A Family of Integral Operators Preserving Subordination and Superordination

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Abstract. The main purpose of the present paper is to investigate some subordination-preserving and superordination-preserving properties of a certain family of integral operators. Several sandwich-type results associated with this family of integral operators are also derived.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

\begin{equation}
  f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\end{equation}

which are analytic in the open unit disk

\[ \mathbb{U} := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \].

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let

\[ \mathcal{H}[a,n] := \{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \} \].

Let $f, g \in \mathcal{A}$, where $f$ is given by (1.1) and $g$ is defined by

\[ g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \]

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Then the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by

\[
(f \ast g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g \ast f)(z).
\]

For two functions \( f \) and \( g \), analytic in \( \mathbb{U} \), we say that the function \( f \) is subordinate to \( g \) in \( \mathbb{U} \), and write

\[ f(z) \prec g(z), \]

if there exists a Schwarz function \( \omega \), which is analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) (\( z \in \mathbb{U} \)) such that

\[ f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \]

Indeed, it is known that

\[ f(z) \prec g(z) \implies f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}). \]

Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence:

\[ f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}). \]

We recall the general Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) defined by (cf., e.g., [30, p. 121 et seq.])

\[
\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^+; \ s \in \mathbb{C} \text{ when } |z| < 1; \ \text{Re}(s) > 1 \text{ when } |z| = 1),
\]

where, as usual,

\[ \mathbb{Z}_0^+ := \{0, \pm 1, \pm 2, \ldots\}; \ \mathbb{N} := \{1, 2, 3, \ldots\}. \]

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) can be found in the recent investigations by, for example, Choi and Srivastava [9], Ferreira and López [11], Garg et al. [12], Lin and Srivastava [13], Lin et al. [14] and Luo and Srivastava [17].

In 2007, Srivastava and Attiya [29] (see also Răducanu and Srivastava [25], Liu [16] and Prajapat and Goyal [24]) introduced and investigated the linear operator

\[ J_{s, b}(f) : \mathcal{A} \rightarrow \mathcal{A} \]

defined in terms of the Hadamard product (or convolution) by

\[
(1.2) \quad J_{s, b}(f)(z) := G_{s, b}(z) \ast f(z) \quad (z \in \mathbb{U}; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^+; \ s \in \mathbb{C}; \ f \in \mathcal{A}),
\]

where, for convenience,

\[
(1.3) \quad G_{s, b}(z) := (1 + b)^s[\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}).
\]

It is easy to observe from (1.2) and (1.3) that

\[
J_{s, b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + b}{k + b}\right)^s a_k z^k.
\]
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Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [4] (see also Darus and Al-Shaqsi [10]) introduced and investigated the integral operator

\[ J_{\lambda, \mu}^{s, b}(f)(z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + b}{k + b} \right)^s \frac{\lambda! (k + \mu - 2)!}{(\mu - 2)! (k + \lambda - 1)!} a_k z^k \quad (z \in \mathbb{U}), \]

where (and throughout this paper unless otherwise mentioned) the parameters \( s, b, \mu \) and \( \lambda \) are constrained as follows:

\[ s \in \mathbb{C}; \quad b \in \mathbb{C} \setminus \mathbb{Z}^-; \quad \mu > 0 \quad \text{and} \quad \lambda > -1. \]

We note that \( J_{s, b}^{1, 2} \) is the Srivastava-Attiya operator, and \( J_{0, b}^{\lambda, \mu} \) is the well-known Choi-Saigo-Srivastava operator (see [8, 15, 28]).

It is readily verified from (1.4) that

\[ z \left( J_{s, b}^{\lambda+1, \mu} f \right)'(z) = (\lambda + 1)J_{s, b}^{\lambda, \mu} f(z) - \lambda J_{s, b}^{\lambda+1, \mu} f(z), \]

\[ z \left( J_{s+1, b}^{\lambda, \mu} f \right)'(z) = (b + 1)J_{s, b}^{\lambda, \mu} f(z) - bJ_{s+1, b}^{\lambda, \mu} f(z), \]

and

\[ z \left( J_{s, b}^{\lambda, \mu} f \right)'(z) = \mu J_{s, b}^{\lambda, \mu+1} f(z) - (\mu - 1)J_{s, b}^{\lambda, \mu} f(z). \]

In the present paper, we aim at proving some subordination-preserving and superordination-preserving properties associated with the operator \( J_{s, b}^{\lambda, \mu} \). Several sandwich-type results involving this operator are also derived (some recent sandwich-type results in analytic function theory can be found in [1–3, 5–7, 22, 26, 27, 31] and the references cited therein).

2. Preliminary results

To derive our main results, we need the following definitions and lemmas.

**Definition 2.1.** [21] A function \( \mathbb{P}(z, t) \) \((z \in \mathbb{U}; \ t \geq 0)\) is said to be a subordination chain if \( \mathbb{P}(., t) \) is analytic and univalent in \( \mathbb{U} \) for all \( t \geq 0 \), \( \mathbb{P}(z, 0) \) is continuously differentiable on \([0, \infty)\) for all \( z \in \mathbb{U} \) and \( \mathbb{P}(z, t_1) \prec \mathbb{P}(z, t_2) \) for all \( 0 \leq t_1 \leq t_2 \).

**Definition 2.2.** [19] Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( \mathbb{U} - E(f) \), where

\[ E(f) = \left\{ \varepsilon \in \partial \mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty \right\}, \]

and such that \( f'(\varepsilon) \neq 0 \) for \( \varepsilon \in \partial \mathbb{U} - E(f) \). The subclass of \( Q \) for which \( f(0) = a \) \((a \in \mathbb{C})\) is denoted by \( Q(a) \).

**Lemma 2.1.** [23] The function \( \mathbb{P}(z, t) : \mathbb{U} \times [0, \infty) \to \mathbb{C} \) of the form

\[ \mathbb{P}(z, t) = a_1(t)z + a_2(t)z^2 + \cdots \quad (a_1(t) \neq 0; \ t \geq 0), \]

\[ z \left( J_{s, b}^{\lambda+1, \mu} f \right)'(z) = (\lambda + 1)J_{s, b}^{\lambda, \mu} f(z) - \lambda J_{s, b}^{\lambda+1, \mu} f(z), \]

\[ z \left( J_{s+1, b}^{\lambda, \mu} f \right)'(z) = (b + 1)J_{s, b}^{\lambda, \mu} f(z) - bJ_{s+1, b}^{\lambda, \mu} f(z), \]

and

\[ z \left( J_{s, b}^{\lambda, \mu} f \right)'(z) = \mu J_{s, b}^{\lambda, \mu+1} f(z) - (\mu - 1)J_{s, b}^{\lambda, \mu} f(z). \]

\[ z \left( J_{s, b}^{\lambda, \mu} f \right)'(z) = \mu J_{s, b}^{\lambda, \mu+1} f(z) - (\mu - 1)J_{s, b}^{\lambda, \mu} f(z). \]
and \( \lim_{t \to \infty} |a_1(t)| = \infty \) is a subordination chain if and only if
\[
\Re \left( \frac{z \partial P}{\partial z} \right) > 0 \quad (z \in U; \ t \geq 0).
\]

**Lemma 2.2.** [18] Suppose that the function \( H : \mathbb{C}^2 \to \mathbb{C} \) satisfies the condition
\[
\Re(H(is, t)) \leq 0
\]
for all real \( s \) and for all
\[
t \leq -\frac{n(1 + s^2)}{2} \quad (n \in \mathbb{N}).
\]
If the function
\[
p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots
\]
is analytic in \( U \) and
\[
\Re \left( H(p(z), zp'(z)) \right) > 0 \quad (z \in U),
\]
then
\[
\Re(p(z)) > 0 \quad (z \in U).
\]

**Lemma 2.3.** [19] Let \( \kappa, \gamma \in \mathbb{C} \) with \( \kappa \neq 0 \) and let \( h \in \mathcal{H}(U) \) with \( h(0) = c \). If
\[
\Re(\kappa h(z) + \gamma) > 0 \quad (z \in U),
\]
then the solution of the following differential equation:
\[
q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; \ q(0) = c)
\]
is analytic in \( U \) and satisfies the inequality given by
\[
\Re(\kappa q(z) + \gamma) > 0 \quad (z \in U).
\]

**Lemma 2.4.** [20] Let \( p \in Q(a) \) and
\[
q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (q \neq a; \ n \in \mathbb{N})
\]
be analytic in \( U \). If \( q \) is not subordinate to \( p \), then there exists two points
\[
z_0 = r_0 e^{i\theta} \in U \quad \text{and} \quad \xi_0 \in \partial U \setminus E(f)
\]
such that
\[
q(U_{r_0}) \subset p(U), \quad q(z_0) = p(\xi_0) \quad \text{and} \quad z_0 q'(z_0) = m \xi_0 p'(\xi_0) \quad (m \geq n).
\]

**Lemma 2.5.** [21] Let \( q \in \mathcal{H}[a, 1] \) and \( \phi : \mathbb{C}^2 \to \mathbb{C} \). Also set
\[
\phi(q(z), zq'(z)) \equiv h(z) \quad (z \in U).
\]
Let
\[
\mathbb{P}(z, t) := \phi(q(z), tzq'(z))
\]
be a subordination chain and \( p \in \mathcal{H}[a, 1] \cap Q(a) \). Then
\[
h(z) \prec \phi(p(z), zp'(z))
\]
implies that
\[
q(z) \prec p(z).
\]
Furthermore, if \( \phi(q(z), zq'(z)) = h(z) \) has a univalent solution \( q \in Q(a) \), then \( q \) is the best subordinant.
3. Main results

We begin by proving our first subordination property given by Theorem 3.1 below.

**Theorem 3.1.** Let \( f, g \in A \) and \( \mu > 0 \). Further let

\[
\text{Re} \left( 1 + \frac{z \varphi''(z)}{\varphi'(z)} \right) > -\varrho \quad \left( z \in \mathbb{U}; \ \varphi(z) := \frac{J_{s, b}^{\lambda, \mu} g(z)}{z} \right),
\]

where

\[
\varrho := \frac{1 + \mu^2 - |1 - \mu^2|}{4\mu}.
\]

Then the subordination

\[
\frac{J_{s, b}^{\lambda, \mu+1} f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu+1} g(z)}{z}
\]

implies that

\[
\frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu} g(z)}{z}.
\]

Furthermore, the function \((J_{s, b}^{\lambda, \mu} g(z))/z\) is the best dominant.

**Proof.** Let the functions \( F, G \) and \( Q \) be defined by

\[
F := \frac{J_{s, b}^{\lambda, \mu} f(z)}{z}, \quad G := \frac{J_{s, b}^{\lambda, \mu} g(z)}{z} \quad \text{and} \quad Q := 1 + \frac{z \varphi''(z)}{\varphi'(z)}.
\]

We assume here, without loss of generality, that \( G \) is analytic and univalent on \( \mathbb{U} \) and

\[ G'(|\zeta|) \neq 0 \quad (||\zeta|| = 1). \]

If not, then we replace \( F \) and \( G \) by \( F(\rho z) \) and \( G(\rho z) \), respectively, with \( 0 < \rho < 1 \). These new functions have the desired properties on \( \mathbb{U} \), and we can use them in the proof of our result. Therefore, the result would follow by letting \( \rho \to 1 \).

We first show that

\[ \text{Re}(Q(z)) > 0 \quad (z \in \mathbb{U}). \]

By virtue of (1.7) and the definitions of \( G \) and \( \varphi \), we know that

\[
\varphi(z) = G(z) + \frac{1}{\mu} z G'(z).
\]

Differentiating both sides of (3.4) with respect to \( z \) yields

\[
\varphi'(z) = \left( 1 + \frac{1}{\mu} \right) G'(z) + \frac{1}{\mu} z G''(z).
\]

Combining (3.3) and (3.5), we easily get

\[
1 + \frac{z \varphi''(z)}{\varphi'(z)} = Q(z) + \frac{z Q'(z)}{Q(z) + \mu} := h(z) \quad (z \in \mathbb{U}).
\]

It follows from (3.1) and (3.6) that

\[
\text{Re} \left( h(z) + \mu \right) > 0 \quad (z \in \mathbb{U}).
\]
Moreover, by Lemma 2.3, we conclude that the differential equation (3.6) has a solution \( Q \in \mathcal{H}(U) \) with \( h(0) = Q(0) = 1 \).

Let
\[
H(u, v) := u + \frac{v}{u + \mu} + \varrho,
\]
where \( \varrho \) is given by (3.2). From (3.6) and (3.7), we obtain
\[
\text{Re} \left( H(Q(z), zQ'(z)) \right) > 0 \quad (z \in U).
\]

To verify the condition that
\[
(3.8) \quad \text{Re}(H(\mu, t)) \leq 0 \quad (s \in \mathbb{R}; \ t \leq -\frac{1 + s^2}{2}),
\]
we proceed as follows:
\[
\text{Re}(H(is, t)) = \text{Re} \left( is + \frac{t}{is + \mu} + \varrho \right) = \frac{\mu t}{|\mu + is|^2} + \varrho \leq -\frac{\Psi(\mu, s)}{2|\mu + is|^2},
\]
where
\[
(3.9) \quad \Psi(\mu, s) := (\mu - 2\varrho)s^2 - 4\varrho s - 2\mu s^2 + \mu.
\]
For \( \varrho \) given by (3.2), we note that the coefficient of \( s^2 \) in the quadratic expression \( \Psi(\mu, s) \) given by (3.9) is positive or equal to zero. Furthermore, we observe that the quadratic expression \( \Psi(\mu, s) \) by \( s \) in (3.9) is a perfect square, which implies that (3.8) holds. Thus, by Lemma 2.2, we conclude that
\[
\text{Re}(Q(z)) > 0 \quad (z \in U).
\]

By the definition of \( Q \), we know that \( \mathcal{G} \) is convex. To prove \( \mathcal{F} \prec \mathcal{G} \), let the function \( \mathbb{P} \) be defined by
\[
(3.10) \quad \mathbb{P}(z, t) := \mathcal{G}(z) + \left( 1 + \frac{1 + t}{\mu} \right) z \mathcal{G}'(z) \quad (z \in U; \ 0 \leq t < \infty).
\]

Since \( \mathcal{G} \) is convex and \( \mu > 0 \), then
\[
\frac{\partial \mathbb{P}(z, t)}{\partial z} \bigg|_{z=0} = \mathcal{G}'(0) \left( 1 + \frac{1 + t}{\mu} \right) \neq 0 \quad (z \in U; \ 0 \leq t < \infty)
\]
and
\[
\text{Re} \left( \frac{z \partial \mathbb{P}(z, t)/\partial z}{\partial \mathbb{P}(z, t)/\partial t} \right) = \text{Re} \left( \mu + (1 + t)Q(z) \right) > 0 \quad (z \in \mathbb{U}).
\]

Therefore, by Lemma 2.1, we deduce that \( \mathbb{P} \) is a subordination chain. It follows from the definition of subordination chain that
\[
\varphi(z) = \mathcal{G}(z) + \frac{1}{\mu} z \mathcal{G}'(z) = \mathbb{P}(z, 0),
\]
and
\[
\mathbb{P}(z, 0) \prec \mathbb{P}(z, t) \quad (0 \leq t < \infty),
\]
which implies that
\[
(3.11) \quad \mathbb{P}(\zeta, t) \notin \mathbb{P}(\mathbb{U}, 0) = \varphi(\mathbb{U}) \quad (\zeta \in \partial \mathbb{U}; \ 0 \leq t < \infty).
\]
If $F$ is not subordinate to $G$, by Lemma 2.4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that
\begin{equation}
F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1 + t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).
\end{equation}

Hence, by virtue of (1.7) and (3.12), we have
\begin{equation}
P(\zeta_0, t) = G(\zeta_0) + \frac{1 + t}{\mu} \zeta_0 G'(\zeta_0) = F(z_0) + \frac{1}{\mu} z_0 F'(z_0) = \frac{J_{s, b}^{\lambda, \mu + 1} f(z_0)}{z_0} \in \varphi(U).
\end{equation}

This contradicts to (3.11). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 3.1.

By similarly applying the method of proof of Theorem 3.1 as well as (1.5) and (1.6), we easily get the following results.

**Corollary 3.1.** Let $f, g \in A$ and $\lambda > -1$. Further let
\begin{equation}
\text{Re} \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\varpi \quad \left( z \in U; \; \chi(z) := \frac{J_{s, b}^{\lambda, \mu} g(z)}{z} \right),
\end{equation}
where
\begin{equation}
\varpi := \frac{1 + (\lambda + 1)^2 - |1 - (\lambda + 1)^2|}{4(\lambda + 1)}.
\end{equation}

Then the subordination
\begin{equation}
\frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu} g(z)}{z}
\end{equation}
implies that
\begin{equation}
\frac{J_{s+1, b}^{\lambda+1, \mu} f(z)}{z} \prec \frac{J_{s+1, b}^{\lambda+1, \mu} g(z)}{z}.
\end{equation}

Furthermore, the function $(J_{s, b}^{\lambda+1, \mu} g(z))/z$ is the best dominant.

**Corollary 3.2.** Let $f, g \in A$ and $b \in \mathbb{R} \setminus \mathbb{Z}_0$ with $b > -1$. Further let
\begin{equation}
\text{Re} \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\vartheta \quad \left( z \in U; \; \chi(z) := \frac{J_{s, b}^{\lambda, \mu} g(z)}{z} \right),
\end{equation}
where
\begin{equation}
\vartheta := \frac{1 + (b + 1)^2 - |1 - (b + 1)^2|}{4(b + 1)}.
\end{equation}

Then the subordination
\begin{equation}
\frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu} g(z)}{z}
\end{equation}
implies that
\begin{equation}
\frac{J_{s+1, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s+1, b}^{\lambda, \mu} g(z)}{z}.
\end{equation}

Furthermore, the function $(J_{s+1, b}^{\lambda, \mu} g(z))/z$ is the best dominant.
If \( f \) is subordinate to \( F \), then \( F \) is superordinate to \( f \). We now derive the following superordination result.

**Theorem 3.2.** Let \( f, g \in A_p \) and \( \mu > 0 \). Further let
\[
\text{Re} \left( 1 + \frac{z \varphi''(z)}{\varphi'(z)} \right) > -\varrho \quad (z \in \mathbb{U}; \, \varphi(z) := \frac{J_{s, \lambda, \mu} f(z)}{z}),
\]
where \( \varrho \) is given by (3.2). If the function \( (J_{s, \lambda, \mu} f(z))/z \) is univalent in \( \mathbb{U} \) and \( (J_{s, \lambda, \mu} f(z))/z \in Q \), then the subordination
\[
\frac{J_{s, \lambda, \mu} g(z)}{z} \prec \frac{J_{s, \lambda, \mu} f(z)}{z}
\]
implies that
\[
\frac{J_{s, \lambda, \mu} g(z)}{z} \prec \frac{J_{s, \lambda, \mu} f(z)}{z}.
\]
Furthermore, the function \( (J_{s, \lambda, \mu} g(z))/z \) is the best subordinant.

**Proof.** Suppose that the functions \( F \) and \( G \) and \( Q \) are defined by (3.3). By applying the similar method as in the proof of Theorem 3.1, we get
\[
\text{Re}(Q(z)) > 0 \quad (z \in \mathbb{U}).
\]
Next, to arrive at our desired result, we show that \( G \prec F \). For this, we suppose that the function \( P \) be defined by (3.10). Since \( \mu > 0 \) and \( G \) is convex, by applying a similar method as in Theorem 3.1, we deduce that \( P \) is subordination chain. Therefore, by Lemma 2.5, we conclude that \( G \prec F \). Moreover, since the differential equation
\[
\varphi(z) = G(z) + \frac{1}{\mu} z G'(z) := \phi(G(z), z G'(z))
\]
has a univalent solution \( G \), it is the best subordinant. This completes the proof of Theorem 3.2.

Applying a similar proof as in Theorem 3.2 and using (1.5) and (1.6), the following results are easily obtained.

**Corollary 3.3.** Let \( f, g \in A \) and \( \lambda > -1 \). Further let
\[
\text{Re} \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\varpi \quad (z \in \mathbb{U}; \, \chi(z) := \frac{J_{s, \lambda, \mu} g(z)}{z}),
\]
where \( \varpi \) is given by (3.13). If the function \( (J_{s, \lambda, \mu} f(z))/z \) is univalent in \( \mathbb{U} \) and \( (J_{s, \lambda, \mu} f(z))/z \in Q \), then the subordination
\[
\frac{J_{s, \lambda, \mu} g(z)}{z} \prec \frac{J_{s, \lambda, \mu} f(z)}{z}
\]
implies that
\[
\frac{J_{s, \lambda, \mu} g(z)}{z} \prec \frac{J_{s, \lambda, \mu} f(z)}{z}.
\]
Furthermore, the function \( (J_{s, \lambda, \mu} g(z))/z \) is the best subordinant.
Corollary 3.4. Let \( f, g \in \mathcal{A} \) and \( b \in \mathbb{R} \setminus \mathbb{Z}_0 \) with \( b > -1 \). Further let

\[
\text{Re} \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\vartheta \quad \left( z \in \mathbb{U}; \chi(z) := \frac{\mathcal{J}^\lambda_{s, b} g(z)}{z} \right),
\]

where \( \vartheta \) is given by (3.14). If the function \((\mathcal{J}^\lambda_{s, b} f)/z\) is univalent in \( \mathbb{U} \) and \((\mathcal{J}^{\lambda+1}_{s+1, b} f)/z \in Q\), then the subordination

\[
\frac{\mathcal{J}^\lambda_{s, b} g(z)}{z} < \frac{\mathcal{J}^\lambda_{s, b} f(z)}{z}
\]

implies that

\[
\frac{\mathcal{J}^{\lambda+1}_{s+1, b} g(z)}{z} < \frac{\mathcal{J}^{\lambda+1}_{s+1, b} f(z)}{z}.
\]

Furthermore, the function \((\mathcal{J}^{\lambda, \mu}_{s, b} g(z))/z\) is the best subordinant.

Corollary 3.5. Let \( f, g_k \in \mathcal{A} \) \((k = 1, 2)\) and \( \mu > 0 \). Further let

\[
\text{Re} \left( 1 + \frac{z \varphi''_k(z)}{\varphi'_k(z)} \right) > -\varpi \quad \left( z \in \mathbb{U}; \varphi_k(z) := \frac{\mathcal{J}^\lambda_{s, b} g_k(z)}{z} \right) \quad (k = 1, 2),
\]

where \( \varpi \) is given by (3.2). If the function \((\mathcal{J}^{\lambda, \mu+1}_{s, b} f)/z\) is univalent in \( \mathbb{U} \) and \((\mathcal{J}^{\lambda, \mu+1}_{s, b} f)/z \in Q\), then the subordination chain

\[
\frac{\mathcal{J}^{\lambda, \mu+1}_{s, b} g_1(z)}{z} < \frac{\mathcal{J}^{\lambda, \mu+1}_{s, b} f(z)}{z} < \frac{\mathcal{J}^{\lambda, \mu+1}_{s, b} g_2(z)}{z}
\]

implies that

\[
\frac{\mathcal{J}^{\lambda}_{s, b} g_1(z)}{z} < \frac{\mathcal{J}^{\lambda}_{s, b} f(z)}{z} < \frac{\mathcal{J}^{\lambda}_{s, b} g_2(z)}{z}.
\]

Furthermore, the functions \((\mathcal{J}^{\lambda, \mu}_{s, b} g_1)/z\) and \((\mathcal{J}^{\lambda, \mu}_{s, b} g_2)/z\) are, respectively, the best subordinant and the best dominant.

Corollary 3.6. Let \( f, g_k \in \mathcal{A} \) \((k = 1, 2)\) and \( \lambda > -1 \). Further let

\[
\text{Re} \left( 1 + \frac{z \chi''_k(z)}{\chi'_k(z)} \right) > -\varpi \quad \left( z \in \mathbb{U}; \chi_k(z) := \frac{\mathcal{J}^{\lambda}_{s, b} g_k(z)}{z} \right) \quad (k = 1, 2),
\]

where \( \varpi \) is given by (3.13). If the function \((\mathcal{J}^{\lambda, \mu}_{s, b} f)/z\) is univalent in \( \mathbb{U} \) and \((\mathcal{J}^{\lambda+1, \mu+1}_{s, b} f)/z \in Q\), then the subordination chain

\[
\frac{\mathcal{J}^{\lambda+1}_{s, b} g_1(z)}{z} < \frac{\mathcal{J}^{\lambda+1}_{s, b} f(z)}{z} < \frac{\mathcal{J}^{\lambda+1}_{s, b} g_2(z)}{z}
\]

implies that

\[
\frac{\mathcal{J}^{\lambda+1}_{s, b} g_1(z)}{z} < \frac{\mathcal{J}^{\lambda+1}_{s, b} f(z)}{z} < \frac{\mathcal{J}^{\lambda+1}_{s, b} g_2(z)}{z}.
\]
Furthermore, the functions \((\mathcal{J}_{s, b}^\lambda \mu g_1)/z\) and \((\mathcal{J}_{s, b}^\lambda \mu g_2)/z\) are, respectively, the best subordinant and the best dominant.

**Corollary 3.7.** Let \(f, g_k \in A\) \((k = 1, 2)\) and \(b \in \mathbb{R} \setminus \mathbb{Z}_0^-\) with \(b > -1\). Further let

\[
\text{Re} \left(1 + \frac{z \chi_k'(z)}{\chi_k(z)}\right) > -\vartheta \quad (z \in \mathbb{U}; \chi_k(z) := \frac{\mathcal{J}_{s, b}^\lambda \mu g_k(z)}{z} \quad (k = 1, 2),
\]

where \(\vartheta\) is given by (3.14). If the function \((\mathcal{J}_{s, b}^\lambda \mu f)/z\) is univalent in \(\mathbb{U}\) and \((\mathcal{J}_{s+1, b}^\lambda \mu f)/z \in Q\), then the subordination chain

\[
\frac{\mathcal{J}_{s, b}^\lambda \mu g_1(z)}{z} \prec \frac{\mathcal{J}_{s, b}^\lambda \mu f(z)}{z} \prec \frac{\mathcal{J}_{s, b}^\lambda \mu g_2(z)}{z}
\]

implies that

\[
\frac{\mathcal{J}_{s+1, b}^\lambda \mu g_1(z)}{z} \prec \frac{\mathcal{J}_{s+1, b}^\lambda \mu f(z)}{z} \prec \frac{\mathcal{J}_{s+1, b}^\lambda \mu g_2(z)}{z}.
\]

Furthermore, the functions \((\mathcal{J}_{s+1, b}^\lambda \mu g_1)/z\) and \((\mathcal{J}_{s+1, b}^\lambda \mu g_2)/z\) are, respectively, the best subordinant and the best dominant.

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**References**


