BULLETIN of the
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(2) (2010), 265–272

A Property on Geodesic Mappings of Pseudo-Symmetric Riemannian Manifolds

[1]Fengyun Fu and [2]Peibiao Zhao
[1,2]Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China
[1]fengyunfu_1984@yahoo.cn, [2]pbzhao@mail.njust.edu.cn

Abstract. This paper discusses the semi-symmetric projective mapping. Some interesting and remarkable results are obtained by virtue of some defined tensors and an important lemma. These verdicts that generalizes the corresponding conclusions posed in the previous papers.

2000 Mathematics Subject Classification: 53C20, 53D11

Key words and phrases: Geodesic mappings, semi-Riemannian manifolds, pseudo-symmetric Riemannian manifolds, curvature tensor.

1. Introduction

The study of the theory of geodesic mappings of affine connected, Riemannian, Kählerian spaces and etc., has been an active field over the past several decades. Many new and interesting results have appeared in [6, 12, 14]. The investigation of geodesic mapping theory for special spaces is an important and active research topic. Geodesic mappings of certain special Riemannian spaces were studied earlier. In particular, the geodesic mapping theory was derived mainly for symmetric [11], recurrent [12], two-symmetric and m-recurrent [7]. Of course, many works here were devoted to the problem of the non-existence in special spaces of geodesic mappings, projective, null-geodesic transformations, and concircular vector fields. For the further and total investigations of geodesic projective mappings of special Riemannian spaces, one can refer to the celebrated works [5, 8, 13, 14] for details. To speak of the fact that the author, recently, even studied the theory of transformations on Carnot Caratheodory spaces, and obtained some interesting results [21].

For the study of semi-symmetric linear connection, as we know, in the early days of 1924, Friedmann and Schouten [4] first introduced the concepts of semi-symmetric linear connection. Afterwards, Yano [15] considered the semi-symmetric metric connection of Riemannian manifolds. The authors in [16–19], in this setting,

Communicated by Lee See Keong.
Received: February 5, 2009; *Revised:* June 16, 2009.

studied and obtained some interesting and remarkable results by using these concepts and the similar approaches.

In this paper, we continue to discuss the problems similarly to those just put forward above, but we consider here in a relatively wide space-pseudo-symmetric Riemannian manifolds. The class of pseudo-symmetric manifolds is essentially wider than that of semi-symmetric manifolds.

Let (M, g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian smooth manifold. We denote by R the curvature tensor of (M, g). If $R \times R = 0$ on M, then the manifold (M, g) is called semi-symmetric [13]. The study of geodesic mappings onto semi-symmetric manifolds leads to the concept of pseudo-symmetric manifolds [3, 8]. At first, we state some terminologies and interesting facts for the semi-symmetric projective mapping via a definition and an important lemma, and pose the semi-symmetric projective metric connection transformations [19]. Secondly, we introduce some tensors and give some results by using these well defined tensors. Finally, we give the interesting theorem with these concepts which includes semi-symmetric manifolds as well as the pseudo-symmetric pseudo-Riemannian manifolds and so on. In particular, we follow the survey paper [6, 8] in which the author posed the following theorem but the proof of this theorem is not published, and offer the similar problem but the conditions here differs heavily from those in [13, 14]. Here we can write the theorem posed in [8] as follows.

Theorem 1.1. *If (M^n, g) is a pseudo-symmetric pseudo-Riemannian manifold admitting a non-trivial geodesic mapping f onto a manifold $(\overline{M}, \overline{g})$, then $(\overline{M}, \overline{g})$ is also a pseudo-symmetric manifold.*

Theorem 1.1 is very important in the research of pseudo-symmetric manifolds. More general results are obtained in [9, 10]. The authors posed out the geodesic mapping [1, 13, 14] for studying the invariance problems. We will use the projective semi-symmetric connection transform [19, 20] to consider the same problem in this paper. We derive Theorem 1.2, given in Section 3, as follows:

Theorem 1.2. *If (M^n, g) is a pseudo-symmetric pseudo-Riemannian manifold admitting a special non-trivial projective semi-symmetric connection mapping f onto a manifold $(\overline{M}, \overline{g})$, then $(\overline{M}, \overline{g})$ is also a pseudo-symmetric manifold.*

Theorem 1.2 which is given and can be regarded as a remarkable result in [1]. Since the approaches are similar to those in [1, 14], we take them as our main reference. All the notations and terminologies offered in this paper are taken from [1, 19, 20].

The organization of this paper is as follows. Section 2 is devoted to introducing some necessary notations and terminologies. Section 3 will state some interesting results, and the proofs of these conclusions are also given in this section.

2. Preliminaries

Let (M^n, g), $n = \dim M \geq 3$, be a connected n-dimensional pseudo-Riemannian smooth manifold of class C^∞. Let $\chi(M)$ denote the set of all the vector fields on M, $\Xi(M)$ be the Lie algebra of vector fields on M, and $V_i \in \chi(M)$ $i = 1, \cdots, n$ be the local orthogonal unit vector fields on M. We denote by $\nabla, \widetilde{R}, R, S$ and κ the

Levi-Civita connection, the curvature tensor, the Riemannian-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. We first define on M the endomorphisms $\widetilde{R}(X,Y)$ and $X \wedge Y$ by $\widetilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, respectively. Furthermore, for a $(0,k)$-type tensor field T on M, we also define the following the $(0, k+2)$-type tensor fields $(R \cdot T)$ and $Q(g, T)$ by the formulas

$$\begin{aligned}(R \cdot T)(X_1, X_2, X_3, X_4; X, Y) = &-T(\widetilde{R}(X,Y)X_1, X_2, X_3, X_4)\\ &- T(X_1, \widetilde{R}(X,Y)X_2, X_3, X_4)\\ &- T(X_1, X_2, \widetilde{R}(X,Y)X_3, X_4)\\ &- T(X_1, X_2, X_3, \widetilde{R}(X,Y)X_4)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}Q(g,T)(X_1, X_2, X_3, X_4; X, Y) = &\,T((X \wedge Y)X_1, X_2, X_3, X_4)\\ &+ T(X_1, (X \wedge Y)X_2, X_3, X_4)\\ &+ T(X_1, X_2, (X \wedge Y)X_3, X_4)\\ &+ T(X_1, X_2, X_3, (X \wedge Y)X_4)\end{aligned} \tag{2.2}$$

where $X, Y, Z, X_1, \cdots, X_4 \in \Xi(M)$.

A semi-Riemannian manifold (M, g) is said to be pseudo-symmetric [2] if at every point of M the following condition is satisfied:

(a) The tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudo-symmetric if and only if $R \cdot R = L_R Q(g, R)$ holds on the set $U_R = \{x \in M | Z(R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R and $Z(R) = R - (\kappa/[n(n-1)])G$ with G defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$.

A semi-Riemannian manifold (M, g) is said to be Ricci-pseudo-symmetric if at every point of M the following condition is satisfied:

(a) The tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold (M, g) is Ricci-pseudo-symmetric if and only if $R \cdot S = L_S Q(g, S)$ holds on the set $U_S = \{x \in M | S - (\kappa/n)g \neq 0 \text{ at } x\}$, where L_S is some function on U_S.

Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two n-dimensional semi-Riemannian manifolds. A diffeomorphism: $f : M \to \widetilde{M}$ which maps geodesic lines into geodesic lines is called a geodesic mapping. It is well known that in a common coordinate system, we can give the curvature tensor and Christoffel symbols [1].

Let D be a linear connection on M. We define the torsion tensor of D by

$$T(X,Y) = D_X Y - D_Y X - [X,Y] \quad \forall X, Y \in \chi(M). \tag{2.3}$$

Suppose that T satisfies the following condition

$$T(X,Y) = \Pi(Y)X - \Pi(X)Y \tag{2.4}$$

and for any $X, Y, Z \in \chi(M)$, there holds $D_X g(Y, Z) = 0$, where Π is of 1-form on M, then D is called the semi-symmetric metric connection on M [18–20]. If there exists the local coordinate system in M such that g, ∇, D, Π have the local expressions, respectively, by $g_{ji}, \{^h_{ji}\}, \Gamma^h_{ji}, P_i$. Then we have $\Gamma^h_{ji} = \{^h_{ji}\} + \delta^h_j P_i - g_{ji}P^h$, where $P_h = g^{hi}P_i$.

If the geodesic with respect to D are always consistent with those of ∇, then D is called the projective equivalent connection with ∇.

If D is both the projective equivalent connection with ∇ and the semi-symmetric, then D is called the projective semi-symmetric connection. As is known to all, if the linear connection has the same geodesic curves as ∇, then it is called the projective transformation of ∇. Analogously, the semi-symmetric connection D has the same geodesic curves as ∇, then it is called the projective semi-symmetric connection transformation.

Let D be the semi-symmetric projective connection transformation of the Riemannian connection ∇, it has the local expression Γ^h_{ji}, then there holds

$$\Gamma^h_{ji} = \{^h_{ji}\} + \delta^h_j\Psi_i + \delta^h_i\Psi_j + \delta^h_j\sigma_i - \delta^h_i\sigma_j \tag{2.5}$$

where Ψ_i, σ_i are the covariant vectors.

From (2.5), we know that the curvature tensor, the Ricci curvature tensor of D are given, respectively, by

$$\widetilde{R}^h_{kji} = R^h_{kji} + \beta_{kj}\delta^h_i + \tau_{ki}\delta^h_j - \tau_{ji}\delta^h_k. \tag{2.6}$$

By virtue of [18–20], we arrive at the following

$$\beta_{ji} + \beta_{ij} = 0, \tag{2.7}$$

$$\tau_{ji} - \tau_{ij} = n\beta_{ij}, \tag{2.8}$$

$$\beta_{ji} = [1/(n+1)](\nabla_i P_j - \nabla_j P_i). \tag{2.9}$$

3. Some results and proofs

According to (2.1) and (2.2) we have the following local expressions

$$\begin{aligned}(R\cdot R)_{hijklm} &= \nabla_m\nabla_l R_{hijk} - \nabla_l\nabla_m R_{hijk}\\ &= -R_{rijk}R^r{}_{hlm} - R_{hrjk}R^r{}_{ilm} - R_{hirk}R^r{}_{jlm} - R_{hijr}R^r{}_{klm},\end{aligned} \tag{3.1}$$

$$\begin{aligned}Q(g,R)_{hijklm} = {}& g_{hm}R_{lijk} + g_{im}R_{hljk} + g_{jm}R_{hilk} + g_{km}R_{hijl}\\ & - g_{hl}R_{mijk} - g_{il}R_{hmjk} - g_{jl}R_{himk} - g_{kl}R_{hijm}.\end{aligned} \tag{3.2}$$

Lemma 3.1. *Let $\Psi : (M, g) \to (M, g)$ be a semi-symmetric projective transformation and let the condition $R \cdot R = LQ(g, R)$ be satisfied on U_R. Then in common coordinate system $\{x^1, \cdots, x^n\}$ on U_R, there holds the following*

$$\begin{aligned}(\widetilde{R}\cdot\widetilde{R})_{hijkm} = {}& \frac{1}{n}\eta(\tilde{g}_{hl}\widetilde{R}_{himk} - \tilde{g}_{hm}\widetilde{R}_{lijk}) + B_{il}\widetilde{R}_{hmjk} - B_{im}\widetilde{R}_{hljk} + B_{jl}\widetilde{R}_{himk}\\ & - B_{jm}\widetilde{R}_{hilk} + B_{ik}\widetilde{R}_{hijm} - B_{km}\widetilde{R}_{hijl} + F_{il}\widetilde{G}_{hmjk} - F_{im}\widetilde{G}_{himk}\\ & - F_{jl}\widetilde{G}_{hilk} + F_{lk}\widetilde{G}_{hijm} - F_{km}\widetilde{G}_{hijl},\end{aligned}$$

where $B_{ij} = -Lg_{ij} + \tau_{ij}$, $F_{ij} = (-1/n)A_{ij} - (1/n(L\tilde{g}^{rs}(\tau_{rs}g_{ij} - g_{rs}\tau_{ij})$, $A_{ij} = \tau_{ir}\widetilde{S}^r_j - \tau_{rs}\widetilde{R}^r_{ij}{}^s$, $\eta = \tilde{g}^{rs}(-g_{rs}L + \tau_{rs})$.

Proof. By using (2.6) and the Ricci identity we have

$$\begin{aligned}
D_mD_l\widetilde{R}^s{}_{ijk} - D_lD_m\widetilde{R}^s{}_{ijk} &= \nabla_m\nabla_lR^s{}_{ijk} - \nabla_l\nabla_mR^s{}_{ijk} + \tau_{mr}R^r{}_{ijk}\delta^s_l - \tau_{lr}R^r{}_{ijk}\delta^s_m \\
&\quad + (\tau_{jr}R^r{}_{mlk} + \tau_{rk}R^r{}_{mjl})\delta^s_i - (\tau_{rk}R^r{}_{mli} + \tau_{ir}R^r{}_{mlk})\delta^s_j \\
&\quad - (\beta_{rj}R^r{}_{mli} + \beta_{ir}R^r{}_{mlj})\delta^s_k + \tau_{li}R^s{}_{mjk} - \tau_{mi}R^s{}_{ljk} \\
&\quad + \tau_{lj}R^s{}_{imk} - \tau_{mj}R^s{}_{ilk} + \tau_{lk}R^s{}_{ijm} - \tau_{mk}R^s{}_{ijl} \\
&\quad + (\tau_{li}\beta_{mj} - \tau_{mi}\beta_{lj} + \tau_{lj}\beta_{im} - \tau_{mj}\beta_{il} - 2\beta_{ml}\beta_{ij})\delta^s_k \\
&\quad - 2\beta_{ml}R^s{}_{ijk} + [\tau_{mk}(\tau_{li} - \tau_{il}) + \tau_{lk}(\tau_{im} - \tau_{mi}) \\
&\quad - 2\beta_{ml}\tau_{ik}]\delta^s_j + [\tau_{mk}(\tau_{jl} - \tau_{lj}) + \tau_{lk}(\tau_{mj} - \tau_{jm}) \\
&\quad + 2\beta_{ml}\tau_{jk}]\delta^s_i.
\end{aligned}$$

If 1-form Π is of closed, that is, we get $\beta_{ij} = 0$, $\tau_{ij} = \tau_{ji}$. Then we find the following formula:

$$\begin{aligned}
D_mD_l\widetilde{R}^s{}_{ijk} - D_lD_m\widetilde{R}^s{}_{ijk} &= \nabla_m\nabla_lR^s{}_{ijk} - \nabla_l\nabla_mR^s{}_{ijk} + \tau_{mr}R^r{}_{ijk}\delta^s_l - \tau_{lr}R^r{}_{ijk}\delta^s_m \\
&\quad + (\tau_{jr}R^r{}_{mlk} + \tau_{rk}R^r{}_{mjl})\delta^s_i - (\tau_{rk}R^r{}_{mli} + \tau_{ir}R^r{}_{mlk})\delta^s_j \\
&\quad - \tau_{li}R^s{}_{mjk} - \tau_{mi}R^s{}_{ljk} + \tau_{lj}R^s{}_{imk} - \tau_{mj}R^s{}_{ilk} \\
&\quad + \tau_{lk}R^s{}_{ijm} - \tau_{mk}R^s{}_{ijl} + [\tau_{mk}(\tau_{li} - \tau_{il}) \\
&\quad + \tau_{lk}(\tau_{im} - \tau_{mi})]\delta^s_j + [\tau_{mk}(\tau_{jl} - \tau_{lj}) + \tau_{lk}(\tau_{mj} - \tau_{jm})]\delta^s_i \\
&= \nabla_m\nabla_lR^s{}_{ijk} - \nabla_l\nabla_mR^s{}_{ijk} + \tau_{mr}R^r{}_{ijk}\delta^s_l - \tau_{lr}R^r{}_{ijk}\delta^s_m \\
&\quad + (\tau_{jr}R^r{}_{mlk} + \tau_{rk}R^r{}_{mjl})\delta^s_i - (\tau_{rk}R^r{}_{mli} + \tau_{ir}R^r{}_{mlk})\delta^s_j \\
&\quad + \tau_{lj}R^s{}_{imk} - \tau_{mj}R^s{}_{ilk} + \tau_{lk}R^s{}_{ijm} - \tau_{mk}R^s{}_{ijl} \\
&\quad + \tau_{li}R^s{}_{mjk} - \tau_{mi}R^s{}_{ljk}.
\end{aligned}$$

In other words, we obtain the following

$$\begin{aligned}
D_mD_l\widetilde{R}^s{}_{ijk} - D_lD_m\widetilde{R}^s{}_{ijk} &= \nabla_m\nabla_lR^s{}_{ijk} - \nabla_l\nabla_mR^s{}_{ijk} + \tau_{mr}R^r{}_{ijk}\delta^s_l - \tau_{lr}R^r{}_{ijk}\delta^s_m \\
&\quad + (\tau_{jr}R^r{}_{mlk} + \tau_{rk}R^r{}_{mjl})\delta^s_i - (\tau_{rk}R^r{}_{mli} + \tau_{ir}R^r{}_{mlk})\delta^s_j \\
&\quad + \tau_{lj}R^s{}_{imk} - \tau_{mj}R^s{}_{ilk} + \tau_{lk}R^s{}_{ijm} - \tau_{mk}R^s{}_{ijl} \\
&\quad - \tau_{li}R^s{}_{mjk} - \tau_{mi}R^s{}_{ljk}.
\end{aligned}$$

Notice that the condition of Lemma 3.1 and (3.1) and (3.2), one has

$$\begin{aligned}
D_mD_l\widetilde{R}^s{}_{ijk} - D_lD_m\widetilde{R}^s{}_{ijk} &= -L(\delta^s_lR_{mijk} - \delta^s_mR_{lijk}) \\
&\quad + L[g_{im}(\widetilde{R}^s{}_{ljk} - \beta_{lj}\delta^s_k - \tau_{lk}\delta^s_j + \tau_{jk}\delta^s_l) \\
&\quad + g_{jm}(\widetilde{R}^s{}_{ilk} - \beta_{il}\delta^s_k - \tau_{ik}\delta^s_l + \tau_{lk}\delta^s_i) \\
&\quad + g_{km}(\widetilde{R}^s{}_{ijl} - \beta_{ij}\delta^s_l - \tau_{il}\delta^s_j + \tau_{jl}\delta^s_i) \\
&\quad - g_{il}(\widetilde{R}^s{}_{mjk} - \beta_{mj}\delta^s_k - \tau_{mk}\delta^s_j + \tau_{jk}\delta^s_m) \\
&\quad - g_{jl}(\widetilde{R}^s{}_{imk} - \beta_{im}\delta^s_k - \tau_{ik}\delta^s_m + \tau_{mk}\delta^s_i) \\
&\quad - g_{kl}(\widetilde{R}^s{}_{ijm} - \beta_{ij}\delta^s_m - \tau_{im}\delta^s_j + \tau_{jm}\delta^s_i)] \\
&\quad + \delta^s_lE_{mijk} - \delta^s_mE_{lijk} + (E_{jmlk} + E_{kmlj})\delta^s_i
\end{aligned}$$

$$
\begin{aligned}
&-(E_{imlk}+E_{kmli})\delta^s_j+\tau_{li}\widetilde{R}^s{}_{mjk}-\tau_{mi}\widetilde{R}^s{}_{ljk}+\tau_{lj}\widetilde{R}^s{}_{imk}\\
&-\tau_{mj}\widetilde{R}^s{}_{ilk}+\tau_{lk}\widetilde{R}^s{}_{ijm}-\tau_{mk}\widetilde{R}^s{}_{ijl}\\
&-2\beta_{ml}\widetilde{R}^s{}_{ijk}-\beta_{rj}\widetilde{R}^r{}_{mli}\delta^s_k-\beta_{ir}\widetilde{R}^r{}_{mlj}\delta^s_k\\
&+n\delta^s_i\beta_{kr}\widetilde{R}^r{}_{mlj}-n\delta^s_j\beta_{kr}\widetilde{R}^r{}_{mli}+n(\tau_{mk}\beta_{li}+\tau_{lk}\beta_{im})\delta^s_j\\
&+n(\tau_{lk}\beta_{mj}+\tau_{mk}\beta_{jl})\delta^s_i+2\beta_{ml}\beta_{ij}\delta^s_k+2\beta_{ml}\tau_{ik}\delta^s_j\\
&-2\beta_{ml}\tau_{jk}\delta^s_i+\beta_{lj}\tau_{mi}\delta^s_k-\beta_{mj}\tau_{li}\delta^s_k+\beta_{il}\tau_{mj}\delta^s_k-\beta_{im}\tau_{lj}\delta^s_k.
\end{aligned}
$$

If 1-form Π is of closed, then we obtain the following

$$
\begin{aligned}
D_mD_l\widetilde{R}^s{}_{ijk}-D_lD_m\widetilde{R}^s{}_{ijk}=&-L(\delta^s_lR_{mijk}-\delta^s_mR_{lijk})\\
&+L[g_{im}(\widetilde{R}^s{}_{ljk}-\tau_{lk}\delta^s_j+\tau_{jk}\delta^s_l)\\
&+g_{jm}(\widetilde{R}^s{}_{ilk}-\tau_{ik}\delta^s_l+\tau_{lk}\delta^s_i)\\
&+g_{km}(\widetilde{R}^s{}_{ijl}-\tau_{il}\delta^s_j+\tau_{jl}\delta^s_i)\\
&-g_{il}(\widetilde{R}^s{}_{mjk}-\tau_{mk}\delta^s_j+\tau_{jk}\delta^s_m)\\
&-g_{jl}(\widetilde{R}^s{}_{imk}-\tau_{ik}\delta^s_m+\tau_{mk}\delta^s_i)\\
&-g_{kl}(\widetilde{R}^s{}_{ijm}-\tau_{im}\delta^s_j+\tau_{jm}\delta^s_i)]\\
&+\delta^s_lE_{mijk}-\delta^s_mE_{lijk}+(E_{ijml}+E_{ijlm})\delta^s_k\\
&-(E_{imlk}+E_{kmli})\delta^s_j+\tau_{li}\widetilde{R}^s{}_{mjk}-\tau_{mi}\widetilde{R}^s{}_{ljk}\\
&+\tau_{lj}\widetilde{R}^s{}_{imk}-\tau_{mj}\widetilde{R}^s{}_{ilk}+\tau_{lk}\widetilde{R}^s{}_{ijm}-\tau_{mk}\widetilde{R}^s{}_{ijl}
\end{aligned}\tag{3.3}
$$

where $E_{mijk}=\tau_{mr}\widetilde{R}^r{}_{ijk}$.

For the convenience, we only consider the case of special projective semi-symmetric connection transformation [16] for Lemma 3.1. From the formula (3.3), one requires the contraction of this with g_{hs} and arrives at

$$
\begin{aligned}
(\widetilde{R}\cdot\widetilde{R})_{hijklm}=&-L(\tilde{g}_{hl}R_{mijk}-\tilde{g}_{hm}R_{lijk})+\tilde{g}_{hl}E_{mijk}-\tilde{g}_{hm}E_{lijk}\\
&+\tilde{g}_{hk}(E_{ijlm}+E_{jilm})-\tilde{g}_{hj}(E_{lmik}+E_{lmki})+\tau_{il}\widetilde{R}_{hmjk}-\tau_{im}\widetilde{R}_{hljk}\\
&+\tau_{jl}\widetilde{R}_{himk}-\tau_{jm}\widetilde{R}_{hilk}+\tau_{kl}\widetilde{R}_{hijm}-\tau_{mk}\widetilde{R}_{hijl}\\
&+L[g_{mi}(\widetilde{R}_{hljk}+\tilde{g}_{hk}\tau_{lj}-\tilde{g}_{hj}\tau_{lk})+g_{mj}(\widetilde{R}_{hilk}+\tilde{g}_{hk}\tau_{il}-\tilde{g}_{hl}\tau_{ik})\\
&+g_{mk}(\widetilde{R}_{hijl}+\tilde{g}_{hl}\tau_{ij}-\tilde{g}_{hj}\tau_{li})-g_{li}(\widetilde{R}_{hmjk}+\tilde{g}_{hk}\tau_{mj}-\tilde{g}_{hj}\tau_{mk})\\
&-g_{jl}(\widetilde{R}_{himk}+\tilde{g}_{hk}\tau_{im}-\tilde{g}_{hm}\tau_{ik})-g_{kl}(\widetilde{R}_{hijm}+\tilde{g}_{hm}\tau_{ij}-\tilde{g}_{hj}\tau_{im})].
\end{aligned}\tag{3.4}
$$

We now consider the symmetrization of (3.4) with respect to h,i and contract the results formula with $\tilde{g}^{hl}$, by a direct computation in a similar way posed in [1] and by using the notation $D_{ji}=g_{js}\tilde{g}^{rs}\tau_{ri}-g_{is}\tilde{g}^{rs}\tau_{rj}$, then we get the following

$$
\begin{aligned}
&(n+1)E_{mijk}+E_{jikm}+E_{kimj}+\tilde{g}_{ik}A_{jm}-\tilde{g}_{ij}A_{km}+\eta\widetilde{R}_{imjk}-nLR_{mijk}\\
&\quad-L[\tilde{g}^{rs}g_{rs}(\tilde{g}_{ik}\tau_{mj}-\tilde{g}_{ij}\tau_{mk})+n(g_{jm}\tau_{ki}-g_{km}\tau_{ji})+\tilde{g}_{ji}D_{km}\\
&\quad+\tilde{g}_{rs}\tau_{rs}(g_{km}\tilde{g}_{ji}-g_{jm}\tilde{g}_{ki})+\tilde{g}_{ki}D_{mj}+\tilde{g}_{mi}D_{kj}]=0.
\end{aligned}
$$

For vanishing the curvature tensor terms offered in this formula above, we permutating the indices m, j, k, and contract indices j, i for connection D, then one gets

$$(2n+1)\tilde{A}_{mk} = -3(n-2)LD_{mk} \tag{3.5}$$

where $\tilde{A}_{mk} = A_{mk} - A_{km}$. Combining this formula with the related results in [1], one can easily derive Lemma 3.1. This completes the proof of Lemma 3.1. ∎

By using Lemma 3.1, we can derive Theorem 3.1 as follows.

Theorem 3.1. *Let $\Psi : (M, g) \to (\widetilde{M}, \tilde{g})$ be a semi-symmetric projective connection mapping of a pseudo-symmetric manifold (M, g) onto a manifold $(\widetilde{M}, \tilde{g})$. Then the manifold $(\widetilde{M}, \tilde{g})$ is a locally pseudo-symmetric manifold.*

Proof. Since we can regard the special semi-symmetric projective connection mapping as the related transformation which keeps unchanged the geodesic lines, in addition, we also obtain the same results for the special semi-symmetric projective connection, which gives for a diffeomorphism f in [1]. Then, we conclude that Theorem 3.1 is tenable. ∎

4. Examples

It is well known that the algebraic characteristics of semi-symmetric spaces is given by the formula $R{\cdot}R = 0$. This paper can be regarded in this setting as an application of the celebrated Osserman conjecture.

Example 4.1. Let a positive function $f(x^1, x^2, \cdots, x^{n+1})$ be defined on $R^{n+1} \setminus \{0\}$ by

$$f(x^1, x^2, \cdots, x^{n+1}) = \left[\sum_{i=1}^{n+1} (x^i)^2\right]^{\frac{1}{2}} - \frac{1}{2}x^1.$$

Define a quadratic form as

$$g_{ij} = \frac{1}{2}\frac{\partial^2 H}{\partial y^i \partial y^j},$$

$i, j \in \{1, 2, \cdots, n+1\}$, where $H = f^2$. By a direction computation, one arrives at the triple (R^{n+1}, H, G) is a pseudo-semi Riemannian manifold.

Example 4.2. Consider a torus $M = \mathbb{C}/\Gamma_1 \times \mathbb{C}/\Gamma_2$, $\gamma = f(z)dz \wedge d\bar{z} + dz \wedge d\bar{w} + dw \wedge d\bar{z}$, where z, w are holomorphic coordinates on each complex plane and f is a smooth real positive valued function on $\mathbb{C}\Gamma_1$. It is easy to see that γ defines an indefinite Kähler metric tensor g on M. Moreover, one can derive that this metric tensor is also Einstein and self-dual.

Acknowledgement. The authors are supported by a Grant-in-Aid for Scientific Research from Nanjing University of Science and Technology (XKF09042, KN11008). The second author is also supported by NNSF of Province and China (BK2006209, 10771102) and would like to thank Professors H. Z. Li, Z. Z. Sun, Z. L. Li, H. Z. Song and X. P. Yang for their encouragement and help.

References

[1] F. Defever and R. Deszcz, A note on geodesic mappings of pseudo-symmetric Riemannian manifolds, *Colloq. Math.* **62** (1991), no. 2, 313–319.

[2] R. Deszcz and W. Grycak, On some class of warped product manifolds, *Bull. Inst. Math. Acad. Sinica* **15** (1987), no. 3, 311–322.

[3] R. Deszcz and M. Hotloś, On geodesic mappings in pseudo-symmetric manifolds, *Bull. Inst. Math. Acad. Sinica* **16** (1988), no. 3, 251–262.

[4] A. Friedmann and J. A. Schouten, Über die Geometrie der halbsymmetrischen Übertragungen, *Math. Z.* **21** (1924), no. 1, 211–223.

[5] J. Mikeš, Geodesic and holomorphically projective mappings of special Riemannian spaces, PhD. Thesis, Odessa Univ., 1979.

[6] J. Mikeš, Geodesic mappings of affine-connected and Riemannian spaces, *J. Math. Sci.* **78** (1996), no. 3, 311–333.

[7] J. Mikeš, Geodesic mappings of semi-symmetric Riemannian spaces, Dep. VINITI, 3924 (1976).

[8] J. Mikeš, Geodesic mappings of special Riemannian spaces, in *Topics in Differential Geometry, Vol. I, II (Debrecen, 1984)*, 793–813, Colloq. Math. Soc. János Bolyai, 46, North-Holland, Amsterdam.

[9] J. Mikeš, Izv. Vyssh. Uchebn. Zaved. Mat. **1994**, no. 2, 37–43; Translation in Russian Math. (Iz. VUZ) **38** (1994), no. 2, 35–41.

[10] J. Mikeš, Zh. Radulovich and M. Khaddad, Izv. Vyssh. Uchebn. Zaved. Mat. **1996**, no. 10, 30–35; Translation in Russian Math. (Iz. VUZ) **40** (1996), no. 10, 28–32 (1997).

[11] N. S. Sinyukov, On geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces, *Dokl. Akad. Nauk SSSR* (N.S.) **98** (1954), 21–23.

[12] N. S. Sinyukov, Geodesic mappings of Riemannian spaces, Tr. III Vse. Mat. S'ezda, **1** (1956), 167–168.

[13] Z. I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,\,Y)\cdot R = 0$. I. The local version, *J. Differential Geom.* **17** (1982), no. 4, 531–582 (1983).

[14] Z. I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. II. Global versions, *Geom. Dedicata* **19** (1985), no. 1, 65–108.

[15] K. Yano, On semi-symmetric metric connection, *Rev. Roumaine Math. Pures Appl.* **15** (1970), 1579–1586.

[16] P. B. Zhao and L. X. Shangguan, On the semi-symmetric connection, *J. of Henan Normal University* **22** (1994), 14–17.

[17] P. B. Zhao, Some properties of the special semi-symmetric connection, *Henan Science*, **16** (1998), 5–7.

[18] P. B. Zhao, The invariant of projective semi-symmetric connection, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **15** (1998), no. 2, 298–302.

[19] P. B. Zhao, Invariant of projective transformation of semi-symmetric metric-recurrent connections and curvature tensor expressions, G*ongcheng Shuxue Xuebao* **17** (2000), no. 2, 105–108.

[20] P. B. Zhao, H. Song and X. Yang, Some invariant properties of semi-symmetric metric recurrent connections and curvature tensor expressions, *Chinese Quart. J. Math.* **19** (2004), no. 4, 355–361.

[21] P. B. Zhao and L. Jiao, Conformal transformations on Carnot-Caratheodory spaces, *Nihonkai Math. J.* **17** (2006), no. 2, 167–185.