \text{subject to } Ex = d,$$

where $x \in \mathbb{R}^n$ is decision variable, G is an $(n \times n)$, real, symmetric, positive definite matrix and the constraints are defined by an $(m \times n)$ full row rank matrix E and an m -dimensional column vector d of right hand side coefficient. If the Hessian matrix G is positive definite, then (2.1) is a strict convex quadratic programming problem and x^* is a unique global solution. We assume that a feasible solution exists and the constraint region is bounded.

The penalty function is defined as

$$P(x) = f(x) + \phi(x),$$

where the penalty term $\phi(x)$ is a function defined on \mathbb{R}^m and satisfies

$$\phi(x) = \begin{cases} 0, & \text{if } Ex - d = 0 \\ \text{nonnegative,} & \text{if } Ex - d \neq 0. \end{cases}$$

A popular choice for $\phi(x)$ has been also called quadratic penalty term and defined as

$$(2.2) \quad \phi(x) = \frac{1}{2} \|Ex - d\|_2^2.$$

In many studies a penalty function of the problem (2.1) has been defined as

$$(2.3) \quad P_{p_k}(x) = f(x) + p_k\phi(x),$$

where p_k is a penalty sequence satisfying $0 < p_k < p_{k+1}$ for $\forall k, p_k \rightarrow \infty$. Indeed, the scalar quantity p_k is called the penalty parameter. Note that the penalty parameter sequence $\{p_k\}$ can be chosen randomly in a rule. It means that you can choose p_k in positive constant which depends on the difficulty of minimizing the penalty function at every iteration. The idea of the penalty function method is to reformulate constrained optimization problem by using a sequence of unconstrained optimization subproblems of the form

$$(2.4) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} P_{p_k}(x).$$

Throughout this paper we will assume that the problem given in (2.4) has a solution for each k . Let us denote the minimizer of $P_{p_k}(x)$ by x_k .

Theorem 2.1. *Let $\{x_k\}$ be a sequence generated by the quadratic penalty method. Then any limit point of the sequence is a solution to (2.1), [12].*

3. A state space modeling with quadratic penalty function

In this section, we aim at solving the equality constrained QP problem given in (2.1). To do that we will use trajectory defined as a system of state space equation.

We will consider the following ordinary differential equations system

$$(3.1) \quad \frac{dx}{dt} = \nabla_x P_{p_k}(x(t)),$$

which is equal to,

$$(3.2) \quad \begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \\ &= \begin{bmatrix} a_{11}(p) & a_{12}(p) & \cdots & a_{1n}(p) \\ a_{21}(p) & a_{22}(p) & \cdots & a_{2n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(p) & a_{n2}(p) & \cdots & a_{nn}(p) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1(p) \\ b_2(p) \\ \vdots \\ b_n(p) \end{bmatrix} u(t). \end{aligned}$$

Equation (3.2) can be called as state space equation or vector matrix differential equations such that

$$(3.3) \quad \begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)u(t) \\ y(t) &= Cx(t), \end{aligned}$$

where $p > 0$ is the large enough penalty parameter, A is the $(n \times n)$ constant matrix, x is the $(n \times 1)$ state vector, B is the $(n \times 1)$ constant matrix, C is the $(n \times n)$ unit

matrix and u is a constant input. We assume that the input u will be chosen as in the following form

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & \text{otherwise.} \end{cases}$$

The transfer function of the system (3.3) corresponding u to y is defined by $Y(s) = G(s)U(s)$, where $U(s)$ and $Y(s)$ are the Laplace transform with respect to $u(t)$ and $y(t)$ with the initial condition $x(0) = 0$, see Kwakernaak [11]. Hence, the transfer function of (3.3) is

$$(3.4) \quad \begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{1}{\det(sI - A)}C [\text{Adj}(sI - A)]B. \end{aligned}$$

Remark 3.1. In this study, we assume that,

- (a) All eigenvalues of $A(p)$ satisfy either $\text{Re } \lambda_i \leq 0$ or $\text{Re } \lambda_i \geq 0$ and every eigenvalue with $\text{Re } \lambda_i = 0$ has an associated Jordan block of order one.
- (b) The linear system (3.3) has a minimal realization.

Definition 3.1. For big enough penalty parameter $p > 0$, the linear system given in (3.3) is asymptotically stable if all eigenvalues of $A(p)$ are in the open left half plane.

From the Definition 3.1 and MATLAB code, we get a linear system from (3.3), which is always stable, by using the transformation below,

- (a) $\text{Re } \lambda[A(p)] \leq 0 \implies \tilde{A}(p) := A(p)$ and $\tilde{B}(p) := B(p)$,
- (b) $\text{Re } \lambda[A(p)] \geq 0 \implies \tilde{A}(p) := -A(p)$ and $\tilde{B}(p) := -B(p)$.

Now we consider the system as follows;

$$(3.5) \quad \begin{aligned} \dot{x}(t) &= \tilde{A}(p)x(t) + \tilde{B}(p)u(t) \\ y(t) &= Cx(t). \end{aligned}$$

Definition 3.2. A point $x_e \in \mathbb{R}^n$ is said to be an equilibrium point of the system of (3.5) if $x(t, x_e) = x_e$ for all $t \geq 0$.

Definition 3.3. A point x_e is said to be stable in the sense of Lyapunov if for any $x(t_0) = x_0$, and any scalar $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x_0 - x_e\| < \delta$, then $\|x(t, x_0) - x_e\| < \varepsilon$ for all $t \geq t_0$.

Definition 3.4. If x_e is a stable equilibrium point and

$$(3.6) \quad \lim_{t \rightarrow \infty} \|x(t, x_0) - x_e\| = 0,$$

then it is said to be asymptotically stable.

Definition 3.5. A system is said to be bounded-input, bounded-output (BIBO) stable if for all t_0 and zero initial conditions at $t = t_0$, every bounded input defined on $[t_0, \infty)$ gives rise to a bounded response on $[t_0, \infty)$.

Theorem 3.1. If the linear system (3.3) is a minimal realization, then bounded-input, bounded output stability implies asymptotic stability [18].

Lemma 3.1. *If $A(p)$ is the $(n \times n)$ stability matrix for big enough penalty parameter p , then there exist positive constants K and k such that*

$$\left\| e^{A(p)t} \right\| \leq K e^{-kt} \quad \text{for all } t \geq 0.$$

Proof. Suppose x is a generalized eigenvector of multiplicity $m : \exists \lambda \in \sigma(A(p))$ (eigenvalue of $A(p)$) such that $(A(p) - \lambda I)^m x = 0$ but $(A(p) - \lambda I)^k x \neq 0$ for $k = 0, 1, \dots, m - 1$. Then

$$e^{A(p)t} x = e^{\lambda I t} e^{(A(p) - \lambda I)t} x = e^{\lambda I t} \sum_{k=0}^{m-1} \frac{(A(p) - \lambda I)^k t^k}{k!} x.$$

Since the exponential series $\sum_{k=0}^{\infty} (t^k/k!)$ is absolutely convergence for every operator t , $t^k e^{-\varepsilon t}$ is bounded in t for any $\varepsilon > 0$. Then, it follows that

$$\begin{aligned} \left\| e^{A(p)t} x \right\| &\leq \bar{K} e^{\operatorname{Re} \lambda t} |\cos(\operatorname{Im} \lambda t) + i \sin(\operatorname{Im} \lambda t)| e^{\varepsilon t} \|x\| \\ &\leq K e^{-kt} \|x\|, \end{aligned}$$

for sufficiently small ε . ■

Theorem 3.2. *Suppose that $P_{p_k}(x)$ is a differentiable function for a large enough penalty parameter $p_k > 0$. Then x^* is a unique optimal solution for problem (2.1) if and only if x^* is the asymptotically stable equilibrium point for the system (3.5).*

Proof. We first prove the necessary condition. Suppose that x^* be a unique optimum of the equality constrained quadratic programming problem as in (2.1). By Theorem 2.1, x^* is also a unique optimum of the quadratic unconstrained optimization problem as in (2.4). By virtue of the equation (3.1) and the Definition 3.1, we get the state space modeling of the unconstrained problem. Then, by using the optimality conditions, we have

$$\nabla_x P_{p_k}(x^*) = 0.$$

In other words, x^* is an equilibrium point of the system (3.5). Let $x(t_0) = x_0$ and $M > 0$ such that $\|u(t)\| < M$. Hence, the solution of the system (3.5) is

$$x(t, x_0) = e^{\tilde{A}(p)(t-t_0)} x_0 + \int_{t_0}^t e^{\tilde{A}(p)(t-\tau)} \tilde{B}(p) u(\tau) d\tau.$$

To show the stability of the solution, we have to satisfy that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|x_0 - x\| < \delta \implies \|x(t, x_0) - x^*\| < \varepsilon, \quad \forall t \geq 0.$$

Therefore,

$$\begin{aligned} \|x(t, x_0) - x^*\| &= \left\| e^{\tilde{A}(p)(t-t_0)} x_0 + \int_{t_0}^t e^{\tilde{A}(p)(t-\tau)} \tilde{B}(p) u(\tau) d\tau - x^* \right\| \\ &\leq \left\| e^{\tilde{A}(p)(t-t_0)} x_0 - x^* \right\| + \left\| \int_{t_0}^t e^{\tilde{A}(p)(t-\tau)} \tilde{B}(p) u(\tau) d\tau \right\| \\ &\leq \left\| e^{\tilde{A}(p)(t-t_0)} (x_0 - x^*) \right\| + \left\| \left(e^{\tilde{A}(p)(t-t_0)} - I_{n \times n} \right) x^* \right\| \end{aligned}$$

$$\begin{aligned}
& + M \left\| \tilde{B}(p) \right\| \int_{t_0}^t \left\| e^{\tilde{A}(p)(t-\tau)} \right\| d\tau \\
& \leq K e^{k(t_0-t)} \|x_0 - x^*\| + K e^{k(t_0-t)} \|x^*\| + \|x^*\| \\
& + MK \left\| \tilde{B}(p) \right\| \int_{t_0}^t e^{k(\tau-t)} d\tau \\
& = K e^{k(t_0-t)} \|x_0 - x^*\| + K e^{k(t_0-t)} \|x^*\| + \|x^*\| \\
& + MK k^{-1} \left\| \tilde{B}(p) \right\| \left(1 - e^{k(t_0-t)} \right) \\
& \leq K \|x_0 - x^*\| + K \|x^*\| + \|x^*\| + MK k^{-1} \left\| \tilde{B}(p) \right\|.
\end{aligned}$$

Now if we choose

$$\delta = K^{-1} \left(\varepsilon - \|x^*\| (K + 1) - MK k^{-1} \left\| \tilde{B}(p) \right\| \right),$$

then x^* becomes a stable equilibrium point of (3.5). Furthermore, we get

$$\begin{aligned}
\|y(t)\| & = \left\| C e^{\tilde{A}(p)(t-t_0)} x_0 + \int_{t_0}^t C e^{\tilde{A}(p)(t-\tau)} \tilde{B}(p) u(\tau) d\tau \right\| \\
& \leq \|C\| \left\| e^{\tilde{A}(p)(t-t_0)} \right\| \|x_0\| + \|C\| \left\| \tilde{B}(p) \right\| M \int_{t_0}^t \left\| e^{\tilde{A}(p)(t-\tau)} \right\| d\tau \\
& \leq K \|C\| \left[\|x_0\| e^{k(t_0-t)} + \left\| \tilde{B}(p) \right\| M \int_{t_0}^t e^{k(\tau-t)} d\tau \right] \\
& = K \|C\| \left[\|x_0\| e^{k(t_0-t)} + k^{-1} \left\| \tilde{B}(p) \right\| M \left(1 - e^{k(t_0-t)} \right) \right],
\end{aligned}$$

and so

$$\|y(t)\| \leq M_1,$$

where

$$M_1 = K \|C\| \left(\|x_0\| + k^{-1} \left\| \tilde{B}(p) \right\| M \right) > 0.$$

Thus, the output $y(t)$ is bounded, that is, x^* is BIBO stable equilibrium point of (3.5) for every initial condition $x(t_0) = x_0$. Finally, by the Theorem 3.1 and Remark 3.1, x^* is asymptotically stable equilibrium point of the system (3.5).

Now the sufficient part will be proved. Suppose that x^* is the asymptotically stable equilibrium point of (3.5). Thereby, for each $\varepsilon > 0$, there is $\delta > 0$ and $t_0 = 0$, we have $x(t_0) = x_0$ such that

$$\|x_0 - x^*\| < \delta \implies \|x(t, x_0) - x^*\| < \varepsilon$$

and $\lim_{t \rightarrow \infty} \|x(t, x_0) - x^*\| = 0$. Then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \|x(t, x_0) - x^*\| & = \left\| \lim_{t \rightarrow \infty} (x(t, x_0) - x^*) \right\| = 0 \\
& \iff \lim_{t \rightarrow \infty} (x(t, x_0) - x^*) = 0 \iff \lim_{t \rightarrow \infty} x(t, x_0) = x^*.
\end{aligned}$$

Hence, we say that x^* is the unique optimal solution of the problem (2.1). ■

In the following part, using the Lemma 3.1 and Theorem 3.2 we developed a solution technique, which is called penalty function with state space modeling (PFSS) to generate a global optimal solution for the problem (2.1). Moreover, we use MATLAB program in order to execute steps in our technique.

The steps of PFSS approach summarize as follow:

Step 1. Given $\varepsilon > 0$, $p_1 > 0$, $k = 1$.

Step 2. Replace constrained problem with unconstrained subproblem by using quadratic penalty term $\phi(x(t))$.

Step 3. Compose state space model of the quadratic penalty function $P_{p_k}(x(t))$ using the ordinary differential equation (3.1).

Step 4. Check the stability of the matrix $A(p)$ using Definition 3.1.

Step 5. Compute transfer function $G(s)$.

Step 6. Use the step response $[c, x, t] = \text{step}(\text{num}, \text{den}, t)$ to find $\min_{x \in \mathbb{R}^n} P_{p_k}(x(t))$ on MATLAB.

Step 7. If $\|Ex_k - d\|_2^2 \leq \varepsilon$, stop; otherwise $p_{k+1} = p_k + 100$, $k = k + 1$, go to Step 5.

4. Numerical results

Example 4.1. Let us consider the problem [13],

$$(4.1) \quad \text{minimize} \quad f(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3 \quad \text{subject to} \quad x_1 - x_2 + 2x_3 - 2 = 0.$$

The optimal solution of the equality constrained optimization problem (4.1) is $x^* = (2.5, -1.5, -1)^T$. Firstly, let us replace equality constrained optimization problem with unconstrained subproblem by using quadratic penalty function (2.3). So we have

$$(4.2) \quad P_{p_k}(x) = x_1^2 - 2x_1 + x_2^2 - x_3^2 + 4x_3 + \frac{1}{2}p(x_1 - x_2 + 2x_3 - 2)^2.$$

Then we obtain the state space modeling of the equation (4.2) by using equation (3.1) and Definition 3.1. Thus,

$$(4.3) \quad \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -2-p & p & -2p \\ p & -2-p & 2p \\ -2p & 2p & 2-4p \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 2p+2 \\ -2p \\ 4p-4 \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

For a big enough penalty parameter p (the value of p should be larger than $p = 1000$), the eigenvalues of $A(p)$ are in the open left half plane, in addition, the eigenvalues of

$A(p)$ are in stable domain. After that we acquire transfer function $G(s)$ as follows;

$$G(s) = \frac{1}{s^3 + 6002s^2 + 15996s + 7992} \begin{pmatrix} 2002s^2 + 18000s + 19992 \\ -2000s^2 - 6000s - 12000 \\ 3996s^2 + 3984s - 8016 \end{pmatrix}.$$

For numerical solution of this problem, we choose zero initial condition. By using the PFSS approach, it can be seen that the optimal solution of x_1 , x_2 and x_3 reach to $x_1^* = 2.5$, $x_2^* = -1.5$ and $x_3^* = -1$ for zero initial point as shown in Figure 1.

The PFSS approach also finds optimal values of the problem (4.1) with different initial points $(1, 1, 1)$, $(-1, 1, -1)$, $(5, 2, -1)$ and $(2, -3, 4)$.

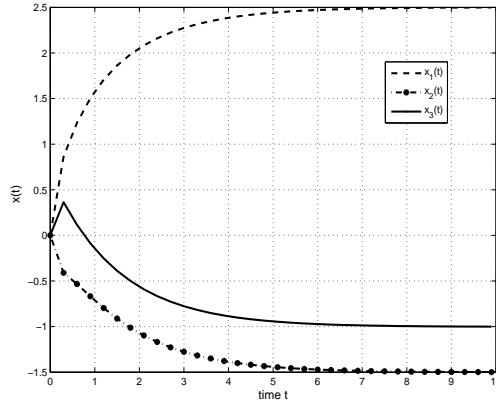


Figure 1. The optimal value of $x(t)$ in Example 4.1

Example 4.2. Consider the following QP problem with equality constrained [9],

$$(4.4) \quad \begin{aligned} & \text{minimize} && f(x) = (x_1 - x_2)^2 + (x_2 + x_3 - 2)^2 + (x_4 - 1)^2 + (x_5 - 1)^2 \\ & \text{subject to} && x_1 + 3x_2 - 4 = 0, \\ & && x_3 + x_4 - 2x_5 = 0, \\ & && x_2 - x_5 = 0. \end{aligned}$$

The theoretical optimal solution of the problem (4.4) is $x^* = (1, 1, 1, 1, 1)^T$. By using the quadratic penalty function (2.3), we get the corresponding unconstrained optimization problem as follow;

$$P_{pk}(x) = (x_1 - x_2)^2 + (x_2 + x_3 - 2)^2 + (x_4 - 1)^2 + (x_5 - 1)^2 + \frac{1}{2}p(x_1 + 3x_2 - 4)^2 + \frac{1}{2}p(x_3 + x_4 - 2x_5)^2 + \frac{1}{2}p(x_2 - x_5)^2.$$

Then we get state space modeling of the problem (4.4) by using the equation (3.1) and Definition 3.1. Hence,

$$(4.5) \quad \begin{aligned} \dot{x}(t) &= A(p)x(t) + B(p)u(t) \\ y(t) &= Cx(t), \end{aligned}$$

where

$$A(p) = \begin{pmatrix} -p-2 & -3p+2 & 0 & 0 & 0 \\ -3p+2 & -10p-4 & 2 & 0 & p \\ 0 & -2 & -p-2 & -p & 2p \\ 0 & 0 & -p & -p-2 & 2p \\ 0 & p & 2p & 2p & -5p-2 \end{pmatrix},$$

$$B(p) = \begin{pmatrix} 4p \\ 12p+4 \\ 4 \\ 2 \\ 2 \end{pmatrix}$$

and $C = I_{5 \times 5}$ identity matrix. For a big enough penalty parameter p (the value of p should be larger than $p = 1000$), the eigenvalues of $A(p)$ are $0, -2, -2, -6$. The transfer function of the state space model (4.5) can be defined as (3.4). As a result, by using the PFSS approach for zero initial conditions, we have seen that the trajectory of $x(t)$ for the state space model (4.5) can be reached to the optimal solution $x^* = (x_1, x_2, x_3, x_4, x_5)^T = (1, 1, 1, 1, 1)^T$, as shown in Figure 2.

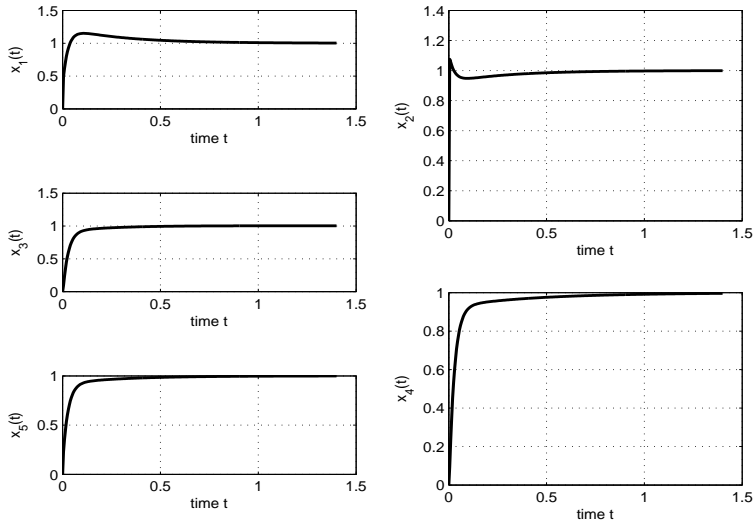


Figure 2. The optimal value of $x(t)$ in Example 4.2

Example 4.3. In this example, we consider the following constrained optimization problem with linear equality constrained which is a special form of the linear quadratic regulator problem [5],

$$(4.6) \quad \begin{aligned} & \text{minimize} && \frac{1}{2} \sum_{i=1}^k (qx_i^2 + rz_i^2) \\ & \text{subject to} && x_i = ax_{i-1} - bz_i, \text{ for } i = 1, 2, \dots, k, \end{aligned}$$

where x_i is the account balance at the end of month i and z_i is our payment in month i . Suppose we currently have a credit card debt of $x_0 = 10000$. Credit card debts are subject to a monthly interest rate 2%, and the account balance is increased by the interest amount every month. Each month, we have the option of reducing the account balance by contributing a payment to the account. Over the next $k = 10$ months, we plan to contribute a payment every month in such a way as to minimize the overall debt level while at the same time minimize the hardship of making monthly payments. We choose $q = 1$, $r = 300$, $a = 1.02$, $b = 1$ and the penalty parameter $p = 10^7$ for solving problem.

Firstly, let us reformulate equality constrained QP problem to unconstrained subproblems by using quadratic penalty function (2.3). So, we have

$$P_{p_k}(x, z) = \frac{1}{2} \sum_{i=1}^{10} (qx_i^2 + rz_i^2) + \frac{1}{2} \sum_{i=1}^{10} p(x_i - 1.02x_{i-1} + z_i)^2.$$

By virtue of the Definition 3.1, the linear ordinary differential equations system (3.5) obtain as following

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A_{11}(p) & A_{12}(p) \\ A_{21}(p) & A_{22}(p) \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} B_{11}(p) \\ B_{21}(p) \end{pmatrix},$$

where

$$A_{11}(p) = \begin{bmatrix} -q - 2.0404p & 1.02p & 0 & \cdots & 0 \\ 1.02p & -q - 2.0404p & 1.02p & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1.02p & -q - 2.0404p & 1.02p \\ 0 & \cdots & 0 & 1.02p & -q - p \end{bmatrix},$$

$$A_{12}(p) = \begin{bmatrix} -p & 1.02p & 0 & \cdots & 0 \\ 0 & -p & 1.02p & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1.02p & -p \end{bmatrix},$$

$$A_{21}(p) = \begin{bmatrix} -p & 0 & \cdots & 0 \\ 1.02p & \ddots & \ddots & \vdots \\ 0 & \ddots & & 0 \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & 1.02p & -p \end{bmatrix},$$

$$A_{22}(p) = \begin{bmatrix} -r - p & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & -r - p \end{bmatrix}$$

are (10×10) dimensional square matrices and

$$B_{11}(p) = \begin{bmatrix} 1.02 \times 10^4 p \\ 0 \\ \vdots \\ 0 \end{bmatrix} 0,$$

$$B_{21}(p) = \begin{bmatrix} 1.02 \times 10^4 p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

are (10×1) column matrices.

For numerical solution of all x_i and z_i ($i = 1, 2, \dots, 10$), we choose zero initial condition. A unique optimal solution for problem (4.6) is illustrated in Figure 3, Figure 4 and Table 1.

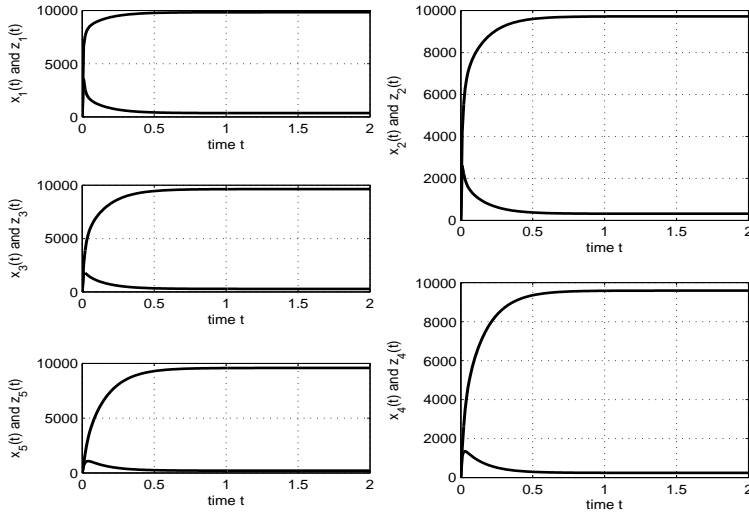
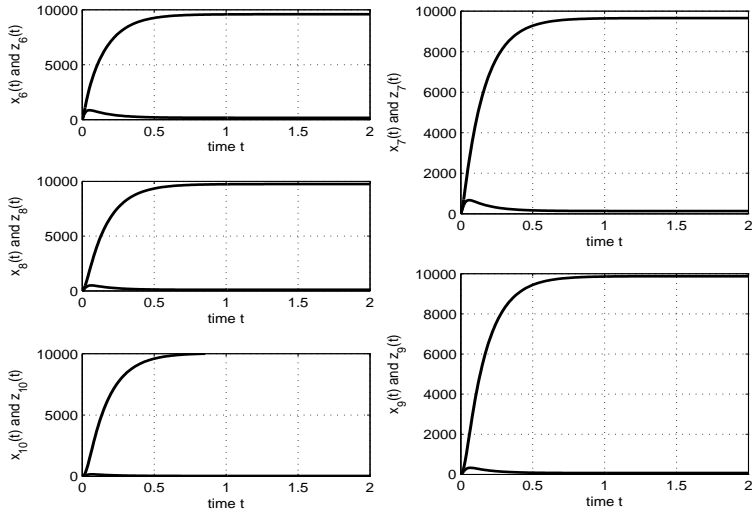


Figure 3. The optimal values of $x(t)$ and $z(t)$ in Example 4.3

Table 1. Numerical Result of Example 4.3

i	1	2	3	4	5	6	7	8	9	10
x_i	9844.7	9725.4	9641.6	9593.2	9579.9	9601.7	9658.6	9750.8	9878.8	10043
z_i	355.34	316.2	278.22	241.25	205.17	169.84	135.13	100.92	67.075	33.476

We use the ordinary differential equations system (3.1) to solve the problem (4.6) and all simulation results show that our optimal solution correspond to the theoretical solution.

Figure 4. The optimal values of $x(t)$ and $z(t)$ in Example 4.3

In Table 2, it shows that the PFSS approach and SQP method are compared with CPU time. Although these two methods are very closed, the PFSS approach results are better than SQP methods.

Table 2. PFSS and SQP methods in comparison with CPU time

Problem	PFSS	SQP
Example 4.1	0.04	0.07
Example 4.2	0.07	0.09
Example 4.3	0.34	0.39

5. Conclusions

In this study, we have presented a dynamical system (state space model) approach for solving equality constrained QP problems with quadratic penalty function method. Necessary and sufficient conditions for the converge of optimal values to the problem have been given under the assumptions. We have also developed an approach to solve problem (2.1) based on the dynamic system modeling. Numerical examples on equality constrained QP problem were performed and the numerical results show that the PFSS approach is useful to find optimal values for this kind of problem with different initial points. Furthermore, computational results shows that the PFSS approach reached the optimal values in less CPU time than the SQP method.

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