

Minimal Sequences and the Kadison-Singer Problem

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Abstract. The Kadison-Singer problem asks: does every pure state on the C^* -algebra $\ell^\infty(\mathbb{Z})$ admit a unique extension to the C^* -algebra $\mathcal{B}(\ell^2(\mathbb{Z}))$? A yes answer is equivalent to several open conjectures including Feichtinger's: every bounded frame is a finite union of Riesz sequences. We prove that for measurable $S \subset \mathbb{T}$, $\{\chi_S e^{2\pi ikt}\}_{k \in \mathbb{Z}}$ is a finite union of Riesz sequences in $L^2(\mathbb{T})$ if and only if there exists a nonempty $\Lambda \subset \mathbb{Z}$ such that χ_Λ is a minimal sequence and $\{\chi_S e^{2\pi ikt}\}_{k \in \Lambda}$ is a Riesz sequence. We also suggest some directions for future research.

2000 Mathematics Subject Classification: Primary: 37B10, 42A55, 46L05

Key words and phrases: Feichtinger conjecture, Riesz sequence, syndetic set, Thue-Morse minimal sequence, Riesz product.

1. Introduction

Recently there has been considerable interest in two deep problems that arose from very separate areas of mathematics. The

Kadison-Singer Problem (KSP): Does every pure state on the C^* -algebra $\ell^\infty(\mathbb{Z})$ admit a unique extension to the C^* -algebra $\mathcal{B}(\ell^2(\mathbb{Z}))$?

arose in the area of operator algebras and has remained unsolved since 1959 [19]. Pure states correspond to points in a topological space, the Stone-Ćech compactification $\beta(\mathbb{Z})$ of \mathbb{Z} , whose construction requires the axiom of choice, and recent work implicates the KSP with set-theoretic foundational issues [28]. The

Feichtinger Conjecture (FC): Every bounded frame can be written as a finite union of Riesz sequences

arose from Feichtinger's work in the area of signal processing involving time-frequency analysis, [10, 17, 18], and has remained unsolved since it was formally stated in the literature in 2005 [7, Conjecture 1.1].

Casazza and Tremain proved [9, Theorem 4.2], that a yes answer to the KSP is equivalent to the FC and Casazza, Fickus, Tremain, and Weber explained many other equivalent conjectures in [8]. In this paper we address the

Communicated by Sriwulan Adji.

Received: August 28, 2009; Revised: December 13, 2009.

Feichtinger Conjecture for Exponentials (FCE): For every non-trivial measurable set $S \subset \mathbb{T}$, the sequence $\{\chi_S e^{2\pi ikt}\}_{k \in \mathbb{Z}}$ is a finite union of Riesz sequences. Although FC implies FCE, and FCE is easily shown to be equivalent to FC for frames of translates, it is unknown if FCE implies FC. Our intuition suggests that FCE is weaker than FC. Our main result relates FCE to the area of **Symbolic Dynamics**:

Theorem 1.1. *For subsets $S \subset \mathbb{T}$ and $\Lambda \subset \mathbb{Z}$ set $B(S, \Lambda) := \{\chi_S e^{2\pi ikt}\}_{k \in \Lambda}$. For every nontrivial measurable $S \subset \mathbb{T}$ the following conditions are equivalent:*

- (1) $B(S, \mathbb{Z})$ is a finite union of Riesz sequences,
- (2) there exists a **syndetic** subset $\Lambda \subset \mathbb{Z}$ such that $B(S, \Lambda)$ is a Riesz sequence,
- (3) there exists a nonempty subset $\Lambda \subset \mathbb{Z}$ such that χ_Λ is a **minimal sequence** and $B(S, \Lambda)$ is a Riesz sequence.

The remainder of this section introduces notation, derives preliminary results, and reviews selected known results. Section 2 derives Theorems 1.1 and 2.1. Section 3 suggests some directions for further research. $\mathbb{N} = \{1, 2, \dots\}$, \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} are the natural, integer, rational, real, and complex numbers, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group, $\mathcal{L}^+(\mathbb{T})$ is the set of Lebesgue measurable $S \subseteq \mathbb{T}$ whose Haar measure $\mu(S) > 0$, and $F_n := \{0, 1, \dots, n - 1\}$. For $Y \subset X$, $X \setminus Y$ is the complement of Y in X and $\chi_Y : X \rightarrow \{0, 1\}$ is the characteristic function of Y . For $S \in \mathcal{L}^+(\mathbb{T})$ and $\Lambda \subset \mathbb{Z}$, P_S, P_Λ are orthogonal projections of $L^2(\mathbb{T})$ onto the closed subspace $\chi_S L^2(\mathbb{T})$, the closed subspace spanned by the sequence $E(\Lambda) := \{e^{2\pi ikt}\}_{k \in \Lambda}$, respectfully.

Lemma 1.1. *The following conditions are equivalent:*

- (1) $\exists \epsilon_1 > 0$ such that $\|P_S P_\Lambda h\| \geq \epsilon_1 \|P_\Lambda h\|$, $h \in L^2(\mathbb{T})$,
- (2) $\exists \epsilon_2 > 0$ such that $\|P_S h\| + \|P_{\mathbb{Z} \setminus \Lambda} h\| \geq \epsilon_2 \|h\|$, $h \in L^2(\mathbb{T})$,
- (3) $\exists \epsilon_3 > 0$ such that $\|P_{\mathbb{Z} \setminus \Lambda} P_{\mathbb{T} \setminus S} h\| \geq \epsilon_3 \|P_{\mathbb{T} \setminus S} h\|$, $h \in L^2(\mathbb{T})$.

Proof. Clearly (2) implies (1) and (3). Let $h \in L^2(\mathbb{T})$. Then $h = h_1 \cos \theta + h_2 \sin \theta$ where $\theta \in [0, \pi/2]$, $h_1 \cos \theta = P_\Lambda h$, $h_2 \sin \theta = P_{\mathbb{Z} \setminus \Lambda} h$, and $\|h_1\| = \|h_2\| = \|h\|$. Hence (1) implies $\|P_S h\| + \|P_{\mathbb{Z} \setminus \Lambda} h\| \geq (\max\{0, \epsilon_1 \cos \theta - \sin \theta\} + \sin \theta) \|h\|$ so (2) holds with $\epsilon_2 = \epsilon_1 (1 + \epsilon_1^2)^{-1/2}$. A similar argument shows that (3) implies (2). ■

Christenson’s book [10] explains frames and Riesz sequences. $B(S, \Lambda)$ is a bounded (below by $\mu(S)$) frame in $P_S L^2(\mathbb{T})$. In Lemma 1.1 condition (1) holds iff $B(S, \Lambda)$ is a Riesz sequence and condition (3) holds iff $R_{\mathbb{T} \setminus S} E(\mathbb{Z} \setminus \Lambda)$ is a frame in $L^2(\mathbb{T} \setminus S)$. Here $R_{\mathbb{T} \setminus S} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T} \setminus S)$ is the restriction operator.

For $\Lambda \subset \mathbb{Z}$ we define lower and upper Beurling densities

$$D^-(\Lambda) = \liminf_{k \rightarrow \infty} \min_{a \in \mathbb{R}} \frac{|\Lambda \cap (a, a + k)|}{k}, \quad D^+(\Lambda) = \limsup_{k \rightarrow \infty} \max_{a \in \mathbb{R}} \frac{|\Lambda \cap (a, a + k)|}{k},$$

lower and upper asymptotic densities

$$d^-(\Lambda) = \liminf_{k \rightarrow \infty} \frac{|\Lambda \cap (-k, k)|}{2k}, \quad d^+(\Lambda) = \limsup_{k \rightarrow \infty} \frac{|\Lambda \cap (-k, k)|}{2k},$$

and if the cardinality $|\Lambda| \geq 2$ we define the separation

$$\Delta(\Lambda) := \min\{|\lambda_2 - \lambda_1| : \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2\}.$$

The following result was inspired by Olevskii and Ulanovskii's paper [26].

Corollary 1.1. *If $B(S, \Lambda)$ is a Riesz sequence then $D^+(\Lambda) \leq \mu(S)$.*

Proof. Since $E(\mathbb{Z} \setminus \Lambda)$ is a frame in $L^2(\mathbb{T} \setminus \Lambda)$ Landau's result [21, Theorem 3], implies $D^-(\mathbb{Z} \setminus \Lambda) \geq \mu(\mathbb{T} \setminus S)$. Therefore $D^+(\Lambda) = 1 - D^{-1}(\mathbb{Z} \setminus \Lambda) \leq 1 - \mu(\mathbb{T} \setminus S) = \mu(S)$. ■

Result 1. Montgomery and Vaughan's result [24, Corollary 2], implies that if S contains an interval having length $T > 1/\Delta(\Lambda)$ then condition (1) in Lemma 2.1 holds with $\epsilon_1 = T - 1/\Delta(\Lambda)$ so $B(S, \Lambda)$ is a Riesz sequence. It follows that if $B(S, \mathbb{Z})$ does not satisfy FCE then there exists a Cantor set $S_c \in \mathcal{L}^+(\mathbb{T})$ such that $S_c \subseteq S$.

Result 2. Casazza, Christiansen, and Kalton [6, Theorem 2.2] showed that for $n \in \mathbb{N}, m \in \mathbb{Z}$, $B(S, n\mathbb{Z} + m)$ is a Riesz basis iff $S + (1/n)F_n = \mathbb{T}$ a.e. This condition never holds if S is a Cantor set.

Result 3. The authors above also showed [6, Theorem 2.4], that for $\Lambda \subseteq \mathbb{N}$, $B(S, \Lambda)$ is a Riesz sequence iff $B(S, \Lambda)$ is a frame.

Result 4. Bourgain and Tzafriri's restricted invertibility result for matrices [3] implies that for every $S \in \mathcal{L}^+(\mathbb{T})$ there exists $\Lambda \subseteq \mathbb{Z}$ such that $d^-(\Lambda) > 0$ and $B(S, \Lambda)$ is a Riesz sequence.

Result 5. Bourgain and Tzafriri's result [4, Theorem 4.1], implies that if χ_S belongs to the Besov space $W_{2,2}^\tau$ for some $\tau > 0$ then $B(S, \mathbb{Z})$ satisfies FCE. Moreover, the proof of their result [4, Corollary 4.2], shows that if S is a Cantor set and $\mathbb{T} \setminus S$ is a union of disjoint open intervals I_n , $n \in \mathbb{N}$ satisfying $\mu(I_n) \leq c2^n$ for some $c > 0$ then $\chi_S \in W_{2,2}^\tau$ for all $\tau \in (0, 1)$.

Result 6. Bownik and Speegle [5, Theorem 4.16], used discrepancy theory to construct $S \in \mathcal{L}^+(\mathbb{T})$ and a class of $\Lambda \subset \mathbb{Z}$ such that $B(S, \Lambda)$ is not a Riesz sequence and related their construction to Gower's results about Szemerédi's Theorem [16].

Result 7. In November 2009 Spielman and Srivastava gave an elementary constructive proof of Bourgain and Tzafriri's restricted invertibility result [29].

2. Minimal sequences

The **symbolic dynamical system** (Ω, σ) , where $\Omega := \{0, 1\}^{\mathbb{Z}}$ has the product topology and $\sigma : \Omega \rightarrow \Omega$ is the shift homeomorphism $\sigma(b)(j) = b(j - 1)$, $b \in \Omega$, belongs to the class of dynamical systems introduced by Bebutov in [1]. Its subsystems (X, σ) correspond to nonempty closed invariant $X \subseteq \Omega$. Elements in Ω are binary sequences and the sets $U_m(b) := \{a \in \Omega : a(k) = b(k), -m < k < m\}$, $b \in \Omega$, $m \in \mathbb{N}$ are a basis for the product topology. Orbits $O(b) := \{\sigma^k(b) : k \in \mathbb{Z}\}$ are (shift) invariant and orbit closures $\overline{O}(b)$ are closed and invariant.

Lemma 2.1. *If $B(S, \Lambda)$ is a Riesz sequence and if b is a nonzero sequence in $\overline{O}(\chi_\Lambda)$ then $B(S, \text{supp}(b))$ is a Riesz sequence.*

Proof. Fix $\epsilon_1 > 0$. Then $B(S, \Lambda)$ satisfies the inequality in condition (1) of Lemma 1.1 iff $B(S, \Lambda_f)$ satisfies this inequality for every finite $\Lambda_f \subseteq \Lambda$. The result then follows from the definition of orbit closure and product topology. ■

A nonempty closed invariant $X \subset \Omega$ is called a minimal set if it is minimal with respect to these properties. Zorn's lemma ensures that every nonempty closed invariant set contains a minimal set. If X is a minimal set and $b \in X$ then $\overline{O}(b) = X$. A **minimal sequence** is a binary sequence b such that $\overline{O}(b)$ is a minimal set.

Definition 2.1. $\Lambda \subset \mathbb{Z}$ is **syndetic** if there exists $n \in \mathbb{N}$ such that $\Lambda + F_n = \mathbb{Z}$, thick if for every $n \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ such that $k + F_n \subset \Lambda$, and piecewise syndetic if $\Lambda = \Lambda_s \cap \Lambda_t$ where Λ_s is syndetic and Λ_t is thick.

Lemma 2.2. If $\mathbb{Z} = \bigcup_{i=1}^n \Lambda_i$ then one of the Λ_i is piecewise syndetic.

Proof. Theorem 1.23 in [13]. ■

Lemma 2.3. If Λ_p is piecewise syndetic then there exists a syndetic set Λ such that $\chi_\Lambda \in \overline{O}(\chi_{\Lambda_p})$.

Proof. $\Lambda_p = \Lambda \cap \Lambda_t$ where Λ is syndetic and Λ_t is thick. Then $\chi_\Lambda \in \overline{O}(\chi_{\Lambda_p})$ follows from the definitions of thick sets, orbit closures, and product topology. ■

Corollary 2.1. For every $S \in \mathcal{L}^+(\mathbb{T})$ the following conditions are equivalent:

- (1) $B(S, \mathbb{Z})$ is a finite union of Riesz sequences,
- (2) there exists a **syndetic** subset $\Lambda \subset \mathbb{Z}$ such that $B(S, \Lambda)$ is a Riesz sequence.

Proof. (2) implies (1): If Λ is syndetic there exists $n \in \mathbb{N}$ with $\Lambda + F_n = \mathbb{Z}$. Then $B(S, \mathbb{Z})$ is the union of the Riesz sequences $B(S, \Lambda + k)$, $k \in F_n$.

(1) implies (2): If $B(S, \mathbb{Z})$ is a finite union of Riesz sets then Lemma 2.2 implies that there exists a piecewise syndetic Λ_p such that $B(S, \Lambda_p)$ is a Riesz sequence. Then Lemma 2.3 implies there exists a syndetic Λ such that $\chi_\Lambda \in \overline{O}(\chi_{\Lambda_p})$. Since $\Lambda = \text{supp}(\chi_\Lambda)$, Lemma 2.1 implies that $B(S, \Lambda)$ is a Riesz sequence. ■

Lemma 2.4. If Λ is syndetic and $b \in \overline{O}(\chi_\Lambda)$ then $\text{supp}(b)$ is syndetic.

Proof. Since Λ is syndetic there exists $n \in \mathbb{N}$ with $\Lambda + F_n = \mathbb{Z}$. Therefore

$$\text{supp}(\sigma^k(\chi_\Lambda)) + F_n = \text{supp}(\chi_\Lambda) + k + F_n = \mathbb{Z}, \quad k \in \mathbb{Z},$$

so the definition of orbit closure implies $\text{supp}(b) + F_n = \mathbb{Z}$ whenever $b \in \overline{O}(\chi_\Lambda)$. ■

For $b \in \Omega$ define the function $\theta_b : \mathbb{Z} \rightarrow \Omega$ by $\theta_b(k) = \sigma^k(b)$, $k \in \mathbb{Z}$.

Definition 2.2. $b \in \Omega$ is almost periodic if $\theta_b^{-1}(U_m(b))$ is syndetic for every $m \in \mathbb{N}$.

Lemma 2.5. A sequence is minimal iff it is almost periodic.

Proof. Gottschalk and Hedlund proved this in [15, Theorems 4.05 and 4.07]. ■

Corollary 2.2. If b is a nonzero minimal sequence then $\text{supp}(b)$ is syndetic.

Proof. Choose $k \in \text{supp}(b)$ and set $m = |k| + 1$. Lemma 2.5 implies that b is almost periodic therefore there exists $n \in \mathbb{N}$ such that $\theta_b^{-1}(U_m(b)) + F_n = \mathbb{Z}$. Therefore $\text{supp}(b) + F_n = \mathbb{Z}$ so $\text{supp}(b)$ is syndetic. ■

Proof of Theorem 1.1. (1) equivalent to (2): This follows from Corollary 2.1. (3) implies (2): Since Λ is nonempty χ_Λ is a nonzero minimal sequence and hence Corollary 2.2 implies that $\Lambda = \text{supp}(\chi_\Lambda)$ is syndetic. (2) implies (3): Zorn's lemma implies that there exists a minimal set $X \subseteq \overline{O}(\chi_\Lambda)$. Then choose $b \in X$. Then b is a minimal sequence. Lemma 2.1 implies that $B(S, \text{supp}(b))$ is a Riesz sequence. Lemma 2.4 implies that $\text{supp}(b)$ is syndetic and hence $\text{supp}(b)$ is nonempty. Then (3) follows from the fact that $\chi_{\text{supp}(b)} = b$. ■

Definition 2.3. A subset $\Lambda \subset \mathbb{Z}$ is a Bohr set if there exists a compact abelian group G , a homomorphism $\psi : \mathbb{Z} \rightarrow G$ with $\psi(\mathbb{Z}) = G$, and a nonempty open subset $U \subset G$ such that $\Lambda = \psi^{-1}(U)$.

If Λ is a Bohr set then χ_Λ is a nonzero minimal sequence. These sets generalize sets having the form $n\mathbb{Z} + m$ where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ and are unions of the Bohr sets defined by Ruzsa [14, Definition 2.5.1], who studied their number theoretic properties. They are named after Harald Bohr, who pioneered the theory of (uniformly) almost periodic functions [2], and are related to the Bohr compactification used by Dutkay, Han, and Jorgensen in their study of spectral pairs [11]. The following extension of Result 2 utilizes spectral properties of Bohr sets.

Theorem 2.1. If S is a Cantor set with $\mu(S) > 0$ and Λ is a Bohr set then $B(S, \Lambda)$ is not a Riesz sequence.

Proof. Without loss of generality we can assume that $\Lambda = \psi^{-1}(U)$ where $U \subseteq G$ is an open set that contains $0 \in G$ and choose an open subset $V \subseteq G$ that contains 0 and satisfies $V - V \subseteq U$. Set $f := \chi_V * \chi_{-V}$ and $g := f \circ \psi \in \ell^\infty(\mathbb{Z})$. Then $\text{supp}(g) \subset \Lambda$ and g equals the Fourier transform $\widehat{\nu}$ of the positive measure ν on \mathbb{T} given by

$$(2.1) \quad \nu = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma) \delta_{\gamma(\psi(1))}, \quad \widehat{f}(\gamma) = |\widehat{\chi}_V(\gamma)|^2$$

where \widehat{G} is the Pontryagin dual of G and $\widehat{f} \in \ell^2(\widehat{G})$ is the Fourier transform of f . Let $\epsilon > 0$. It suffices to construct $h \in L^2(\mathbb{T})$ such that $\|P_S(\nu * h)\| < \epsilon \|\nu * h\|$ since $P_\Lambda(\nu * h) = \nu * h$. Partition $\widehat{G} = \Gamma_1 \cup \Gamma_2$ where Γ_1 is finite, let ν_i be the component of ν supported on $\Gamma_i, i = 1, 2$, and set $\alpha := \sum_{\gamma \in \Gamma_2} \widehat{f}(\gamma)$ and $\beta := \sum_{\gamma \in \Gamma_1} \widehat{f}(\gamma)^2$. Since S is nowhere dense $\text{supp}(\nu_1) + S \neq \mathbb{T}$ so there exists $h \in L^2(\mathbb{T})$ such that $\|h\| = 1$ and $\text{supp}(h)$ is contained in an arc $I \subset \mathbb{T}$ that is disjoint from $\text{supp}(\nu_1) + S$ and such that the intervals $I + \gamma, \gamma \in \Gamma_1$ are mutually disjoint. Then $\|P_S(\nu * h)\| = \|P_S(\nu_2 * h)\| \leq \|\nu_2 * h\| \leq \alpha$, and $\|\nu * h\| \geq \|\nu_1 * h\| = \beta$. The result follows by choosing Γ_1 so $\alpha < \epsilon\beta$ which is possible since as Γ_1 increases $\alpha \rightarrow 0$ and $\beta \rightarrow f(1) > 0$. ■

Bohr minimal sequences are simple. We discuss methods to construct more sophisticated minimal sequences. For nonempty invariant $X, Y \subseteq \Omega$, a function $\zeta : X \rightarrow Y$ is equivariant if $\zeta \circ \sigma = \sigma \circ \zeta$. For $m \in \mathbb{N}$ every function $c : \{0, 1\}^{\{-m+1, \dots, m-1\}} \rightarrow \{0, 1\}$ defines the function $\zeta_c : \Omega \rightarrow \Omega$ by

$$(2.2) \quad \zeta_c(b)(k) = c(R_{\{-m+1, \dots, m-1\}}(\sigma^k(b))), \quad b \in \Omega, k \in \mathbb{Z}.$$

Furthermore, for every nonempty closed invariant $X \subseteq \Omega$ the restriction $\zeta_c : X \rightarrow \Omega$ is continuous and equivariant and every continuous equivariant $\zeta : X \rightarrow \Omega$ equals ζ_c for some c . Equivariant images of minimal sets and sequences are minimal.

The Thue-Morse minimal sequence $b = \cdots 10010110.0110100110010110 \cdots$, introduced in [25, 30], can be constructed using substitutions $0 \rightarrow 01$ and $1 \rightarrow 10$. Its orbit closure $X = \overline{O}(b)$ admits a unique invariant ergodic probability measure λ [20]. The spectrum of the unitary operator $(U_\sigma f)(x) = f(\sigma(x))$, $f \in L^2(X, \lambda)$ admits a Riesz product representation, has no point components, and is supported on a dense set of measure zero [23, 27].

3. Research directions

We suggest three questions, related to the material in this paper, as directions towards a solution of the FCE. In this section we assume that $S \in \mathcal{L}^+(\mathbb{T})$ is a Cantor set such that $\chi_S \notin W_{2,2}^\tau$ for all $\tau > 0$ and that χ_Λ is a nonzero minimal sequence. We let $\mathcal{M}(\Lambda)$, $\mathcal{P}(\Lambda)$ denote the set of measures, pseudomeasures, respectively, on \mathbb{T} whose Fourier transforms are supported on Λ , (see [22, (4.2)]).

Question 1. What properties of a pair (S, Λ) , determine whether or not $B(S, \Lambda)$ is a Riesz sequence? Such properties include the rate of decay of the restriction of $\widehat{\chi}_S$ to Λ , and the sumsets $S + \text{supp}(\nu)$ where $\nu \in \mathcal{M}(\Lambda)$ or $\nu \in \mathcal{P}(\Lambda)$. Of particular interest are pairs where χ_Λ is a substitution minimal sequence because their spectral properties have been intensively studied [27].

Question 2. What is the spectrum of $P_S + P_\Lambda$? Condition (2) in Lemma 1.1 implies that $B(S, \Lambda)$ is a Riesz sequence iff this spectrum is bounded below by a positive number. Let $\mathcal{A}(P_S, P_\Lambda)$ denote the C^* -subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by P_S and P_Λ . A standard result [12, (8.5.5)], shows that $\mathcal{A}(P_S, P_\Lambda)$ equals a homomorphic image of a specific crossed-product C^* -algebra and implies that $\mathcal{A}(P_S, P_\Lambda)$ is determined by the spectrum of $P_S + P_\Lambda$.

Question 3. What are the spectrums of submatrices of the Laurent operators $L_S : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ defined by $L_S f = \widehat{\chi}_S * f$? Spielman and Srivastava's algorithm [29] may provide an efficient method to compute these spectrums. Of particular interest are Cantor sets having the form $S = \bigcap_{n \in \mathbb{N}} S_n$ where each S_n is obtained by deleting a large number of equally spaced, equal length open arcs from \mathbb{T} . This construction was suggested to the author by Alexander Olevskii as a method of constructing Cantor sets S such that $\widehat{\chi}_S$ decays slowly and is easily computable.

Acknowledgment. We thank Eric Weber for introducing the beguiling Kadison-Singer problem and Peter Casazza, Ilya Chrishtal, Ole Christiansen, Palle Jorgensen, Anders Mouritzen, and Alexander Olevskii for their advice and encouragement.

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