BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33***(2) (2010), 281–294*

On the Existence of T-Direction and Nevanlinna Direction of K-Quasi-Meromorphic Mapping Dealing with Multiple Values

[1]HONG-YAN XU AND [2]TANG-SEN ZHAN

[1,2]Department of Informatics and Engineering, Jingdezhen Ceramic Institute,
Jingdezhen, Jiangxi, 333403, China
[1]xhyhhh@126.com, [2]ztangsen@yahoo.com.cn

Abstract. In this paper, by using Ahlfors' theory of covering surfaces, we prove that for quasi-meromorphic mapping f satisfying

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = +\infty,$$

there exists at least one T-direction of f dealing with multiple values. We also prove that there exists at least one Nevanlinna direction of f dealing with multiple values which is also T-direction of f dealing with multiple values under the same condition.

2000 Mathematics Subject Classification: 30D60

Key words and phrases: K-quasi-meomrophic mapping, multiple value, Nevanlinna direction.

1. Introduction, definitions and results

In 1997, the value distribution theory of meromorphic functions due to Nevanlinna (see [4, 9] for standard references) was extended to the corresponding theory of quasi-meromorphic mappings by Sun and Yang [1, 7]. The singular direction for f is one of the main objects studied in the theory of value distribution of quasi-meromorphic mappings. In [7], Sun and Yang obtained an existence theorem of the Borel direction through the filling disc theorem of quasi-meromorphic mappings. Later, several types of singular directions have been introduced in the literature. In 1999, Chen and Sun [1] defined Nevanlinna directions of quasi-meromorphic mappings on the complex plane and proved that there exists at least one Nevanlinna direction for quasi-meromorphic mappings of infinite order and it is also one Borel direction with respect to the type function. In 2004, Liu and Yang [6] studied

Communicated by Lee See Keong.
Received: June 28, 2008; *Revised:* July 11, 2009.

the connections between the Julia direction and the Nevanlinna direction of quasi-meromorphic mappings by applying a fundamental inequality of an angular domain of quasi-meromorphic mappings.

For a meromorphic function f, Zheng [11] introduced a new singular direction called a T-direction conjectured that a transcendental meromorphic function f must have at least one T-direction and proved that

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty.$$

This result was later proved by Guo, Zheng and Ng [2] by using Ahlfors-Shimizu character $T(r,\Omega)$ of a meromorphic function in an angular domain Ω. Zheng [12] raised the open problem that there are similar results for algebroid functions. Xuan [8] investigated the above problem and proved the existence of T-direction of algebroid function dealing with multiple values. Thus a natural question is: Are there similar results for K-quasi-meromorphic mappings? Recently, Li and Gu [5] proved that for a K-quasi-meromorphic mapping f satisfying

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty,$$

there exists at least one Nevanlinna direction. However, it was not discussed whether there exists one Nevanlinna direction dealing with its multiple values or not. In this paper we investigate this very problem. In the following part, some definitions and notations are given, which can be found in [7].

Definition 1.1. [7] *Let f be a complex and continuous functions in a region D. If for any rectangle $R=\{x+iy;\ a<x<b,\ c<y<d\}$ in D, $f(x+iy)$ is an absolutely continuous function of y for almost every $x\in(a,b)$, and $f(x+iy)$ is an absolutely continuous function of x for almost every $y\in(c,d)$, then f is said to be absolutely continuous on lines in the region D. We also call that f is ACL in D.*

Definition 1.2. [7, Definition 1.1] *Let f be a homemorphism from D to D'. If*

(i) *f is ACL in D,*
(ii) *there exists $K\geq 1$ such that $f(z)=u(x,y)+iv(x,y)$ satisfies $|f_z|+|f_{\bar z}|\leq K(|f_z|-|f_{\bar z}|)$ a. e. in D, then f is called an univalent K-quasiconformal mapping in D. If D' is a region on Riemann sphere V, then f is named an univalent K-quasi-meromorphic mapping in D.*

Definition 1.3. [7, Definition 1.2] *Let f be a complex and continuous function in the region D. For every point z_0 in D, if there is a neighborhood $U(\subset D)$ and a positive integer n depending on z_0, such that*

$$F(z)=\begin{cases}(f(z))^{\frac{1}{n}}, & f(z_0)=\infty,\\ (f(z)-f(z_0))^{\frac{1}{n}}+f(z_0), & f(z_0)\neq\infty.\end{cases}$$

is an univalent K-quasi-meromorphic mapping, then f is named n-valent K-quasi-meromorphic mapping at point z_0. If f is n-valent K-quasi-meromorphic at every point of D, then f is called a K-quasi-meromorphic mapping in D.

Let V be the Riemann sphere whose diameter is 1. For any complex number a, let $n(r,a)$ be the number of zero points of $f(z)-a$ in disc $|z|<r$, counted according

to their multiplicities, $\bar{n}^{l)}(r,a)$ be the number of distinct zeros of $f(z)-a$ with multiplicity $\leq l$ in disc $|z|<r$. Let F_r be the covering surface $f(z)=u(x,y)+iv(x,y)$ on sphere V and $S(r,f)$ be the average covering times of F_r to V,

$$S(r,f)=\frac{|F_r|}{|V|}=\frac{1}{\pi}\int_0^r\int_0^{2\pi}\frac{|f_z|^2-|f_{\bar{z}}|^2}{(1+|f|^2)^2}rd\varphi dr,$$

where $|F_r|$ and $|V|$ are the areas of F_r and V respectively,

$$T(r,f)=\int_0^r\frac{S(r,f)}{r}dr,$$

$$N(r,a)=\int_0^r\frac{n(t,a)-n(0,a)}{t}dt+n(0,a)\log r,$$

$$\bar{N}^{l)}(r,a)=\int_0^r\frac{\bar{n}^{l)}(t,a)-\bar{n}^{l)}(0,a)}{t}dt+\bar{n}^{l)}(0,a)\log r.$$

Let $\Omega(\varphi_1,\varphi_2)=\{z\in\mathbb{C}:\varphi_1<\arg z<\varphi_2\}$ $(0\leq\varphi_1<\varphi_2\leq 2\pi)$, we denote

$$S(r,\varphi_1,\varphi_2;f)=\frac{|F_r|}{|V|}=\frac{1}{\pi}\int_0^r\int_{\varphi_1}^{\varphi_2}\frac{|f_z|^2-|f_{\bar{z}}|^2}{(1+|f|^2)^2}rd\varphi dr,$$

$$T(r,\varphi_1,\varphi_2;f)=\int_0^r\frac{S(r,\varphi_1,\varphi_2;f)}{r}dr,$$

when $\varphi_1=0$, $\varphi_2=2\pi$, we note $S(r,0,2\pi;f)=S(r,f)$, $T(r,0,2\pi;f)=T(r,f)$.

For any complex number a, let $n(r,\varphi_1,\varphi_2;a)$ be the number of zero points of $f(z)-a$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$, counted according to their multiplicities, $\bar{n}^{l)}(r,\varphi_1,\varphi_2;a)$ be the number of distinct zeros of $f(z)-a$ with multiplicity $\leq l$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$. We define

$$N(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{n(t,\varphi_1,\varphi_2;a)-n(0,\varphi_1,\varphi_2;a)}{t}dt+n(0,\varphi_1,\varphi_2;a)\log r,$$

$$\bar{N}^{l)}(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{\bar{n}^{l)}(t,\varphi_1,\varphi_2;a)-\bar{n}^{l)}(0,\varphi_1,\varphi_2;a)}{t}dt+\bar{n}^{l)}(0,\varphi_1,\varphi_2;a)\log r.$$

Next we give the definitions concerning the Nevanlinna direction of K-quasi-meromorphic mappings dealing with multiple values.

Definition 1.4. *Let f be a K-quasi-meromorphic mapping and $l(\geq 3)$ be a positive integer. Then we call $\Theta^{l)}(a,\varphi_0)$ the deficiency of the value a in the direction $\Delta(\varphi_0)$: $\arg z=\varphi_0, 0\leq\varphi_0<2\pi$. We call a the deficiency value of f in the direction $\Delta(\varphi_0)$ if $\Theta^{l)}(a,\varphi_0)>0$, where*

$$\Theta^{l)}(a,\varphi_0)=1-\limsup_{\varepsilon\to+0}\limsup_{r\to\infty}\frac{\bar{N}^{l)}(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;a)}{T(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;f)}.$$

Definition 1.5. *We call $\Delta(\varphi_0):\arg z=\varphi_0$ the Nevanlinna direction of f dealing with multiple values if*

$$\sum_{a\in\mathbb{C}\cup\{\infty\}}\Theta^{l)}(a,\varphi_0)\leq\frac{2(l+1)}{l}$$

holds for any finitely many deficient value a, where $l(\geq 3)$ is a positive integer.

Definition 1.6. *Let f be the K-quasi-meromorphic mapping, and $l(l \geq 3)$ be a positive integer. If, for arbitrary $\varepsilon > 0$ $(0 < \varepsilon < \frac{\pi}{2})$, we have that*

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \varphi_0 - \varepsilon, \varphi_0 + \varepsilon, a)}{T(r, f)} > 0$$

holds for any complex value a except at most 2 possible exceptions, then the half line $B : \arg z = \varphi_0 (0 \leq \varphi_0 \leq 2\pi)$ is called a T-direction of f dealing with multiple values.

Now, we will give an existence theorem of T-direction of K-quasi-meromorphic mapping f dealing with multiple values as follows.

Theorem 1.1. *Let f be the K-quasi-meromorphic mapping and $l(\geq 3)$ be a positive integer. If*

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

then there exists at least one T-direction dealing with multiple values of f.

A latest result of Zhang [10] shows the connection between T-direction and Borel direction of meromorphic function f. For K-quasi-meromorphic mapping f satisfying

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

we raise an interesting problem: What is the relationship between Nevanlinna direction and T-direction of f dealing with multiple values?

In this paper, we investigate the problem and obtain the following result:

Theorem 1.2. *Let f be the K-quasi-meromorphic mapping and $l(\geq 3)$ be a positive integer. If $f(z)$ satisfies*

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

There exists at least one direction which is both one Nevanlinna direction of f for multiple values and one T-direction of f for multiple values.

2. Some lemmas

Let F be a finite covering surface of F_1, F is bounded by a finite number of analytic closed Jordan curves, its boundary is denoted by ∂F. We call the part of ∂F, which lies the interior of F_1, the relative boundary of F, and denote its length by L. Let D be a domain of F_1, its boundary consists of finite number of points or analytic closed Jordan curves, and $F(D)$ be the part of F, which lies above D. We denote the area of $F, F_1, F(D)$ and D by $|F|, |F_1|, |F(D)|$ and $|D|$, respectively. We call

$$S = \frac{|F|}{|F_1|}, \qquad S(D) = \frac{|F(D)|}{|D|}$$

the mean covering numbering of F relative to F_1, D, respectively.

Lemma 2.1. [2, Lemma 4] *Let F be a simply connected finite covering surface on the unit sphere V, $D_j (j = 1, 2, \ldots, q)$ be $q(\geq 3)$ disjoint disks with radius $\delta(> 0)$, and $n_j^{l)}$ be the number of simply connected islands in $F(D_j)$, which consist of not more than l sheets, then*

$$\sum_{\nu=1}^{q} n_{\nu}^{l)} > \left(q - 2 - \frac{2}{l}\right) S - \frac{C}{\delta^3} L,$$

where $C = 960 + 2\pi q$ and $l \geq 3$ is a positive integer.

Lemma 2.2. [5, Lemma 2.2] *Let $f(z)$ be a K-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$, $a_1, \ldots, a_q (q \leq 3)$ are distinct points on the unit sphere V and the spherical distance of any two points is no smaller than $\gamma \in (0, 1/2)$. Let $F_0 = V \setminus \{a_1, a_2, \ldots, a_q\}$, $D = \Omega(r, \varphi_0 - \varphi, \varphi_0 + \varphi) \cap \{z : |z| > 1\} \setminus \{f^{-1}(a_1), f^{-1}(a_2), \ldots, f^{-1}(a_1)\}$ and $D_r = D \cap \{z : |z| < r\}(r > 1)$, $F_r = f(D_r) \subset V$, then for any positive number φ satisfying $0 < \varphi < \delta$, we have*

$$\begin{aligned} L(\partial f(D_r)) \leq & \sqrt{2K}\pi \left[\frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi}\right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}} \\ & + \sqrt{2K\delta r}\mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) + \sqrt{2K\delta}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta). \end{aligned} \tag{2.1}$$

where F_r is the covering surface of F_0 and $L(\partial f(D_r))$ is the length of the relative boundary of F_r relative to F_0, and

$$\mu(r, \varphi_0 - \delta, \varphi_0 + \delta) = \int_{\varphi_0 - \delta}^{\varphi_0 + \delta} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f(re^{i\varphi}|^2)^2} r d\varphi.$$

Lemma 2.3. *Let $f(z)$ be a K-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$, $a_1, \ldots, a_q (q \leq 3)$ are distinct points on the unit sphere V and the spherical distance of any two points is no small than $\gamma \in (0, 1/2)$. Then*

$$\begin{aligned} & \left(q - 2 - \frac{2}{l}\right) S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \leq \sum_{j=1}^{q} \bar{n}^{l)}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2\gamma^{-6}\pi^2 K}{(q - 2 - \frac{2}{l})(\delta - \varphi)} \log r \\ & \quad + \left(q - 2 - \frac{2}{l}\right) S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) + 2C\gamma^{-3}\delta^{\frac{1}{2}} K^{\frac{1}{2}} r^{\frac{1}{2}} \mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) \\ & \quad + 2C\gamma^{-3}\delta^{\frac{1}{2}} K^{\frac{1}{2}} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & \left(q - 2 - \frac{2}{l}\right) T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \leq \sum_{j=1}^{q} \bar{N}^{l)}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2\gamma^{-6}\pi^2 K}{(q - 2 - \frac{2}{l})(\delta - \varphi)} (\log r)^2 \end{aligned}$$

(2.3)

$$+\left(q-2-\frac{2}{l}\right)T(1,\varphi_0-\varphi,\varphi_0+\varphi;f)+\left(q-2-\frac{2}{l}\right)S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\log r$$
$$+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\log r+\lambda(r,\varphi_0-\delta,\varphi_0+\delta)$$

for any $\varphi, 0<\varphi<\delta$, *where* C *is a constant depending only on* $\{a_1,a_2,\dots,a_q\}$.

$$\lambda(r,\varphi_0-\delta,\varphi_0+\delta)=2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\int_1^r\left(\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right)^{\frac{1}{2}}dr,$$

$$\mu(r,\varphi_0-\delta,\varphi_0+\delta)=\int_{\varphi_0-\delta}^{\varphi_0+\delta}\frac{|f_z|^2-|f_{\bar z}|^2}{(1+|f(re^{i\varphi})|^2)^2}rd\varphi,$$

$$\lambda(r,\varphi_0-\delta,\varphi_0+\delta\le 2C\gamma^{-3}\delta^{\frac{1}{2}}\pi^{\frac{1}{2}}K^{\frac{1}{2}}(T(r,\varphi_0-\delta,\varphi_0+\delta;f))^{\frac{1}{2}})$$
$$\times\log T(r,\varphi_0-\delta,\varphi_0+\delta;f) \tag{2.4}$$

outside a set E_δ *of* r *at most, where* E_δ *consists of a series of intervals and satisfies* $\int_{E_\delta}(r\log r)^{-1}dr<+\infty$.

Proof. Under the condition of Lemma 2.2 and Lemma 2.3, we have

$$S(D_r)=S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f). \tag{2.5}$$

Using Lemma 2.1, we easily obtain

$$\left(q-2-\frac{2}{l}\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]$$
$$\le\sum_{j=1}^{q}\bar n^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+C\gamma^{-3}L(\partial(D_r)). \tag{2.6}$$

where C is a constant depending only on $\{a_1,a_2,\dots,a_q\}$.

Taking (2.1) into (2.6), we have

$$\left(q-2-\frac{2}{l}\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]$$
$$-\sum_{j=1}^{q}\bar n^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac{1}{2}}(r,\varphi_0-\delta,\varphi_0+\delta)$$
$$-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)$$

(2.7)

$$\le C\gamma^{-3}\sqrt{2K}\pi\left[\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac{1}{2}}(\log r)^{\frac{1}{2}}.$$

We denote

$$A(r,\varphi)=\left(q-2-\frac{2}{l}\right)[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]$$
$$-\sum_{j=1}^{q}\bar n^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac{1}{2}}(r,\varphi_0-\delta,\varphi_0+\delta)$$

$$-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta). \tag{2.8}$$

By (2.7) and (2.8), we have

$$A(r,\varphi)\le C\gamma^{-3}\sqrt{2K}\pi\left[\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac{1}{2}}$$
$$\times(\log r)^{\frac{1}{2}}. \tag{2.9}$$

From (2.8) we verify that $A(r,\varphi)$ is an increasing function of φ. Thus, there exists $\delta_0>0$, such that $A(r,\varphi)\le 0$ for $0<\varphi\le\delta_0$ and $A(r,\varphi)>0$ for $\varphi>\delta_0$.

We shall consider two cases in the following:

Case 1. For $\varphi>\delta_0$, by (2.9) we have

$$[A(r,\varphi)]^2$$
$$\le 2C^2\gamma^{-6}K\pi^2\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\log r. \tag{2.10}$$

By (2.8) we have

$$\frac{dA(r,\varphi)}{d\varphi}$$
$$=\left(q-2-\frac{2}{l}\right)\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}. \tag{2.11}$$

From (2.10) and (2.11) we have

$$[A(r,\varphi)]^2\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}}\cdot\frac{dA(r,\varphi)}{d\varphi},$$

i.e.,

$$d\varphi\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}}\cdot\frac{dA(r,\varphi)}{[A(r,\varphi)]^2}.$$

Integrating two sides of the inequality leads to

$$\delta-\varphi=\int_\varphi^\delta d\varphi\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}}\int_\varphi^\delta\frac{dA(r,\varphi)}{[A(r,\varphi)]^2}\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}}\cdot\frac{1}{A(r,\varphi)}.$$

Thus

$$A(r,\varphi)\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}. \tag{2.12}$$

Case 2. Because $A(r,\varphi)\le 0$ when $0<\varphi\le\delta_0$, the above inequality also holds.

By Case 1 and Case 2, we can easily get

$$A(r,\varphi)\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)},$$

for any $\varphi, 0<\varphi<\delta$. Combining with the definition of $A(r,\varphi)$, we can easily get (2.2).

By dividing r and then integrating from 1 to r on two sides of (2.2) we get

$$\begin{aligned}
&\left(q-2-\frac{2}{l}\right)T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)\\
&\le \sum_{j=1}^{q}\bar{N}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+\frac{2C^2\gamma^{-6}\pi^2K}{(q-2-2/l)\,(\delta-\varphi)}(\log r)^2\\
&\quad+\left(q-2-\frac{2}{l}\right)T(1,\varphi_0-\varphi,\varphi_0+\varphi;f)+\left(q-2-\frac{2}{l}\right)S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\log r\\
&\quad+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\log r+2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\int_1^r\left[\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right]^{\frac{1}{2}}dr.
\end{aligned}\tag{2.13}$$

From the definitions of $S(r,\varphi_1,\varphi_2;f)$, $\mu(r,\varphi_0-\delta,\varphi_0+\delta)$ and $\lambda(r,\varphi_0-\delta,\varphi_0+\delta)$, and Schwarz's inequality we get

$$\begin{aligned}
(\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2&=4C^2\gamma^{-6}\delta K\left[\int_1^r\left(\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right)^{\frac{1}{2}}dr\right]^2\\
&\le 4C^2\gamma^{-6}\delta K\int_1^r\mu(r,\varphi_0-\delta,\varphi_0+\delta)dr\int_1^r r^{-1}dr\\
&\le 4C^2\gamma^{-6}\pi\delta K\log r\int_1^r dS(r,\varphi_0-\delta,\varphi_0+\delta;f)\\
&\le 4C^2\gamma^{-6}\pi\delta KS(r,\varphi_0-\delta,\varphi_0+\delta;f)\log r\\
&=4C^2\gamma^{-6}\pi\delta K\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{dr}r\log r.
\end{aligned}\tag{2.14}$$

Choosing $r_0, r_0>0$ such that $T(r_0,\varphi_0-\delta,\varphi_0+\delta;f)>1$, and setting $E_\delta=\{r_0<r<\infty:(\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2>4C^2\gamma^{-6}\pi\delta KT(r,\varphi_0-\delta,\varphi_0+\delta;f)(\log T(r,\varphi_0-\delta,\varphi_0+\delta;f))^2\}$, thus we have

$$\begin{aligned}
\int_{E_\delta}\frac{dr}{r\log r}&\le\int_{E_\delta}\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{T(r,\varphi_0-\delta,\varphi_0+\delta;f)(\log T(r,\varphi_0-\delta,\varphi_0+\delta;f))^2}\\
&\le[\log T(r_0,\varphi_0-\delta,\varphi_0+\delta;f)]^{-1}<+\infty.
\end{aligned}\tag{2.15}$$

Then when $r>r_0$ and $r\notin E_\delta$, (2.3) holds. Thus, we completed the proof of Lemma 2.3. ∎

Lemma 2.4. *Let $F(r)$ be a positive nondecreasing function defined for $1<r<+\infty$ and satisfy*

$$\limsup_{r\to\infty}\frac{F(r)}{(\log r)^2}=+\infty.\tag{2.16}$$

Then, for any subset $E\subset(1,+\infty)$ satisfying

$$\int_E\frac{dr}{r\log r}<\frac{1}{\kappa}(\kappa\ge 2),$$

$$\limsup_{r\to\infty,r\in(1,+\infty)\setminus E}\frac{F(r)}{(\log r)^2}=+\infty.$$

Proof. Otherwise, there exists some set $E \subset (1, +\infty)$ with

$$\int_E \frac{dr}{r \log r} := t < \frac{1}{\kappa},$$

such that

$$\limsup_{r\to\infty, r\in(1,+\infty)\setminus E} \frac{F(r)}{(\log r)^2} < +\infty. \tag{2.17}$$

For any $\{r'_n\} \subset (1, +\infty), \{r'_n\} \to \infty, m \geq 2$, we have

$$\begin{aligned}\int_{[r'_n, m(r'_n)^2]\setminus E} \frac{dr}{r\log r} &\geq \int_{[r'_n, m(r'_n)^2]} \frac{dr}{r\log r} - \int_E \frac{dr}{r\log r}\\ &= \int_{\log r'_n}^{\log m + 2\log r'_n} \frac{dr}{r} - t\\ &\geq \frac{\log m + \log r'_n}{\log m + 2\log r'_n} - t \geq \frac{1}{\kappa} - t > 0.\end{aligned}$$

Then there exists $r''_n \in [r'_n, m(r'_n)^2] \setminus E$. By the nondecreasing property of $F(r)$ we have

$$\frac{F(r''_n)}{(\log r''_n)^2} \geq \frac{F(r'_n)}{[\log(mr'_n)^2]^2} = \frac{F(r'_n)}{(\log r'_n)^2(2 + \log m/\log r'_n)^2}.$$

Combining this with (2.17) we have

$$\begin{aligned}\limsup_{r'_n\to\infty} \frac{F(r'_n}{(\log r'_n)^2} &= 4\limsup_{r'_n\to\infty} \frac{F(r'_n)}{(\log r'_n)^2(2 + \log m/\log r'_n)^2}\\ &\leq 4\limsup_{r''_n\to\infty} \frac{F(r''_n}{(\log r''_n)^2}\\ &\leq 4 \limsup_{r\to\infty, r\in(1,+\infty)\setminus E} \frac{F(r)}{(\log r)^2} < +\infty.\end{aligned}$$

Since $\{r'_n\}$ is arbitrary, the above inequality contradicts (2.16). So Lemma 2.4 holds. ∎

Lemma 2.5. *Let $f(z)$ be the K-quasi-meromorphic mapping, $l(\geq 3)$ be a positive integer number and $m(m > 1)$ be a positive integer. Put $\varphi_0 = 0, \varphi_1 = 2\pi/m, \ldots, \varphi_{m-1} = (m-1)(2\pi/m)$. Let*

$$\Delta(\varphi_i) = \left\{ z \mid |\arg z - \varphi_i| < \frac{3\pi}{m} \right\} \quad (0 \leq i \leq m-1).$$

Then there exists an angular domain $\Delta(\varphi_i)$ among these m angular domains $\Delta(\varphi_i)$ $(i = 0, 1, \ldots, m-1)$ such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\varphi_i), a)}{T(r, f)} > 0$$

for any value of a with 2 possible exceptions.

Proof. Suppose that the conclusion is false. Then for every $\Delta(\varphi_i)$ ($i = 0, 1, \ldots, m-1$), there exists 3 exceptional values $\{a_j\}_{j=1}^{3}$ such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\varphi_i), a_j)}{T(r, f)} = 0. \tag{2.18}$$

Let β be any positive integer. Put $\varphi_{i,k} = 2\pi i/m + 2k\pi/\beta m$, $0 \le i \le m-1, 0 \le k \le \beta - 1$. For any given number $r > 1$, writing

$$\Delta_{i,k}(r) = \{z||z| < r, \varphi_{i,k} < \varphi_{i,k+1}\}.$$

Then

$$\{|z| < r\} = \sum_{k=0}^{\beta-1} \sum_{i=0}^{m-1} \Delta_{i,k}(r).$$

Put

$$\overline{\Delta}_i = \left\{ z| \frac{\varphi_{i,0} + \varphi_{i,1}}{2} \le \arg z \le \frac{\varphi_{i+1,0} + \varphi_{i+1,1}}{2} \right\},$$
$$\Delta_i^0 = \{z|\varphi_{i,0} < \arg z < \varphi_{i+1,1}\}, \quad 0 \le i \le m-1.$$

From Lemma 2.3 we have

$$\left(1 - \frac{2}{l}\right) S(r, \overline{\Delta}_i, f) \le \sum_{j=1}^{3} \overline{n}^{l)}(r, \Delta_i^0, a_i^j) + O(\log r) + h_i r^{\frac{1}{2}} \mu^{\frac{1}{2}}(r, \varphi_{i,0}, \varphi_{i+1,1}),$$

where h_i is a constant depending only on $\{a_1, a_2, a_3\}$.

Add from $i = 0$ to $m-1$ and divide both sides of this inequality by r and integrate both sides from 1 to r, then we obtain the following inequality

$$\begin{aligned}\left(1 - \frac{2}{l}\right) T(r, f) \le& \sum_{j=1}^{3} \sum_{i=0}^{m-1} \overline{N}^{l)}(r, \Delta_i^0, a_i^j) + O((\log r)^2) \\ &+ \sum_{i=0}^{m-1} \lambda(r, \varphi_{i,0}, \varphi_{i+1,1}),\end{aligned} \tag{2.19}$$

where

$$\lambda(r, \varphi_{i,0}, \varphi_{i+1,1}) \le h_i \left[\frac{2\pi}{m}(1 + \frac{1}{\beta})\right]^{\frac{1}{2}} (T(r, \varphi_{i,0}, \varphi_{i+1,1}; f))^{\frac{1}{2}} \log T(r, \varphi_{i,0}, \varphi_{i+1,1}; f)$$

at most outside a set E_1 of r, where E_i satisfies

$$\int_{E_i} (r \log r)^{-1} dr < +\infty (i = 0, 1, \ldots, m-1).$$

For any $i \in \{0, 1, \ldots, m-1\}$ and $\kappa \ge 2$, we can choose $r_i > 0$ such that $T(r_i, \varphi_{i,0}, \varphi_{i+1,1}; f) > e^{\kappa m}$. Then from the proof of Lemma 2.3 we have

$$\int_{E_i} \frac{1}{r \log r} dr \le \frac{1}{\log T(r, \varphi_{i,0}, \varphi_{i+1,1}; f)} < \frac{1}{\kappa m} < \frac{1}{\kappa}.$$

Put $E = \cup_{i=0}^{m-1} E_i$, then

$$\int_E \frac{1}{r \log r} dr \le \sum_{i=0}^{m-1} \int_{E_i} \frac{1}{r \log r} dr \le m \max_{0 \le i \le m-1} \int_{E_i} \frac{1}{r \log r} dr < m \cdot \frac{1}{\kappa m} < \frac{1}{\kappa}.$$

By applying Lemma 2.4 to this set E and $T(r,f)$, we obtain that

$$\limsup_{r\to\infty, r\in(1,\infty)\setminus E} \frac{T(r,f)}{(\log r)^2} = +\infty.$$

There exists $\{r_n\} \in (r,+\infty)\setminus E$,

$$\lim_{n\to\infty} \frac{T(r_n,f)}{(\log r_n)^2} = +\infty.$$

For this sequence $\{r_n\}$, by (2.19) we have

$$\begin{aligned}\left(1-\frac{2}{l}\right)T(r_n,f) \le \sum_{j=1}^{3}\sum_{i=0}^{m-1}\overline{N}^{l)}(r_n,\Delta_i^0,a_i^j) + O((\log r_n)^2)\\ + \sum_{i=0}^{m-1}\lambda(r_n,\varphi_{i,0},\varphi_{i+1,1}).\end{aligned} \tag{2.20}$$

From (2.18), by dividing both sides of (2.20) by $T(r_n,f)$ and letting $n\to\infty$, we obtain $1-(2/l)\le 0$. This contradicts $l\ge 3$. So Lemma 2.5 holds. ∎

3. Proof of theorems

Proof of Theorem 1.1. By Lemma 2.5, for any given positive integer m, there exists

$$\Delta_m = \left\{z \,\middle|\, |\arg z - \theta_m| < \frac{3\pi}{m}\right\}$$

such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r,\Delta(\theta_m),a)}{T(r,f)} > 0,$$

for any value of a with 2 possible exceptions at most. By choosing a subsequence, we can assume that $\theta_m \to \theta_0$ when $m\to\infty$. Then $B : \arg z = \theta_0$ has the properties of Theorem 1.1. ∎

Proof of Theorem 1.2. Suppose $\delta\in(0,2\pi)$, we can choose $r_0>0$ such that $T(r_0,\varphi_0-\delta,\varphi+\delta;f) > e^{\kappa}$. Then from the proof of Lemma 2.3, we have

$$\int_{E_\delta} \frac{1}{r\log r}dr \le \frac{1}{\log T(r_0,\varphi_0-\delta,\varphi+\delta;f)} < \frac{1}{\kappa}.$$

By applying Lemma 2.4 to this set E_δ and $T(r,f)$, we obtain that

$$\limsup_{r\to\infty, r\in(1,\infty)\setminus E_\delta} \frac{T(r,f)}{(\log r)^2} = +\infty.$$

There exists $\{r_n\} \in (r,+\infty)\setminus E_\delta$,

$$\lim_{n\to\infty} \frac{T(r_n,f)}{(\log r_n)^2} = +\infty. \tag{3.1}$$

By using the finite covering theorem at $[0,2\pi]$, we obtain that there exists some φ_0 such that $\varphi_0\in[0,2\pi]$ and

$$\limsup_{n\to\infty} \frac{T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)}{T(r_n,f)} > 0 \tag{3.2}$$

for an arbitrary $\varphi, 0<\varphi<\varphi_0$. Hence, we claim that the direction $\Delta(\varphi_0):\arg z=\varphi_0$ is the Nevanlinna direction of $f(z)$ dealing with multiple values which is also the T-direction of $f(z)$ dealing with multiple values.

First, we prove that the direction $\Delta(\varphi_0):\arg z=\varphi_0$ is one Nevanlinna direction of $f(z)$ dealing with multiple values.

Otherwise, for an arbitrary positive number $\varepsilon_0>0$, there exists some complex numbers $\{a_j\}(j=1,2,\dots,q)(q\geq 3)$ such that the following inequality holds:

$$\frac{l}{l+1}\sum_{j=1}^{q}\Theta^{l)}(a_j,\varphi_0)>2+2\varepsilon_0.$$

From Definition 1.5, we have

$$\limsup_{\varphi\to+0}\limsup_{r\to+\infty}\sum_{j=1}^{q}\frac{\overline{N}^{l)}(r,\varphi_0-\varphi,\varphi_0+\varphi;a_j)}{T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)}<q-\frac{2(l+1)}{l}-\frac{2(l+1)\varepsilon_0}{l}.$$

Therefore, there exists some φ' such that $\varphi'>0$ and the following inequality holds:

$$\limsup_{r\to+\infty}\sum_{j=1}^{q}\frac{\overline{N}^{l)}(r,\varphi_0-\varphi,\varphi_0+\varphi;a_j)}{T(r,\varphi_0-\varphi,\varphi_0+\varphi;f)}<q-\frac{2(l+1)}{l}-\frac{2(l+1)\varepsilon_0}{l} \tag{3.3}$$

for an arbitrary φ, $0<\varphi<\varphi'$.

For any φ, $0<\varphi<\varphi'$, we define an increasing function as follows:

$$T(\varphi)=\limsup_{n\to+\infty}\frac{T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)}{T(r_n,f)}. \tag{3.4}$$

From (3.2) we deduce $T(\varphi)>0$. So we have $0<T(\varphi)\leq 1$. Since the increasing of $T(\varphi)$ in interval $[0,\varphi']$ and the continuous theorem for monotonous functions, we see that all discontinuous points of $T(\varphi)$ constitute a countable set at most. Then, by Lemma 2.3, we can get

$$\begin{aligned}&\left(q-2-\frac{2}{l}\right)T(r_n,\varphi_0-\varphi,\varphi_0+\varphi;f)\\ &\leq\sum_{j=1}^{q}\bar{N}^{l)}(r_n,\varphi_0-\delta,\varphi_0+\delta;a_j)+O(\log r_n)^2\\ &\quad+O((T(r_n,\varphi_0-\delta,\varphi_0+\delta;f))^{\frac{1}{2}}\log T(r_n,\varphi_0-\delta,\varphi_0+\delta;f))\end{aligned} \tag{3.5}$$

for $0<\varphi<\delta<\varphi'$ and $r_n\notin E_\delta$.

From (3.3), (3.5) and the definition of $T(\varphi)$ we can obtain

$$\left(q-\frac{2(l+1)}{l}\right)T(\varphi)<\left(q-\frac{2(l+1)}{l}-\frac{2(l+1)\varepsilon_0}{l}\right)T(\delta). \tag{3.6}$$

From (3.2) we get

$$T(\varphi)\to T(\delta),\qquad \varphi\to\delta. \tag{3.7}$$

By combining (3.6) with (3.7), we can obtain $T(\delta)=0$. This contradicts $T(\delta)>0$. Then $\Delta(\varphi_0):\arg z=\varphi_0$ is the Nevanlinna direction of $f(z)$ dealing with multiple values.

Second, we claim that $\Delta(\varphi_0) : \arg z = \varphi_0$ is the T-direction of $f(z)$ dealing with multiple values. Otherwise, there exists $\varepsilon_0 > 0$ and 3 complex numbers a_1, a_2, a_3 satisfying

$$\lim_{r\to\infty} \frac{\overline{N}^{l)}(r, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; a_j)}{T(r, f)} = 0 \qquad (j = 1, 2, 3).$$

Then we have

$$\lim_{n\to\infty} \frac{\overline{N}^{l)}(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; a_j)}{T(r_n, f)} = 0 \qquad (j = 1, 2, 3). \tag{3.8}$$

For $\varphi \in (0, \varepsilon_0)$, we can define $T(\varphi)$ in a similar way of (3.4), then we have $0 < T(\varphi) \le 1$. By Lemma 2.3, for the above $\{r_n\} \subset (1, +\infty)\backslash E_\delta$ and $0 < \varphi < \varphi' < \delta$, we can get

$$\begin{aligned}
&\left(1 - \frac{2}{l}\right) T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\
&\le \sum_{j=1}^{3} \overline{N}^{l)}(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; a_i^j) + O((\log r_n)^2) \\
&\quad + O((T(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; f))^{\frac{1}{2}} \log T(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; f)).
\end{aligned} \tag{3.9}$$

By (3.1), (3.8), (3.9) and $l \ge 3$, we can obtain $T(\varphi) \le 0$. This contradicts $T(\varphi) > 0$. Therefore $\Delta(\varphi_0) : \arg z = \varphi_0$ is the T-direction of $f(z)$ dealing with multiple values. Thus, we complete the proof of Theorem 1.2. ∎

Acknowledgement. The authors want to thanks the referee for their thorough review. The research was supported by the Soft Scientific and Technological Research Projects of Jiangxi Province ([2008]147) and the Youth Foundation of Education Department of Jiangxi (No. GJJ10223) of China.

References

[1] T. W. Chen and D. C. Sun, Singular directions of quasi-meromorphic mappings, *Acta Math. Sci. Ser. A Chin. Ed.* **19** (1999), no. 4, 472–478.

[2] Z. S. Gao, Multiple values of quasiconformal mappings, *J. Math.* (*Wuhan*) **19** (1999), no. 2, 121–126.

[3] H. Guo, J. H. Zheng and T. W. Ng, On a new singular direction of meromorphic functions, *Bull. Austral. Math. Soc.* **69** (2004), no. 2, 277–287.

[4] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[5] C. H. Li and Y. X. Gu, A fundamental inequality for K-quasimeromorphic mappings in an angular domain and its application, *Acta Math. Sinica* (Chin. Ser.) **49** (2006), no. 6, 1279–1286.

[6] M. S. Liu and Y. Yang, The Nevanlinna direction and Julia direction of quasimeromorphic mappings, *Acta Math. Sci. Ser. A Chin. Ed.* **24** (2004), no. 5, 578–582.

[7] D. Sun and L. Yang, Value distribution of K-quasi-meromorphic mappings, *Sci. China Ser. A* **27** (1997), no. 2, 132–139.

[8] Z.-X. Xuan, On the existence of T-direction of algebroid functions: A problem of J. H. Zheng, *J. Math. Anal. Appl.* **341** (2008), no. 1, 540–547.

[9] L. Yang, *Value Distribution Theory*, Translated and revised from the 1982 Chinese original, Springer, Berlin, 1993.

[10] Q. D. Zhang, T direction and Borel direction of meromorphic functions of finite and positive order, *Acta Math. Sinica* (Chin. Ser.) **50** (2007), no. 2, 413–420.
[11] J. Zheng, On transcendental meromorphic functions with radially distributed values, *Sci. China Ser. A* **47** (2004), no. 3, 401–416.
[12] J. Zheng, *On value distribution of meromorphic functions with respect to arguments*, preprint.