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Using Implicit Relations to Prove Unified Fixed Point Theorems in Metric and 2-Metric Spaces

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Abstract. The purpose of this paper is to prove some common fixed point theorems in metric and 2-metric spaces for two pairs of weakly compatible mappings satisfying an implicit relation which unify and generalize most of the existing relevant fixed point theorems. As an application of our main result, a Bryant type fixed point theorem is also derived besides furnishing two illustrative examples.

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1. Introduction

The concept of 2-metric spaces was initiated and developed (to a considerable extent) by Gähler in a series of papers [8–10] and by now there exists considerable literature on this topic. In this course of development, a number of authors have studied various aspects of metric fixed point theory in the setting of 2-metric spaces which are generally motivated by the corresponding existing concepts already known for ordinary metric spaces. Iséki [17] (also see [18]) appears to be the first mathematician who studied fixed point theorems in the setting of 2-metric spaces. Since then a multitude of results on fixed points have been proved in 2-metric spaces which include Cho *et al.* [4], Imdad *et al.* [15], Murthy *et al.* [28], Naidu and Prasad [31], Pathak *et al.* [34] and the references cited therein. The authors of the articles [3, 15, 28, 31, 34, 38, 39] also utilized the concepts of weakly commuting mappings, compatible mappings, compatible mappings of type (A) and (P) and weakly compatible mappings of type (A) to prove fixed point theorems in 2-metric spaces.

Jungck [21] introduced the notion of weakly compatible mappings in ordinary metric spaces which is proving handy to prove common fixed point theorems with minimal commutativity requirement. In recent years, using this idea several general

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common fixed point theorems have been proved in metric and 2-metric spaces which include Popa [35], Imdad *et al.* [16] and Abu-Donia and Atia [1] and others.

Recently, Popa [35] utilized implicit relations to prove results on common fixed points which are proving fruitful as they cover several definitions in one go. In this paper, we utilize slightly modified form of implicit relation of Popa [35] to prove our results in the setting of 2-metric spaces. Here, for a change, we prove our results in 2-metric spaces and then use them to indicate their analogous in metric spaces which are refined versions of several recent known results (e.g. [5, 7, 12–14, 24, 29]).

2. Preliminaries

Let X be a nonempty set. A real valued function d on X^3 is said to a 2-metric if,

(M_1) to each pair of distinct points x, y in X, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(M_2) $d(x, y, z) = 0$ when at least two of x, y, z are equal,

(M_3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(M_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

The function d is called a 2-metric on the set X whereas the pair (X, d) stands for 2-metric space. Geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y and z.

It has been known since Gähler [8] that a 2-metric d is a non-negative continuous function in any one of its three arguments but it need not be continuous in two arguments. A 2-metric d is said to be continuous if it is continuous in all of its arguments. Throughout this paper d stands for a continuous 2-metric.

Definition 2.1. *A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim x_n = x$, if $\lim d(x_n, x, z) = 0$ for all $z \in X$.*

Definition 2.2. *A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be Cauchy sequence if $\lim d(x_n, x_m, z) = 0$ for all $z \in X$.*

Definition 2.3. *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Remark 2.1. In general a convergent sequence in a 2-metric space (X, d) need not be Cauchy but every convergent sequence is a Cauchy sequence whenever 2-metric d is continuous. A 2-metric d on a set X is said to be weakly continuous if every convergent sequence under d is Cauchy (see [31]).

Definition 2.4. [28] *Let S and T be mappings from a 2-metric space (X, d) into itself. The mappings S and T are said to be compatible if $\lim d(STx_n, TSx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.*

Definition 2.5. [1] *A pair of self mappings S and T of a 2-metric space (X, d) is said to be weakly compatible if $Sx = Tx$ (for some $x \in X$) implies $STx = TSx$.*

Definition 2.6. [28] *Let (S, T) be a pair of self mappings of a 2-metric space (X, d). The mappings S and T are said to be compatible of type (A) if $\lim d(TSx_n, SSx_n, z) = \lim d(STx_n, TTx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.*

Definition 2.7. [34] *Let (S,T) be a pair of self mappings of a 2-metric space (X,d). Then the pair (S,T) is said to be weakly compatible of type (A) if*

$$\lim d(STx_n, TTx_n, z) \leq \lim d(TSx_n, TTx_n, z)$$

and

$$\lim d(TSx_n, SSx_n, z) \leq \lim d(STx_n, SSx_n, z)$$

for all $z \in X$, where $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.

In view of Proposition 2.4 of [34], every pair of compatible mappings of type (A) is weakly compatible mappings of type (A) whereas in view of Proposition 2.9 of [34], every pair of compatible mappings of type (A) is weak compatible.

The purpose of this paper is three fold which can be described as follows.

(i) We slightly modify the implicit relation of Popa [35] so that contraction conditions obtained involving functional inequalities (e.g. Husain and Sehgal [13]) can also be covered.

(ii) Using modified implicit relation and weak compatibility some common fixed point theorems are proved in 2-metric spaces and use them to indicate their metric analogue.

(iii) As an application of our main result, a Bryant [2] type generalized common fixed point theorem has been proved besides deriving related results and furnishing illustrative examples.

3. Implicit relations

Let $\mathcal{F}$ be the set of all continuous functions $F : \Re_+^6 \to \Re$ satisfying the following conditions:

(F_1) F is non-increasing in variables t_5 and t_6.

(F_2) there exists $h \in (0,1)$ such that for $u, v \geq 0$ with

$$(F_a) : F(u, v, v, u, u+v, 0) \leq 0 \quad \text{or} \quad (F_b) : F(u, v, u, v, 0, u+v) \leq 0$$

implies $u \leq h.v$.

(F_3) $F(u, u, 0, 0, u, u) > 0 \ \forall \ u > 0$.

The following examples of such functions F satisfying F_1, F_2 and F_3 are available in [35] with verifications and other details.

Example 3.1. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}, \text{ where } k \in (0,1).$$

Example 3.2. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - t_1(\alpha t_2 + \beta t_3 + \gamma t_4) - \eta t_5 t_6,$$

where $\alpha > 0$; $\beta, \gamma, \eta \geq 0$, $\alpha + \beta + \gamma < 1$ and $\alpha + \eta < 1$.

Example 3.3. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2,$$

where $\alpha > 0$; $\beta, \gamma, \eta \geq 0$, $\alpha + \beta < 1$ and $\alpha + \gamma + \eta < 1$.

Example 3.4. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^3 - \alpha \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1 + t_2 + t_3 + t_4}, \text{ where } \alpha \in (0, 1).$$

Example 3.5. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1 + t_3^2 + t_4^2}, \text{ where } \alpha > 0, \beta \geq 0 \text{ and } \alpha + \beta < 1.$$

Here one may further notice that some other well known contraction conditions (cf. [12, 14, 19]) can also be deduced as particular cases to implicit relation of Popa [35]. In order to strengthen this view point we add some more examples to this effect and utilize them to demonstrate that how this implicit relation can cover several other known contractive conditions and is also good enough to yield further unknown natural contractive conditions as well.

Example 3.6. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - a_1 \dfrac{t_3^2 + t_4^2}{t_3 + t_4} - a_2 t_2 - a_3(t_5 + t_6), \text{ if } t_3 + t_4 \neq 0 \\ t_1, \text{ if } t_3 + t_4 = 0 \end{cases}$$

where $a_i \geq 0$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

F_1: Obvious.
$F_2(F_a)$: Let $u > 0$. $F(u, v, v, u, u+v, 0) = u - a_1(v^2 + u^2)/(v + u) - a_2 v - a_3(u+v) \leq 0$. If $u \geq v$, then $u \leq (a_1 + a_2 + 2a_3)u < u$ which is a contradiction. Hence $u < v$ and $u \leq hv$ where $h \in (0, 1)$.
(F_b): Similar argument as in (F_a).
F_3: $F(u, u, 0, 0, u, u) = u > 0$ for all $u > 0$.

We also add the following two examples without verification.

Example 3.7. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - \alpha t_2 - \dfrac{\beta t_3 t_4 + \gamma t_5 t_6}{t_3 + t_4}, \text{ if } t_3 + t_4 \neq 0 \\ t_1, \text{ if } t_3 + t_4 = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ such that $1 < 2\alpha + \beta < 2$.

Example 3.8. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6 \text{ where } \sum_{i=1}^{5} a_i < 1.$$

Example 3.9. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \alpha \Big[\beta \max\Big\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\Big\} + (1 - \beta)\Big[\max\Big\{t_2^2, t_3 t_4, t_5 t_6, \frac{t_3 t_6}{2}, \frac{t_4 t_5}{2}\Big\}\Big]^{\frac{1}{2}}\Big],$$

where $\alpha \in (0, 1)$ and $0 \leq \beta \leq 1$.

Example 3.10. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\left\{\frac{t_3 t_5}{2}, \frac{t_4 t_6}{2}\right\} - \gamma t_5 t_6$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$.

In what follows, we notice that Husain and Sehgal [13] type contraction conditions (e.g. [7, 24, 30, 31, 37]) can be deduced from similar implicit relations in addition to all earlier ones if we slightly modify (F_1) as follows:
(F_1') F is decreasing in variables $t_2, \ldots, t_6$.
Hereafter, let $F : \Re_+^6 \to \Re$ be a continuous function which satisfy the conditions F_1', F_2 and F_3 and let Ψ be the family of such functions F. We employ such implicit relation to prove our results in this paper. Before we proceed further, let us furnish some examples to highlight the utility of the modifications instrumented herein.

Example 3.11. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)$$

where $\phi : \Re^+ \to \Re^+$ is an increasing upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$.

F_1': Obvious.
$F_2(F_a)$: Let $u > 0$. $F(u, v, v, u, u + v, 0) = u - \phi(\max\{v, v, u, (u + v)/2\}) \leq 0$. If $u \geq v$, then $u \leq \phi(u) < u$ which is a contradiction. Hence $u < v$ and $u \leq hv$ where $h \in (0, 1)$.
(F_b): Similar argument as in (F_a).
F_3: $F(u, u, 0, 0, u, u) = u - \phi(\max\{u, 0, 0, (u + u)/2\}) = u - \phi(u) > 0$ for all $u > 0$.

Example 3.12. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - \phi(t_2, t_3, \ldots, t_6)$$

where $\phi : \Re_+^5 \to \Re$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Example 3.13. Define $F(t_1, t_2, \ldots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \ldots, t_6) = t_1^2 - \phi(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5)$$

where $\phi : \Re_+^5 \to \Re$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq 3$.

Here it may be noticed that all earlier mentioned examples continue to enjoy the format of modified implicit relation as adopted herein.

4. Common fixed point theorems

The following proposition notes that in the following specific setting the common fixed point of the involved four mappings is always unique provided it exists.

Proposition 4.1. *Let (X, d) be a 2-metric space and let $A, B, S, T : X \to X$ be four mappings satisfying the condition*

$$F(d(Ax, By, a), d(Sx, Ty, a), d(Sx, Ax, a), \tag{4.1}$$
$$d(Ty, By, a), d(Sx, By, a), d(Ty, Ax, a)) \leq 0$$

for all $x, y \in X$ and for all $a \in X$, where F enjoys the property (F_3). Then A, B, S and T have at most one common fixed point.

Proof. Let on contrary that A, B, S and T have two common fixed points u and v such that $u \neq v$. Then by (4.1), we have

$$F(d(Au, Bv, a), d(Su, Tv, a), d(Su, Au, a), d(Tv, Bv, a), d(Su, Bv, a), d(Tv, Au, a)) \leq 0$$

or $F(d(u, v, a), d(u, v, a), 0, 0, d(u, v, a), d(u, v, a)) \leq 0$, for all $a \in X$ which contradicts (F_3), yielding thereby $u = v$. ∎

Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying the following condition:

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X). \tag{4.2}$$

Since $A(X) \subseteq T(X)$, for arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for the point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2};\ n = 0, 1, 2, \ldots. \tag{4.3}$$

Lemma 4.1. *If A, B, S and T be mappings from a 2-metric space (X, d) into itself which satisfy conditions (4.1) and (4.2), then*

(a) $d(y_n, y_{n+1}, y_{n+2}) = 0$ *for every $n \in N$;*
(b) $d(y_i, y_j, y_k) = 0$ *for $i, j, k \in N$, where $\{y_n\}$ is a sequence described by (4.3).*

Proof. (a) From (4.1), we have

$$F(d(Ax_{2n+2}, Bx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Tx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Ax_{2n+2}, y_{2n}),$$
$$d(Tx_{2n+1}, Bx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Bx_{2n+1}, y_{2n}), d(Tx_{2n+1}, Ax_{2n+2}, y_{2n})) \leq 0$$

or $$F(d(y_{2n+2}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n+2}, y_{2n}),$$
$$d(y_{2n}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}, y_{2n})) \leq 0$$

or $$F(d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, 0) \leq 0$$

or $$F(d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, d(y_{2n+2}, y_{2n+1}, y_{2n})) \leq 0$$

yielding thereby $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$ (due to F_b). Similarly, using (F_a) we can show that $d(y_{2n+1}, y_{2n}, y_{2n-1}) = 0$. Thus it follows that $d(y_n, y_{n+1}, y_{n+2}) = 0$ for every $n \in N$.

(b) For all $a \in X$, let us write $d_n = d(y_n, y_{n+1}, a),\ n = 0, 1, 2, \ldots$. First we shall prove that $\{d_n\}$ is a non-decreasing sequence in $\Re^+$. From (4.1), we have

$$F(d(Ax_{2n}, Bx_{2n+1}, a), d(Sx_{2n}, Tx_{2n+1}, a), d(Sx_{2n}, Ax_{2n}, a), d(Tx_{2n+1}, Bx_{2n+1}, a), d(Sx_{2n}, Bx_{2n+1}, a), d(Tx_{2n+1}, Ax_{2n}, a)) \leq 0,$$

or $$F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n+1}, a), d(y_{2n}, y_{2n}, a)) \leq 0,$$

or $$F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}, a) + d(y_{2n+1}, y_{2n}, a), 0) \leq 0$$

or $$F(d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a) + d(y_{2n}, y_{2n+1}, a), 0) \leq 0$$

implying thereby $d_{2n} \leq hd_{2n-1} < d_{2n-1}$ (due to (F_a)). Similarly using (F_b), we have $d_{2n+1} \leq hd_{2n}$. Thus $d_{n+1} < d_n$ for $n = 0, 1, 2, \ldots$. Now proceeding on the lines of the proof of Lemma 3.2 [34, p. 355], we can show that $d(y_i, y_j, y_k) = 0$ for $i, j, k \in N$. ∎

Lemma 4.2. *Let $\{y_n\}$ be a sequence in a 2-metric space (X, d) described by (4.3), then $\lim d(y_n, y_{n+1}, a) = 0$ for all $a \in X$.*

Proof. As in Lemma 4.1, we have $d_{2n+1} \leq hd_{2n}$ and $d_{2n} \leq hd_{2n-1}$. Therefore, we obtain $d_n \leq h^n d_0$. Hence $\lim d(y_n, y_{n+1}, a) = \lim d_n = 0$. ∎

Lemma 4.3. *Let A, B, S and T be mappings from a 2-metric space (X, d) into itself satisfying (4.1) and (4.2). Then the sequence $\{y_n\}$ described by (4.3) is a Cauchy sequence.*

Proof. Since $\lim d(y_n, y_{n+1}, a) = 0$ by Lemma 4.2, it is sufficient to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in X. Then for every $\epsilon > 0$ there exists $a \in X$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $k \leq n_k < m_k$ with $d(y_{2n_k-1}, y_{2m_k}, a) \geq \epsilon$ and $d(y_{2n_k}, y_{2m_k-2}, a) < \epsilon$. Now proceeding on the lines of the proof of Lemma 1.3 [4] (or Lemma 3.3 [34]), we obtain $\lim d(y_{2n_k}, y_{2m_k}, a) = \epsilon$, $\lim d(y_{2n_k}, y_{2m_k-1}, a) = \epsilon$, $\lim d(y_{2n_k+1}, y_{2m_k}, a) = \epsilon$ and $\lim d(y_{2n_k+1}, y_{2m_k-1}, a) = \epsilon$. Now using (4.1), we have

$$F(d(Ax_{2m_k}, Bx_{2n_k+1}, a), d(Sx_{2m_k}, Tx_{2n_k+1}, a), d(Sx_{2m_k}, Ax_{2m_k}, a), d(Bx_{2n_k+1}, Tx_{2n_k+1}, a), d(Sx_{2m_k}, Bx_{2n_k+1}, a), d(Tx_{2n_k+1}, Ax_{2m_k}, a) \leq 0$$

or $$F(d(y_{2m_k}, y_{2n_k+1}, a), d(y_{2m_k-1}, y_{2n_k}, a), d(y_{2m_k-1}, y_{2m_k}, a), d(y_{2n_k}, y_{2n_k+1}, a), d(y_{2m_k-1}, y_{2n_k+1}, a), d(y_{2n_k}, y_{2m_k}, a) \leq 0.$$

Letting $n \to \infty$, we have

$$F(\epsilon, \epsilon, 0, 0, \epsilon, \epsilon) \leq 0$$

which is a contradiction to (F_3). Therefore $\{y_{2n}\}$ is a Cauchy sequence. ∎

Theorem 4.1. *Let A, B, S and T be self mappings of a 2-metric space (X, d) satisfying the conditions (4.1) and (4.2). If one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspace of X, then*

(c) *the pair* (A, S) *has a point of coincidence,*
(d) *the pair* (B, T) *has a point of coincidence.*

Moreover, A, S, B *and* T *have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. Let $\{y_n\}$ be the sequence defined by (4.3). By Lemma 4.3, $\{y_n\}$ is a Cauchy sequence in X. Suppose that $S(X)$ is a complete subspace of X, then the subsequence $\{y_{2n+1}\}$ which is contained in $S(X)$ must have a limit z in $S(X)$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore $\{y_n\}$ also converges implying thereby the convergence of the subsequence $\{y_{2n}\}$, i.e. $\lim Ax_{2n} = \lim Bx_{2n+1} = \lim Tx_{2n+1} = \lim Sx_{2n+2} = z$. Let $u \in S^{-1}(z)$, then $Su = z$. If $Au \neq z$, then using (4.1), we have

$$F(d(Au, Bx_{2n-1}, a), d(Su, Tx_{2n-1}, a), d(Su, Au, a), d(Tx_{2n-1}, Bx_{2n-1}, a), d(Su, Bx_{2n-1}, a), d(Tx_{2n-1}, Au, a)) \leq 0$$

which on letting $n \to \infty$, reduces to

$$F(d(Au, z, a), d(z, z, a), d(z, Au, a), d(z, z, a), d(z, z, a), d(z, Au, a)) \leq 0$$

or

$$F(d(Au, z, a), 0, d(z, Au, a), 0, 0, d(z, Au, a)) \leq 0$$

implying thereby $d(z, Au, a) = 0$ for all $a \in X$ (due to F_b). Hence $z = Au = Su$.

Since $A(X) \subseteq T(X)$, there exists $v \in T^{-1}(z)$ such that $Tv = z$. By (4.1), we have

$$F(d(Au, Bv, a), d(Su, Tv, a), d(Su, Au, a), d(Tv, Bv, a), d(Su, Bv, a), d(Tv, Au, a)) \leq 0$$

$$\text{or} \quad F(d(z, Bv, a), 0, 0, d(z, Bv, a), d(z, Bv, a), 0) \leq 0$$

yielding thereby $d(z, Bv, a) = 0$ for all $a \in X$ (due to (F_a)). Therefore $z = Bv$. Hence $Au = Su = Bv = Tv = z$ which establishes (c) and (d).

If one assumes that $T(X)$ is a complete subspace of X, then analogous arguments establish (c) and (d). The remaining two cases also pertain essentially to the previous cases. Indeed, if $A(X)$ is complete, then $z \in A(X) \subseteq T(X)$. Similarly if $B(X)$ is complete, then $z \in B(X) \subseteq S(X)$. Thus in all cases, (c) and (d) are completely established.

Since A and S are weakly compatible and $Au = Su = z$, then $ASu = SAu$ which implies $Az = Sz$. By (4.1), we have

$$F(d(Az, Bv, a), d(Sz, Tv, a), d(Sz, Az, a), d(Tv, Bv, a), d(Sz, Bv, a), d(Tv, Az, a)) \leq 0$$

$$\text{or} \quad F(d(Az, z, a), d(Az, z, a), 0, 0, d(Az, z, a), d(Az, z, a)) \leq 0$$

a contradiction to (F_3) if $d(Az, z, a) > 0$. Hence $z = Az = Sz$.

Since B and T are weakly compatible and $Bv = Tv = z$, then $BTv = TBv$ which implies $Bz = Tz$. Again by (4.1), we have

$$F(d(Az, Bz, a), d(Sz, Tz, a), d(Sz, Az, a), d(Tz, Bz, a), d(Sz, Bz, a), d(Tz, Az, a)) \leq 0$$

$$\text{or} \quad F(d(z, Bz, a), d(z, Bz, a), 0, 0, d(z, Bz, a), d(z, Bz, a)) \leq 0$$

a contradiction to (F_3) if $d(z, Bz, a) > 0$. Hence $z = Bz = Tz$. Therefore, $z = Az = Sz = Bz = Tz$ which shows that z is a common fixed point of the mappings A, B, S and T. In view of Proposition 4.1, z is the unique common fixed point of the mappings A, B, S and T. ∎

Corollary 4.1. *The conclusions of Theorem 4.1 remain true if (for all $x, y, a \in X$) implicit relation* (4.1) *is replaced by any one of the following.*

(a_1)

$$d(Ax, By, a) \leq k \max\Big\{d(Sx, Ty, a), d(Sx, Ax, a), d(Ty, By, a), \frac{1}{2}(d(Sx, By, a) + d(Ty, Ax, a))\Big\}, \text{ where } k \in (0,1).$$

(a_2)

$$d^2(Ax, By, a) \leq d(Ax, By, a)[\alpha d(Sx, Ty, a) + \beta d(Sx, Ax, a) + \gamma d(Ty, By, a)] + \eta d(Sx, By, a).d(Ty, Ax, a)$$

where $\alpha > 0$; $\beta, \gamma, \eta \geq 0$, $\alpha + \beta + \gamma < 1$ *and* $\alpha + \eta < 1$.

(a_3)

$$d^3(Ax, By, a) \leq \alpha d^2(Ax, By, a)d(Sx, Ty, a) + \beta d(Ax, By, a)d(Sx, Ax, a)d(Ty, By, a) + \gamma d^2(Sx, By, a)d(Ty, Ax, a) + \eta d(Sx, By, a)d^2(Ty, Ax, a)$$

where $\alpha > 0$; $\beta, \gamma, \eta \geq 0$, $\alpha + \beta < 1$ *and* $\alpha + \gamma + \eta < 1$.

(a_4)

$$d^3(Ax, By, a) \leq \alpha \frac{d^2(Sx, Ax, a)d^2(Ty, By, a) + d^2(Sx, By, a)d^2(Ty, Ax, a)}{1 + d(Sx, Ty, a) + d(Sx, Ax, a) + d(Ty, By, a)}$$

where $\alpha \in (0, 1)$.

(a_5)

$$d^2(Ax, By, a) \leq \alpha d^2(Sx, Ty, a) + \beta \frac{d(Sx, By, a)d(Ty, Ax, a)}{1 + d^2(Sx, Ax, a) + d^2(Ty, By, a)}$$

where $\alpha > 0, \beta \geq 0$ *and* $\alpha + \beta < 1$.

(a_6)

$$d(Ax, By, a) \leq a_1 \frac{d^2(Sx, Ax, a) + d^2(Ty, By, a)}{d(Sx, Ax, a) + d(Ty, By, a)} + a_2 d(Sx, Ty, a) + a_3(d(Sx, By, a) + d(Ty, Ax, a))$$

where $a_i \geq 0$ *with at least one* a_i *non zero and* $a_1 + a_2 + 2a_3 < 1$.

(a_7)

$$d(Ax,By,a) \leq \alpha d(Sx,Ty,a) + \frac{\beta d(Sx,Ax,a)d(Ty,By,a) + \gamma d(Sx,By,a)d(Ty,Ax,a)}{d(Sx,Ax,a) + d(Ty,By,a)}$$

where $\alpha, \beta, \gamma \geq 0$ *such that* $1 < 2\alpha + \beta < 2$.

(a_8)

$$d(Ax,By,a) \leq a_1 d(Sx,Ty,a) + a_2 d(Sx,Ax,a) + a_3 d(Ty,By,a) + a_4 d(Sx,By,a) + a_5 d(Ty,Ax,a),$$

where $\sum_{i=1}^{5} a_i < 1$.

(a_9)

$$d(Ax,By,a) \leq \alpha\Big[\beta \max\Big\{d(Sx,Ty,a), d(Sx,Ax,a), d(Ty,By,a), \frac{1}{2}(d(Sx,By,a) + d(Ty,Ax,a))\Big\} + (1-\beta)\Big[\max\Big\{d^2(Sx,Ty,a), d(Sx,Ax,a)d(Ty,By,a), d(Sx,By,a)d(Ty,Ax,a), \frac{d(Sx,Ax,a)d(Ty,Ax,a)}{2}, \frac{d(Ty,By,a)(d(Sx,By,a)}{2}\Big\}\Big]^{\frac{1}{2}}\Big]$$

where $\alpha \in (0,1)$ *and* $0 \leq \beta \leq 1$.

(a_{10})

$$d^2(Ax,By,a) \leq \alpha \max\{d^2(Sx,Ty,a), d^2(Sx,Ax,a), d^2(Ty,By,a)\} + \beta \max\left\{\frac{d(Sx,Ax,a)d(Sx,By,a)}{2}, \frac{d(Ty,By,a)d(Ty,Ax,a)}{2}\right\} + \gamma d(Sx,By,a)d(Ty,Ax,a)$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + \gamma < 1$.

(a_{11})

$$d(Ax,By,a) \leq \phi(\max\{d(Sx,Ty,a), d(Sx,Ax,a), d(Ty,By,a), \frac{1}{2}[d(Sx,By,a) + d(Ty,Ax,a)]\})$$

where $\phi : \Re^+ \to \Re^+$ *is an upper semicontinuous and increasing function with* $\phi(0) = 0$ *and* $\phi(t) < t$ *for each* $t > 0$.

(a_{12})

$$d(Ax,By,a) \leq \phi(d(Sx,Ty,a), d(Sx,Ax,a), d(Ty,By,a), d(Sx,By,a), d(Ty,Ax,a))$$

where $\phi : \Re_+^5 \to \Re$ *is an upper semicontinuous and nondecreasing function in each coordinate variable such that* $\phi(t,t,\alpha t,\beta t,\gamma t) < t$ *for each* $t > 0$ *and* $\alpha, \beta, \gamma \geq 0$ *with*

$\alpha + \beta + \gamma \leq 3$.

(a_{13})

$$d^2(Ax, By, a) \leq \phi(d^2(Sx, Ty, a), d(Sx, Ax, a)d(Ty, By, a),$$
$$d(Sx, By, a)d(Ty, Ax, a), d(Sx, Ax, a)d(Ty, Ax, a),$$
$$d(Ty, By, a)d(Sx, By, a))$$

where $\phi : \Re^5_+ \to \Re$ *is an upper semicontinuous and nondecreasing function in each coordinate variable such that* $\phi(t, t, \alpha t, \beta t, \gamma t) < t$ *for each* $t > 0$ *and* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma \leq 3$.

Proof. The proof follows from Theorem 4.1 and Examples 3.1–3.13. ∎

Remark 4.1. The majority of results corresponding to various above contraction conditions present generalized and improved versions of numerous existing results which include Cho [3], Constantin [6], Gajić [11], Imdad *et al.* [15], Iséki *et al.* [18], Khan and Fisher [23], Murthy *et al.* [28], Naidu and Prasad [31], Singh *et al.* [36] and others whereas some of these present 2-metric space version of certain existing results of literature (e.g. Chugh and Kumar [5], Imdad and Ali [14], Jeong and Rhoades [19], Hardy and Rogers [12], Lal *et al.* [26] and others) besides yielding some results which are seeming new to the literature (e.g. $(a_2), (a_3)$, (a_4) and (a_5)).

The following example illustrates Theorem 4.1.

Example 4.1. Let $X = \{a, b, c, d\}$ be a finite subset of $\Re^2$ equipped with natural area function on X^3 where $a = (0, 0)$, $b = (4, 0)$, $c = (8, 0)$ and $d = (0, 1)$. Then clearly (X, d) is a 2-metric space. Define the self mappings A, B, S and T on X as follows.

$$Aa = Ab = Ad = a,\ Ac = b,\ Sa = Sb = a,\ Sc = c,\ Sd = b$$
$$Ba = Bb = Bc = a,\ Bd = b \quad \text{and} \quad Ta = Tb = a,\ Tc = b,\ Td = c.$$

Notice that $A(X) = \{a, b\} \subset \{a, b, c\} = T(X)$ and $B(X) = \{a, b\} \subset \{a, b, c\} = S(X)$. Also $A(X), T(X), B(X)$ and $S(X)$ are complete subspaces of X. The pair (A, S) is weakly compatible but not commuting as $ASc \neq SAc$ whereas the pair (B, T) is commuting and hence weakly compatible. Define $F(t_1, t_2, \ldots, t_6) : \Re^+_6 \to \Re^+$ as

$$F(t_1, t_2, \ldots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}.$$

Then by a routine calculation, one can verify that the condition (4.1) is satisfied with $k = \frac{1}{2}$. Thus all the conditions of Theorem 4.1 are satisfied and $a = (0, 0)$ is a unique common fixed point of A, B, S and T. Here one may notice that both the pairs have two points of coincidence, namely $a = (0, 0)$ and $b = (4, 0)$.

For a mapping $T : (X, d) \to (X, d)$, we denote $F(T) = \{x \in X : x = Tx\}$.

Theorem 4.2. *Let* A, B, S *and* T *be mappings from a 2-metric space* (X, d) *into itself. If inequality* (4.1) *holds for all* $x, y, a \in X$, *then*

$$(F(S) \cap F(T)) \cap F(A) = (F(S) \cap F(T)) \cap F(B).$$

Proof. Let $x \in (F(S) \cap F(T)) \cap F(A)$. Then using (4.1), we have

$F(d(Ax, Bx, a), d(Sx, Tx, a), d(Sx, Ax, a), d(Tx, Bx, a), d(Sx, Bx, a), d(Tx, Ax, a)) \leq 0$

or

$$F(d(x, Bx, a), 0, 0, d(x, Bx, a), d(x, Bx, a), 0) \leq 0$$

yielding thereby $d(x, Bx, a) = 0$, $\forall\ a \in X$ (due to F_a). Hence $x = Bx$. Thus $(F(S) \cap F(T)) \cap F(A) \subset (F(S) \cap F(T)) \cap F(B)$. Similarly, using (F_b) we can show that $(F(S) \cap F(T)) \cap F(B) \subset (F(S) \cap F(T)) \cap F(A)$. ∎

Theorems 4.1 and 4.2 imply the following one.

Theorem 4.3. *Let A, B and $\{T_i\}_{i \in N \cup \{0\}}$ be mappings of a 2-metric space (X, d) into itself such that*

(e) $T_0(X) \subseteq A(X)$ *and* $T_i(X) \subseteq B(X)$,

(f) *the pairs* (T_0, B) *and* $(T_i, A)(i \in N)$ *are weakly compatible,*

(g) *the inequality*

$$F(d(T_0x, T_iy, a), d(Ax, By, a), d(Ax, T_0x, a), d(By, T_iy, a), d(Ax, T_iy, a), d(By, T_0x, a)) \leq 0$$

for each $x, y, a \in X$, $\forall\ i \in N$, where $F \in \Psi$ (or $\mathcal{F}$).

Then A, B and $\{T_i\}_{i \in N \cup \{0\}}$ have a unique common fixed point in X provided one of $A(X), B(X)$ or $T_0(X)$ is a complete subspace of X.

Next, as an application of Theorem 4.1, we prove a Bryant [2] type generalized common fixed point theorem for four finite families of self mappings which runs as follows:

Theorem 4.4. *Let $\{A_1, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_p\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self-mappings on a 2-metric space (X, d) with $A = A_1A_2 \ldots A_m$, $B = B_1B_2 \ldots B_n$, $S = S_1S_2 \ldots S_p$ and $T = T_1T_2 \ldots T_q$ so that A, B, S and T satisfy* (4.1) *and* (4.2). *If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then*

(h) *the pair (A, S) has a point of coincidence,*

(i) *the pair (B, T) has a point of coincidence.*

Moreover, if $A_iA_j = A_jA_i$, $S_kS_l = S_lS_k$, $B_rB_s = B_sB_r$, $T_tT_u = T_uT_t$, $A_iS_k = S_kA_i$ and $B_rT_t = T_tB_r$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1, 2, \ldots, n\}$ and $t, u \in I_4 = \{1, 2, \ldots, q\}$, then (for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$) A_i, B_r, S_k and T_t have a common fixed point.

Proof. The conclusions (h) and (i) are immediate as A, B, S and T satisfy all the conditions of Theorem 4.1. In view of pairwise commutativity of various pairs of the families $\{A, S\}$ and $\{B, T\}$, the weak compatibility of pairs (A, S) and (B, T) are immediate. Thus all the conditions of Theorem 4.1 (for mappings A, B, S and T) are satisfied ensuring the existence of a unique common fixed point, say z. Now, one needs to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} A(A_iz) &= ((A_1A_2 \ldots A_m)A_i)z = (A_1A_2 \ldots A_{m-1})((A_mA_i)z) \\ &= (A_1 \ldots A_{m-1})(A_iA_mz) = (A_1 \ldots A_{m-2})(A_{m-1}A_i(A_mz)) \end{aligned}$$

$$= (A_1 \dots A_{m-2})(A_i A_{m-1}(A_m z))$$
$$= \dots = \dots = \dots = \dots = \dots = \dots = \dots = \dots = \dots$$
$$= A_1 A_i (A_2 A_3 A_4 \dots A_m z) = A_i A_1 (A_2 A_3 \dots A_m z) = A_i (Az) = A_i z.$$

Similarly, one can show that

$$A(S_k z) = S_k(Az) = S_k z, \quad S(S_k z) = S_k(Sz) = S_k z,$$
$$S(A_i z) = A_i(Sz) = A_i z, \quad B(B_r z) = B_r(Bz) = B_r z,$$
$$B(T_t z) = T_t(Bz) = T_t z, \quad T(T_t z) = T_t(Tz) = T_t z,$$

and

$$T(B_r z) = B_r(Tz) = B_r z,$$

which show that (for all i, k, r and t) $A_i z$ and $S_k z$ are other fixed points of the pair (A, S) whereas $B_r z$ and $T_t z$ are other fixed points of the pair (B, T). Now in view of uniqueness of the fixed point of A, B, S and T (for all i, k, r and t), one can write

$$z = A_i z = S_k z = B_r z = T_t z,$$

which shows that z is a common fixed point of A_i, S_k, B_r and T_t for all i, k, r and t. ∎

By setting $A_1 = A_2 = \dots A_m = A$, $B_1 = B_2 \dots B_n = B$, $S_1 = S_2 = \dots = S_p = S$ and $T_1 = T_2 = \dots = T_q = T$, one deduces the following corollary for various iterates of A, B, S and T which can also be viewed as partial generalization of Theorem 4.1.

Corollary 4.2. *Let (A, S) and (B, T) be two commuting pairs of self mappings of a 2-metric space (X, d) such that $A^m(X) \subseteq T^q(X)$ and $B^n(X) \subseteq S^p(X)$ which satisfy*

$$\begin{aligned} &F(d(A^m x, B^n y, a), d(S^p x, T^q y, a), d(S^p x, A^m x, a), d(T^q y, B^n y, a), \\ &d(S^p x, B^n y, a), d(T^q y, A^m x, a)) \leq 0 \end{aligned} \tag{4.4}$$

for all $x, y \in X$ and for all $a \in X$, where $F \in \Psi$ (or $\mathcal{F}$). If one of $A^m(X)$, $T^q(X)$, $B^n(X)$ or $S^p(X)$ is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Remark 4.2. A result similar to Corollary 4.1 involving various iterates of mappings corresponding to Corollary 4.2 can also be derived. Due to repetition, the details are avoided.

Next, we furnish an example which establishes the utility of Corollary 4.2 over Theorem 4.1.

Example 4.2. Consider $X = \{a, b, c, d\}$ is a finite subset of $\Re^2$ with $a = (0, 0)$, $b = (1, 0)$, $c = (2, 0)$ and $d = (0, 1)$ equipped with natural area function on X^3. Define self mappings A, B, S and T on X as follows.

$$Aa = Ab = Ad = a, \ Ac = b \qquad Sa = Sb = a, \ Sc = Sd = b$$
$$Ba = Bb = Bc = a, \ Bd = c \quad \text{and} \quad Ta = Tb = Tc = Td = a.$$

Notice that $A^2(X) = \{a\} = T^1(X)$ and $B^2(X) = \{a\} = S^2(X)$ and the pairs (A, S) and (B, T) are commuting. Define $F(t_1, t_2, \dots, t_6) : \Re_6^+ \to \Re^+$ as

$$F(t_1, t_2, \dots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}.$$

Then it is straightforward to verify that contraction condition (4.4) is satisfied for A^2, B^2, S^2 and T^1 as $d(A^2x, B^2y, z) = d(a, a, z) = 0$ for all $x, y, z \in X$. Thus all the conditions of Corollary 4.2 are satisfied for A^2, B^2, S^2 and T^1 and hence in view of Corollary 4.2, the mappings A, B, S and T have a unique common fixed point.

However, Theorem 4.1 is not applicable in the context of this example, as $A(X) = \{a, b\} \nsubseteq \{a\} = T(X)$ and $B(X) = \{a, c\} \nsubseteq \{a, b\} = S(X)$. Moreover, the contraction condition (4.1) is not satisfied for A, B, S and T. To substantiate this, consider the case when $x = c$ and $y = a$, then one gets

$$1 \leq k \max\{1, 0, 0, 0, 1\} = k$$

which is a contradiction to the fact that $k < 1$. Thus, in all, Corollary 4.2 is genuinely different to Theorem 4.1.

We now indicate how the metric space versions of various earlier obtained results involving functional inequalities can be deduced from metric space version of Theorem 4.1 which remain unaccommodated in earlier general common fixed point theorems contained in Imdad *et al.* [16]. Hereafter, unless otherwise stated, A, B, S and T are self mappings of the metric space (X, d). Generally, it may be pointed out that Theorems 4.1–4.4 remain true if one replaces 2-metric space X with a metric space X retaining rest of the hypotheses. As a sample, we state the metric space version of Theorem 4.1 without proof whose is essentially the same as that of Theorem 2.1 of Imdad *et al.* [16] except using $F \in \Psi$ instead of $F \in \mathcal{F}$.

Theorem 4.5. *Let A, B, S and T be self mappings of a metric space (X, d) with $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ satisfying the condition*

$$(4.5) \qquad F(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0$$

for all $x, y \in X$ where $F \in \Psi$. If one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspace of X, then

(j) *the pair (A, S) has a point of coincidence,*
(k) *the pair (B, T) has a point of coincidence.*

Moreover, A, S, B and T have a unique common fixed point provided the pairs (A, S) and (B, T) are weakly compatible.

Proof. Proof follows on the lines of Imdad *et al.* [16], hence it is omitted. ∎

Remark 4.3. The modified implicit relation enables us to derive the refined and sharpened versions of some fixed point theorems involving functional inequalities contained in Chugh and Kumar [5], Daneš [7], Husain and Sehgal [13], Khan and Imdad [24], Naidu [29], Naidu and Prasad [30], Singh and Meade [37] whereas results due to Gajić [11], Imdad and Ali [14], Jeong and Rhoades [19], Jungck [20], Kang and Kim [22], Mudgal and Vats [27], Pant [32, 33] and some others can also be deduced from Theorem 4.5 which were not covered by Corollary 2.1 of Imdad *et al.* [16].

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