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A Family of Integral Operators Preserving Subordination and Superordination

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Abstract. The main purpose of the present paper is to investigate some subordination-preserving and superordination-preserving properties of a certain family of integral operators. Several sandwich-type results associated with this family of integral operators are also derived.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a,n] := \{\, \mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \ \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

Let $f, \ g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

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Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g*f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We recall the general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [30, p. 121 *et seq.*])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \ \text{ when } \ |z| < 1;\ \operatorname{Re}(s) > 1 \ \text{ when } \ |z| = 1),$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \qquad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\};\ \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by, for example, Choi and Srivastava [9], Ferreira and López [11], Garg *et al.* [12], Lin and Srivastava [13], Lin *et al.* [14] and Luo and Srivastava [17].

In 2007, Srivastava and Attiya [29] (see also Răducanu and Srivastava [25], Liu [16] and Prajapat and Goyal [24]) introduced and investigated the linear operator

$$\mathcal{J}_{s,\, b}(f) :\ \mathcal{A} \longrightarrow \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{s,\, b}(f)(z) := G_{s,\, b}(z) * f(z) \qquad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ f \in \mathcal{A}), \tag{1.2}$$

where, for convenience,

$$G_{s,\, b}(z) := (1+b)^s [\Phi(z, s, b) - b^{-s}] \qquad (z \in \mathbb{U}). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$\mathcal{J}_{s,\, b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^s a_k z^k.$$

Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [4] (see also Darus and Al-Shaqsi [10]) introduced and investigated the integral operator

$$\mathcal{J}_{s,\ b}^{\lambda,\ \mu}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{s} \frac{\lambda!(k+\mu-2)!}{(\mu-2)!(k+\lambda-1)!}\, a_k z^k \qquad (z \in \mathbb{U}), \tag{1.4}$$

where (and throughout this paper unless otherwise mentioned) the parameters $s,\ b,\ \mu$ and λ are constrained as follows:

$$s \in \mathbb{C};\quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\quad \mu > 0 \text{ and } \lambda > -1.$$

We note that $\mathcal{J}_{s,\ b}^{1,\ 2}$ is the Srivastava-Attiya operator, and $\mathcal{J}_{0,\ b}^{\lambda,\ \mu}$ is the well-known Choi-Saigo-Srivastava operator (see [8, 15, 28]).

It is readily verified from (1.4) that

$$z\left(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f\right)'(z) = (\lambda+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - \lambda \mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z), \tag{1.5}$$

$$z\left(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f\right)'(z) = (b+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - b \mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z), \tag{1.6}$$

and

$$z\left(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f\right)'(z) = \mu \mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z) - (\mu-1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z). \tag{1.7}$$

In the present paper, we aim at proving some subordination-preserving and superordination-preserving properties associated with the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$. Several sandwich-type results involving this operator are also derived (some recent sandwich-type results in analytic function theory can be found in [1–3, 5–7, 22, 26, 27, 31] and the references cited therein).

2. Preliminary results

To derive our main results, we need the following definitions and lemmas.

Definition 2.1. [21] *A function $\mathbb{P}(z,t)$ $(z \in \mathbb{U};\ t \geq 0)$ is said to be a subordination chain if $\mathbb{P}(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $\mathbb{P}(z,0)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $\mathbb{P}(z,t_1) \prec \mathbb{P}(z,t_2)$ for all $0 \leq t_1 \leq t_2$.*

Definition 2.2. [19] *Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where*

$$E(f) = \left\{\varepsilon \in \partial\mathbb{U} :\ \lim_{z\to\varepsilon} f(z) = \infty\right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$. The subclass of Q for which $f(0) = a$ $(a \in \mathbb{C})$ is denoted by $Q(a)$.

Lemma 2.1. [23] *The function $\mathbb{P}(z,t) :\ \mathbb{U} \times [0,\infty) \to \mathbb{C}$ of the form*

$$\mathbb{P}(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \qquad (a_1(t) \neq 0;\ t \geq 0),$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\mathrm{Re}\left(\frac{z\,\partial\mathbb{P}/\partial z}{\partial\mathbb{P}/\partial t}\right) > 0 \qquad (z \in \mathbb{U};\ t \geq 0).$$

Lemma 2.2. [18] *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the condition*

$$\mathrm{Re}(H(is,t)) \leq 0$$

for all real s *and for all*

$$t \leq -\frac{n\left(1+s^2\right)}{2} \qquad (n \in \mathbb{N}).$$

If the function

$$p\,(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ *and*

$$\mathrm{Re}\,(H(p\,(z), zp'(z))) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\mathrm{Re}(p\,(z)) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 2.3. [19] *Let* $\kappa,\ \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$*. If*

$$\mathrm{Re}(\kappa h(z) + \gamma) > 0 \qquad (z \in \mathbb{U}),$$

then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \qquad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ *and satisfies the inequality given by*

$$\mathrm{Re}(\kappa q(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 2.4. [20] *Let* $p \in Q(a)$ *and*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (q \neq a;\ n \in \mathbb{N})$$

be analytic in $\mathbb{U}$*. If* q *is not subordinate to* p*, then there exists two points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \text{and} \quad \xi_0 \in \partial\mathbb{U}\backslash E(f)$$

such that

$$q(\mathbb{U}_{r_0}) \subset p\,(\mathbb{U}), \quad q(z_0) = p\,(\xi_0) \quad \text{and} \quad z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geq n).$$

Lemma 2.5. [21] *Let* $q \in \mathcal{H}[a,1]$ *and* $\phi : \mathbb{C}^2 \to \mathbb{C}$*. Also set*

$$\phi\,(q(z), zq'(z)) \equiv h(z) \qquad (z \in \mathbb{U}).$$

Let

$$\mathbb{P}(z,t) := \phi\,(q(z), tzq'(z))$$

be a subordination chain and $p \in \mathcal{H}[a,1] \cap Q(a)$*. Then*

$$h(z) \prec \phi\,(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\phi\,(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in Q(a)$*, then* q *is the best subordinant.*

3. Main results

We begin by proving our first subordination property given by Theorem 3.1 below.

Theorem 3.1. *Let $f,\ g \in \mathcal{A}$ and $\mu > 0$. Further let*

$$\mathrm{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho \qquad \left(z \in \mathbb{U};\ \varphi(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}g(z)}{z}\right), \tag{3.1}$$

where

$$\varrho := \frac{1+\mu^2-|1-\mu^2|}{4\mu}. \tag{3.2}$$

Then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}g(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z))/z$ is the best dominant.

Proof. Let the functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{Q}$ be defined by

$$\mathcal{F} := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}, \quad \mathcal{G} := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z} \quad \text{and} \quad \mathcal{Q} := 1+\frac{z\mathcal{G}''(z)}{\mathcal{G}'(z)}. \tag{3.3}$$

We assume here, without loss of generality, that $\mathcal{G}$ is analytic and univalent on $\overline{\mathbb{U}}$ and

$$\mathcal{G}'(\zeta) \neq 0 \qquad (|\zeta| = 1).$$

If not, then we replace $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}(\rho z)$ and $\mathcal{G}(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, and we can use them in the proof of our result. Therefore, the result would follow by letting $\rho \to 1$.

We first show that

$$\mathrm{Re}(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

By virtue of (1.7) and the definitions of $\mathcal{G}$ and φ, we know that

$$\varphi(z) = \mathcal{G}(z) + \frac{1}{\mu}\, z\mathcal{G}'(z). \tag{3.4}$$

Differentiating both sides of (3.4) with respect to z yields

$$\varphi'(z) = \left(1+\frac{1}{\mu}\right)\mathcal{G}'(z) + \frac{1}{\mu} z\mathcal{G}''(z). \tag{3.5}$$

Combining (3.3) and (3.5), we easily get

$$1+\frac{z\varphi''(z)}{\varphi'(z)} = \mathcal{Q}(z) + \frac{z\mathcal{Q}'(z)}{\mathcal{Q}(z)+\mu} := \mathbb{h}(z) \qquad (z \in \mathbb{U}). \tag{3.6}$$

It follows from (3.1) and (3.6) that

$$\mathrm{Re}\,(\mathbb{h}(z)+\mu) > 0 \qquad (z \in \mathbb{U}). \tag{3.7}$$

Moreover, by Lemma 2.3, we conclude that the differential equation (3.6) has a solution $\mathcal{Q} \in \mathcal{H}(\mathbb{U})$ with $\mathbb{h}(0) = \mathcal{Q}(0) = 1$.

Let

$$H(u,v) := u + \frac{v}{u+\mu} + \varrho,$$

where ϱ is given by (3.2). From (3.6) and (3.7), we obtain

$$\mathrm{Re}\left(H(\mathcal{Q}(z), z\mathcal{Q}'(z))\right) > 0 \qquad (z \in \mathbb{U}).$$

To verify the condition that

$$\mathrm{Re}(H(is,t)) \leq 0 \qquad \left(s \in \mathbb{R};\ t \leq -\frac{1+s^2}{2}\right), \tag{3.8}$$

we proceed it as follows:

$$\mathrm{Re}(H(is,t)) = \mathrm{Re}\left(is + \frac{t}{is+\mu} + \varrho\right) = \frac{t\mu}{|\mu + is|^2} + \varrho \leq -\frac{\Psi(\mu,s)}{2\,|\mu+is|^2},$$

where

$$\Psi(\mu,s) := (\mu - 2\varrho)s^2 - 4\varrho\mu s - 2\varrho\mu^2 + \mu. \tag{3.9}$$

For ϱ given by (3.2), we note that the coefficient of s^2 in the quadratic expression $\Psi(\mu,s)$ given by (3.9) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\Psi(\mu,s)$ by s in (3.9) is a perfect square, which implies that (3.8) holds. Thus, by Lemma 2.2, we conclude that

$$\mathrm{Re}(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

By the definition of $\mathcal{Q}$, we know that $\mathcal{G}$ is convex. To prove $\mathcal{F} \prec \mathcal{G}$, let the function $\mathbb{P}$ be defined by

$$\mathbb{P}(z,t) := \mathcal{G}(z) + \left(\frac{1+t}{\mu}\right) z\mathcal{G}'(z) \qquad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{3.10}$$

Since $\mathcal{G}$ is convex and $\mu > 0$, then

$$\frac{\partial \mathbb{P}(z,t)}{\partial z}\Big|_{z=0} = \mathcal{G}'(0)\left(1 + \frac{1+t}{\mu}\right) \neq 0 \qquad (z \in \mathbb{U};\ 0 \leq t < \infty)$$

and

$$\mathrm{Re}\left(\frac{z\,\partial\mathbb{P}(z,t)/\partial z}{\partial\mathbb{P}(z,t)/\partial t}\right) = \mathrm{Re}\left(\mu + (1+t)\mathcal{Q}(z)\right) > 0 \qquad (z \in \mathbb{U}).$$

Therefore, by Lemma 2.1, we deduce that $\mathbb{P}$ is a subordination chain. It follows from the definition of subordination chain that

$$\varphi(z) = \mathcal{G}(z) + \frac{1}{\mu} z\mathcal{G}'(z) = \mathbb{P}(z,0),$$

and

$$\mathbb{P}(z,0) \prec \mathbb{P}(z,t) \qquad (0 \leq t < \infty),$$

which implies that

$$\mathbb{P}(\zeta,t) \notin \mathbb{P}(\mathbb{U},0) = \varphi(\mathbb{U}) \qquad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{3.11}$$

If $\mathcal{F}$ is not subordinate to $\mathcal{G}$, by Lemma 2.4, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\mathcal{F}(z_0) = \mathcal{G}(\zeta_0) \quad \text{and} \quad z_0\mathcal{F}'(z_0) = (1+t)\zeta_0\mathcal{G}'(\zeta_0) \quad (0 \leq t < \infty). \tag{3.12}$$

Hence, by virtue of (1.7) and (3.12), we have

$$\mathbb{P}(\zeta_0, t) = \mathcal{G}(\zeta_0) + \frac{1+t}{\mu}\zeta_0\mathcal{G}'(\zeta_0) = \mathcal{F}(z_0) + \frac{1}{\mu}z_0\mathcal{F}'(z_0) = \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}f(z_0)}{z_0} \in \varphi(\mathbb{U}).$$

This contradicts to (3.11). Thus, we deduce that $\mathcal{F} \prec \mathcal{G}$. Considering $\mathcal{F} = \mathcal{G}$, we see that the function $\mathcal{G}$ is the best dominant. This completes the proof of Theorem 3.1. ∎

By similarly applying the method of proof of Theorem 3.1 as well as (1.5) and (1.6), we easily get the following results.

Corollary 3.1. *Let $f,\ g \in \mathcal{A}$ and $\lambda > -1$. Further let*

$$\operatorname{Re}\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi \qquad \left(z \in \mathbb{U};\ \chi(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}\right),$$

where

$$\varpi := \frac{1 + (\lambda+1)^2 - \left|1 - (\lambda+1)^2\right|}{4(\lambda+1)}. \tag{3.13}$$

Then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}g(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}g(z))/z$ is the best dominant.

Corollary 3.2. *Let $f,\ g \in \mathcal{A}$ and $b \in \mathbb{R} \setminus \mathbb{Z}_0^-$ with $b > -1$. Further let*

$$\operatorname{Re}\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right) > -\vartheta \qquad \left(z \in \mathbb{U};\ \chi(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}\right),$$

where

$$\vartheta := \frac{1 + (b+1)^2 - \left|1 - (b+1)^2\right|}{4(b+1)}. \tag{3.14}$$

Then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}g(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}g(z))/z$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result.

Theorem 3.2. *Let $f,\ g \in \mathcal{A}_p$ and $\mu > 0$. Further let*

$$\operatorname{Re}\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho \qquad \left(z\in\mathbb{U};\ \varphi(z):=\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}g(z)}{z}\right),$$

where ϱ is given by (3.2). If the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}f)/z$ is univalent in $\mathbb{U}$ and $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f)/z \in Q$, then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}g(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}f(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z))/z$ is the best subordinant.

Proof. Suppose that the functions $\mathcal{F}$ and $\mathcal{G}$ and $\mathcal{Q}$ are defined by (3.3). By applying the similar method as in the proof of Theorem 3.1, we get

$$\operatorname{Re}(\mathcal{Q}(z)) > 0 \qquad (z\in\mathbb{U}).$$

Next, to arrive at our desired result, we show that $\mathcal{G} \prec \mathcal{F}$. For this, we suppose that the function $\mathbb{P}$ be defined by (3.10). Since $\mu > 0$ and $\mathcal{G}$ is convex, by applying a similar method as in Theorem 3.1, we deduce that $\mathbb{P}$ is subordination chain. Therefore, by Lemma 2.5, we conclude that $\mathcal{G} \prec \mathcal{F}$. Moreover, since the differential equation

$$\varphi(z) = \mathcal{G}(z) + \frac{1}{\mu} z\mathcal{G}'(z) := \phi\left(\mathcal{G}(z), z\mathcal{G}'(z)\right)$$

has a univalent solution $\mathcal{G}$, it is the best subordinant. This completes the proof of Theorem 3.2. ∎

Applying a similar proof as in Theorem 3.2 and using (1.5) and (1.6), the following results are easily obtained.

Corollary 3.3. *Let $f,\ g\in\mathcal{A}$ and $\lambda > -1$. Further let*

$$\operatorname{Re}\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\varpi \qquad \left(z\in\mathbb{U};\ \chi(z):=\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z}\right),$$

where ϖ is given by (3.13). If the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f)/z$ is univalent in $\mathbb{U}$ and $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f)/z \in Q$, then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}g(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}g(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}f(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu}g(z))/z$ is the best subordinant.

Corollary 3.4. *Let $f,\ g \in \mathcal{A}$ and $b \in \mathbb{R} \setminus \mathbb{Z}_0^-$ with $b > -1$. Further let*

$$\operatorname{Re}\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right) > -\vartheta \qquad \left(z \in \mathbb{U};\ \chi(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g(z)}{z}\right),$$

where ϑ is given by (3.14). If the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f)/z$ is univalent in $\mathbb{U}$ and $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f)/z \in Q$, then the subordination

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g(z)}{z} \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z)}{z}.$$

Furthermore, the function $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g(z))/z$ is the best subordinant.

Combining the above-mentioned subordination and superordination results involving the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$, the following "sandwich-type results" are derived.

Corollary 3.5. *Let $f,\ g_k \in \mathcal{A}$ $(k = 1,\ 2)$ and $\mu > 0$. Further let*

$$\operatorname{Re}\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\varrho \qquad \left(z \in \mathbb{U};\ \varphi_k(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} g_k(z)}{z} \quad (k = 1,\ 2)\right),$$

where ϱ is given by (3.2). If the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f)/z$ is univalent in $\mathbb{U}$ and $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f)/z \in Q$, then the subordination chain

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} g_1(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} g_2(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_1(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_2(z)}{z}.$$

Furthermore, the functions $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_1)/z$ and $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_2)/z$ are, respectively, the best subordinant and the best dominant.

Corollary 3.6. *Let $f,\ g_k \in \mathcal{A}$ $(k = 1,\ 2)$ and $\lambda > -1$. Further let*

$$\operatorname{Re}\left(1 + \frac{z\chi_k''(z)}{\chi_k'(z)}\right) > -\varpi \qquad \left(z \in \mathbb{U};\ \chi_k(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_k(z)}{z} \quad (k = 1,\ 2)\right),$$

where ϖ is given by (3.13). If the function $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f)/z$ is univalent in $\mathbb{U}$ and $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f)/z \in Q$, then the subordination chain

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_1(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ p} g_2(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} g_1(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} g_2(z)}{z}.$$

Furthermore, the functions $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} g_1)/z$ *and* $(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} g_2)/z$ *are, respectively, the best subordinant and the best dominant.*

Corollary 3.7. *Let* $f,\ g_k \in \mathcal{A}$ $(k = 1,\ 2)$ *and* $b \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *with* $b > -1$. *Further let*

$$\mathrm{Re}\left(1 + \frac{z\chi_k''(z)}{\chi_k'(z)}\right) > -\vartheta \qquad \left(z \in \mathbb{U};\ \chi_k(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_k(z)}{z} \quad (k = 1,\ 2)\right),$$

where ϑ *is given by* (3.14). *If the function* $(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f)/z$ *is univalent in* $\mathbb{U}$ *and* $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f)/z \in Q$, *then the subordination chain*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_1(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} g_2(z)}{z}$$

implies that

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g_1(z)}{z} \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g_2(z)}{z}.$$

Furthermore, the functions $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g_1)/z$ *and* $(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} g_2)/z$ *are, respectively, the best subordinant and the best dominant.*

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