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Boundedness for Multilinear Singular Integral Operators on Morrey Spaces

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Abstract. In this paper, we prove the boundedness for some multilinear operators related to certain singular integral operators on the Morrey spaces by using a sharp inequality of the multilinear operators.

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1. Preliminaries and theorem

Fix $\lambda \geq 0$. For $1 \leq p < \infty$, set

$$||f||_{L^{p,\lambda}} = \sup_{x \in R^n,\ d>0} \left(\frac{1}{d^\lambda} \int_{B(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $B(x,d) = \{y \in R^n : |x-y| < d\}$. The Morrey spaces is defined by (see [14, 15])

$$L^{p,\lambda}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\lambda}} < \infty\}.$$

In this paper, we shall study some singular integral operators as following.

Definition 1.1. *Fix $\varepsilon > 0$ and $0 \leq \delta < n$. Let S be the Schwartz space of test functions, and S' be its dual, that is S' is the set of all tempered distributions on S (see* [19, p.88–89]*). Suppose $T : S \to S'$ be a linear operator and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that*

$$T(f)(x) = \int_{R^n} K(x,y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x,y)| \leq C|x-y|^{-n+\delta}$$

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and

$$|K(y,x)-K(z,x)|+|K(x,y)-K(x,z)|\le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z|\le|x-z|$. *Let* m_j *be some positive integers(*$j=1,\cdots,l$*),* $m_1+\cdots+m_l=m$ *and* A_j *be some functions on* R^n *(*$j=1,\cdots,l$*), set* $A=(A_1,\cdots,A_l)$. *The multilinear operator related to* T *is defined by*

$$T^A(f)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f(y)dy,$$

where

$$R_{m_j+1}(A_j;x,y)=A_j(x)-\sum_{|\alpha|\le m_j}\frac{1}{\alpha!}D^{\alpha}A_j(y)(x-y)^{\alpha}.$$

Note that when $m=0$, $T^A(f)(x)=\int_{R^n}\prod_{j=1}^{l}(A_j(x)-A_j(y))K(x,y)f(y)dy$, that is T^A is just the multilinear commutator of T and A (see [18]). While when $m>0$, T^A is a non-trivial generalizations of the commutator. It is well-known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3–5, 8]). Cohen and Gosselin (see [3–5]) obtained the $L^p(p>1)$ boundedness of the multilinear singular integral operator. Hu and Yang (see [10]) proved a sharp estimate for the multilinear singular integral operators. In [16–18], the authors prove some sharp estimate for the multilinear singular integral commutator and obtained the L^p $(p>1)$ boundedness for the multilinear operator. As the Morrey space may be considered as an extension of the Lebesgue space (Morrey space $L^{p,\lambda}$ becomes Lebesgue space L^p when $\lambda=0$), it is natural and important to study the boundedness of the multilinear singular integral operator on the Morrey Spaces $L^{p,\lambda}$ with $\lambda>0$. The purpose of this paper has twofold, first, we establish a sharp inequality for the multilinear singular integral operators, and second, we prove the boundedness for the multilinear singular integral operators on the Morrey spaces by using the sharp inequality.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [9])

$$f^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO}=||f^{\#}||_{L^{\infty}}$. For $0\le\eta<1$, let M_η be the fractional maximal operator defined by

$$M_\eta(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1-\eta}}\int_Q|f(y)|dy.$$

We denote by $M(f)(x) = M_\eta(f)(x)$ if $\eta = 0$, which is the Hardy-Littlewood maximal operator. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

We shall prove the following theorem.

Theorem 1.1. *Let $1 < p < n/\delta$, $0 < \lambda < n - p\delta$, $1/q = 1/p - \delta/(n-\lambda)$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Suppose that T^A is the multilinear operator as Definition 1.1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $1 < u < n/\delta$ and $1/v = 1/u - \delta/n$. Then, for any $f \in L^{p,\lambda}(R^n)$,*

$$||T^A(f)||_{L^{q,\lambda}} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) ||f||_{L^{p,\lambda}}.$$

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 2.1. [5] *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2. [2, 6] *Let $1 < p < \infty$, $0 < \lambda < n$ and $f \in L^{p,\lambda}(R^n)$. Then the following estimates hold:*

(a) $||M(f)||_{L^{p,\lambda}} \le C||f^\#||_{L^{p,\lambda}}$;

(b) $||M_\eta(f)||_{L^{q,\lambda}} \le C||f||_{L^{p,\lambda}}$ *for* $0 < \eta < (n-\lambda)/np$ *and* $1/q = 1/p - n\eta/(n-\lambda)$.

Lemma 2.3. *Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Suppose that T^A is the multilinear operator as Definition 1.1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $1 < u < n/\delta$ and $1/v = 1/u - \delta/n$. Then there exists a constant $C > 0$ such that for any $f \in L^{p,\lambda}(R^n)$, $1 < r < p < n/\delta$, $0 < \lambda < n - p\delta$ and a.e. $\tilde{x} \in R^n$,*

$$(T^A(f))^\#(\tilde{x}) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

Proof. It suffices to prove for some constant C_0 and $f \in L^{p,\lambda}(R^n)$, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_0| dx \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$
\begin{aligned}
T^A(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) f(y) dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) f_2(y) dy \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) f_1(y) dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) K(x,y) f_1(y) dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y) K(x,y) f_1(y) dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} K(x,y) f_1(y) dy,
\end{aligned}
$$

then

$$
\begin{aligned}
&\left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| \\
&\le \left|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) f_1(y) dy\right| \\
&\quad + \left|\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) K(x,y) f_1(y) dy\right| \\
&\quad + \left|\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y) K(x,y) f_1(y) dy\right| \\
&\quad + \left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} K(x,y) f_1(y) dy\right| \\
&\quad + \left|T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)\right| \\
&:= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\end{aligned}
$$

thus,

$$
\frac{1}{|Q|} \int_Q \left|T^A(f)(x) - T^{\tilde{A}}(f_2)(x_0)\right| dx
$$

$$\leq \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx$$
$$+ \frac{1}{|Q|}\int_Q I_5(x)dx$$
$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 2.1, we get

$$R_m(\tilde{A}_j; x, y) \leq C|x-y|^m \sum_{|\alpha_j|=m} ||D^{\alpha_j} A_j||_{BMO},$$

thus, by the (L^r, L^q)-boundedness of T with $1 < r < n/\delta$ and $1/q = 1/r - \delta/n$, we obtain

$$I_1 \leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) \frac{1}{|Q|}\int_Q |T(f_1)(x)|dx$$
$$\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) \left(\frac{1}{|Q|}\int_{R^n} |T(f_1)(x)|^q dx\right)^{1/q}$$
$$\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) |Q|^{-1/q} \left(\int_{R^n} |f_1(x)|^r dx\right)^{1/r}$$
$$\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) \left(\frac{1}{|Q|^{1-r\delta/n}}\int_{\tilde{Q}} |f(x)|^r dx\right)^{1/r}$$
$$\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO}\right) M_{\delta,r}(f)(\tilde{x}).$$

For I_2, denoting $r = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/s = 1/p - \delta/n$, we have, by Hölder's inequality,

$$I_2 \leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|dx$$
$$\leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|^s dx\right)^{1/s}$$
$$\leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/s}\left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) f_1(x)|^p dx\right)^{1/p}$$
$$\leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pq'} dx\right)^{1/pq'}$$
$$\times \left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}}\int_{\tilde{Q}} |f(x)|^{pq} dx\right)^{1/pq}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Similarly, for I_4, denoting $r=pq_3$ for $1<p<n/\delta$, $q_1,q_2,q_3>1$, $1/q_1+1/q_2+1/q_3=1$ and $1/s=1/p-\delta/n$, we obtain

$$\begin{aligned}
I_4 &\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|dx\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|^s dx\right)^{1/s}\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/s}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)f_1(x)|^p dx\right)^{1/p}\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(x)|^{pq_1}dx\right)^{1/pq_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(x)|^{pq_2}dx\right)^{1/pq_2}\\
&\quad\times\left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}}\int_{\tilde{Q}}|f(x)|^{pq_3}dx\right)^{1/pq_3}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,p}(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)\\
&=\int_{R^n}\left(\frac{K(x,y)}{|x-y|^m}-\frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)f_2(y)dy\\
&\quad+\int_{R^n}\left(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\right)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)f_2(y)dy\\
&\quad+\int_{R^n}\left(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\right)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)f_2(y)dy\\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)\right.\\
&\quad\left.-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\right]D^{\alpha_1}\tilde{A}_1(y)f_2(y)dy
\end{aligned}$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) \right.$$
$$\left. - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] D^{\alpha_2}\tilde{A}_2(y) f_2(y)dy$$
$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y) \right]$$
$$\times D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) f_2(y)dy$$
$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 2.1 and the following inequality (see [19])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)||b||_{BMO} \quad \text{for} \quad Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|)$$
$$\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on K,

$$|I_5^{(1)}| \le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) \prod_{j=1}^2 |R_{m_j}(\tilde{A}_j; x, y)||f_2(y)|dy$$
$$\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} \right.$$
$$\left. + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f(y)|dy$$
$$\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^\infty k^2 (2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f(y)|dy$$
$$\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^\infty k^2 (2^{-k} + 2^{-\varepsilon k})$$
$$\times \left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}} \int_{2^k\tilde{Q}} |f(y)|dy \right)^{1/r}$$
$$\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 2.1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\begin{aligned}|I_5^{(2)}|&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|dy\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).\end{aligned}$$

Similarly,

$$|I_5^{(3)}|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(4)}$, we get

$$\begin{aligned}|I_5^{(4)}|&\le C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\&\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy\\&\quad+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}\\&\quad\times|D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy\\&\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\\&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}}\int_{2^k\tilde{Q}}|f(y)|^r dy\right)^{1/r}\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).\end{aligned}$$

Similarly,

$$|I_5^{(5)}|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(6)}$, taking $q_1, q_2 > 1$ such that $1/r + 1/q_1 + 1/q_2 = 1$, then

$$|I_5^{(6)}| \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left| \frac{(x-y)^{\alpha_1+\alpha_2} K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} K(x_0,y)}{|x_0-y|^m} \right|$$
$$\times |D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f_2(y)|dy$$
$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}} \int_{2^k\tilde{Q}} |f(y)|^r dy \right)^{1/r}$$
$$\times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(y)|^{q_2} dy \right)^{1/q_2}$$
$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

Thus

$$|I_5| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

This completes the proof of Lemma 2.3. ∎

3. Proof of theorem

Proof of Theorem 1.1. Taking $1 < r < \min(p, (n-\lambda)/p\delta)$ in Lemma 2.3, by Lemma 2.2, we obtain

$$\begin{aligned}
||T^A(f)||_{L^{q,\lambda}} &\le C||M(T^A(f))||_{L^{q,\lambda}} \\
&\le C||(T^A(f))^{\#}||_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) ||M_{\delta,r}(f)||_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) ||(M_{r\delta/n}(|f|^r))^{1/r}||_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) ||M_{r\delta/n}(|f|^r)||_{L^{q/r,\lambda}}^{1/r} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) |||f|^r||_{L^{p/r,\lambda}}^{1/r} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) ||f||_{L^{p,\lambda}}.
\end{aligned}$$

This completes the proof of the theorem. ∎

4. Applications

In this section, we shall apply the results of the paper to the Calderón-Zygmund singular integral operator and fractional integral operator.

Application 1. Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [19, 20]), then the kernel K of T satisfies the conditions of Definition 1.1 with $\delta = 0$, and T is bounded on $L^p(R^n)$ for any $1 < p < \infty$. The multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x,y) f(y) dy.$$

Then Theorem 1.1 and Lemma 2.3 hold for T^A with $\delta = 0$.

Application 2. Fractional integral operator with rough kernel.

For $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [1, 8])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y) dy.$$

The multilinear operator related to T_δ is defined by

$$T_\delta^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y) dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ and $\Omega \in Lip_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\varepsilon$. In this time, the kernel satisfies the conditions of Definition 1.1. When $\Omega \equiv 1$, T_δ is the Riesz potentials. Then Theorem 1.1 and Lemma 2.3 hold for T_δ^A.

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