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Lie Group Analysis of Magnetohydrodynamic Flow and Mass Transfer of a Visco-Elastic Fluid Over a Porous Stretching Sheet

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Abstract. An analysis is carried out to study the momentum and mass transfer characteristics in a visco-elastic fluid flow over a porous stretching sheet in the presence of a transverse magnetic field. The flow is generated solely due to the linear stretching of the sheet. The symmetry groups obtained using a special form of Lie group transformations viz. Scaling group of transformations, reduce the momentum equation and the concentration conservation equation into fourth order and second order ordinary differential equations respectively. Closed form analytical solutions have been derived for non-dimensional concentration and mass flux profiles in the form of confluent hyper geometric (Kummer's) functions, for two different cases of the boundary conditions, namely

(1) Prescribed Sheet Concentration (PSC) and
(2) Prescribed Mass Flux (PMF).

The main emphasis of this paper is to derive the final equations using the scaling group of transformations and to study the effects of the visco-elastic parameter, suction/blowing parameter, magnetic parameter, concentration and mass flux parameters and Schmidt number on the mass transfer characteristics. It has been observed that, for the case of suction and for the values of the parameters considered, an ideal combination to obtain a reduced concentration boundary layer thickness would be to choose smaller values of the visco-elastic and magnetic parameters and relatively larger value for the Schmidt number. This is seen to be more significant in the PMF case. An increase in the concentration and mass flux parameters has shown a steep decrease in the concentration boundary layer thickness.

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1. Introduction

In the recent past, considerable research has been reported on the momentum, heat and mass transfer characteristics of a visco-elastic fluid in the boundary layer formed by its flow over a stretching sheet. The study of mass transfer carried special importance in energy equation. Abel *et al.* [1] carried out an analysis to study the influence of momentum, mass and heat transfer characteristics on the flow of a visco-elastic fluid (Walter's liquid–B model) past a stretching sheet in the presence of a transverse magnetic field. An analysis was made by Acharya *et al.* [2] to determine the heat and mass transfer occurring in the laminar boundary layer on a linearly accelerating surface with temperature dependent heat source subjected to suction or blowing. Anuradha *et al.* [3] studied the effects of radiation and variable thermal conductivity on the Magnetohydrodynamic (MHD) flow of a visco-elastic fluid and heat transfer over a stretching porous sheet. The Prandtl boundary layer theory was extended for an idealized elastico-viscous liquid by Beard and Walters [4]. Hayat [5, 6] discussed the influence of thermal radiation on the MHD flow of a second grade fluid. Mukhopadhyay *et al.* [8] used the Lie group transformations to study the free convective boundary layer flow and heat transfer of a fluid with variable viscosity over a porous stretching vertical sheet in the presence of thermal radiation. Prasad *et al.* [9] studied the diffusion of chemically reactive species of a non-Newtonian fluid immersed in a porous medium over a stretching sheet. Radwan *et al.* [10] considered a steady two dimensional laminar flow and studied mass transfer over a stretching sheet with variable concentration in a transverse magnetic field. Rajgopal *et al.* [11] analysed the flow characteristics and heat transfer behaviours only in the visco-elastic fluid flow of the type Walters' liquid B and second order fluid. Rollins and Vajravelu [12] investigated the heat transfer characteristics in a visco-elastic fluid over a continuous impermeable linearly stretching sheet. Sonth *et al.* [13] studied a mathematical analysis of heat and mass phenomenon in a visco-elastic fluid flow over an accelerating stretching sheet in the presence of heat source/sink, viscous dissipation and suction/blowing. Sujit Kumar Khan *et al.* [7] studied the momentum, heat and mass transfer characteristics of a visco-elastic fluid flow over a porous sheet.

In all the above mentioned works, a suitable similarity transformation has been used in order to reduce the highly non-linear partial differential equations into ordinary differential equations. In the present study, symmetry methods are applied to the boundary value problem. The main advantage of this method is that it can be successfully applied to non-linear differential equations. The symmetries of differential equations are those continuous groups of transformations under which the differential equations remain invariant. Using these symmetry groups, fourth order and second order ordinary differential equations corresponding to the momentum and concentration conservation equations are derived. The ordinary differential equations so obtained are compared with those [13] obtained using a similarity transformation. These are found to match in the absence of a magnetic field. Closed form analytical solutions for non-dimensional concentration and mass flux profiles in the form of confluent hyper geometric (Kummer's) functions are derived for two different cases:

(1) Prescribed Sheet Concentration (PSC) and
(2) Prescribed Mass Flux (PMF).

By giving special values to the concentration parameter and the mass flux parameter, the results are compared with those of Sonth *et al.* [13] and are found to agree in the absence of the magnetic field. The velocity profiles are not described in detail as the main emphasis of this paper is to derive the final equations using the scaling group of transformations and to study the effects of the visco-elastic parameter, suction/blowing parameter, magnetic parameter, concentration and mass flux parameters and Schmidt number on the mass transfer characteristics. The effects of the flow parameters on the concentration distribution of the flow field have been studied with the help of graphs and tables.

2. Formulation of the problem

Consider a steady two dimensional boundary layer flow of an electrically conducting incompressible visco-elastic fluid in a region $y > 0$ over a stretching porous sheet. The flow is caused by the linear stretching of the sheet.

The following assumptions are made:

(1) The fluid properties are assumed to be constant.
(2) The flow is exposed to the influence of a transverse uniform magnetic field of strength B_0.
(3) The magnetic Reynolds number is considered to be small and so, the induced magnetic field is negligible.
(4) The concentration of diffusing species is very small in comparison to other chemical species.
(5) No chemical reactions take place in the fluid.

With the above assumptions, the governing basic boundary layer equations for momentum and mass transfer in Walters' liquid B [4], take the following form.
Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{2.1}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} - k_0\left[u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right] - \frac{\sigma B_0^2 u}{\rho}; \tag{2.2}$$

Concentration conservation equation:

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2}, \tag{2.3}$$

where u and v are the velocity components in the x and y directions, ν, ρ, σ, k_0 are respectively the kinematic viscosity, fluid density, electrical conductivity and coefficient of visco-elasticity, c represents the concentration and D is the diffusion coefficient of the fluid. The necessary boundary conditions are

$$\begin{aligned} &\text{At} \quad y = 0, \quad u = bx, \quad v = v_w \\ &c = c_w = c_\infty + Ax^r \qquad \text{PSC case} \end{aligned} \tag{2.4}$$

$$-D\frac{\partial c}{\partial y} = Ex^s \qquad \text{PMF case.}$$

$$\begin{aligned} &\text{As} \quad y \to \infty, u \to 0, u_y \to 0 \\ &c \to c_\infty \end{aligned} \tag{2.5}$$

where c_w and c_∞ denote the concentration at the sheet level and far away from the sheet respectively, b is the constant stretching rate, r is the concentration parameter, s is the mass flux parameter, v_w represents suction velocity across the stretching sheet when $v_w < 0$ and blowing velocity when $v_w > 0$. When $v_w = 0$, the sheet is impermeable. A and E are constants which depend on the properties of the fluid.

3. Method of solution

Introduce the following relations for u, v and c as follows:

$$\begin{aligned} &u = \frac{\partial \phi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial x}, \\ &\chi = \frac{c - c_\infty}{c_w - c_\infty} \qquad \text{PSC case} \\ &g = \frac{c - c_\infty}{c_w - c_\infty} \qquad \text{PMF case} \end{aligned} \tag{3.1}$$

where ϕ is the stream function, χ and g are the non-dimensional concentration parameters for the PSC and PMF cases respectively.

Using (3.1) in (2.2) and (2.3),

$$\begin{aligned} \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial y^2} = v\frac{\partial^3 \phi}{\partial y^3} - k_0\left[\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^3 \phi}{\partial y^3} + \frac{\partial \phi}{\partial y}\frac{\partial^4 y}{\partial x \partial y^3} - \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial \phi}{\partial x}\frac{\partial^4 \phi}{\partial y^4}\right] \\ - \frac{\sigma B_0^2}{\rho}\frac{\partial \phi}{\partial y}, \end{aligned} \tag{3.2}$$

$$\frac{r}{x}\chi\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y}\frac{\partial \chi}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial \chi}{\partial y} = D\frac{\partial^2 \chi}{\partial y^2} \qquad \text{PSC case,} \tag{3.3}$$

$$\frac{\partial \phi}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial g}{\partial y} = D\frac{\partial^2 g}{\partial y^2} \qquad \text{PMF case.} \tag{3.4}$$

The boundary conditions (2.4) and (2.5) are rewritten in the following form

$$\begin{aligned} &\text{At} \quad y = 0, \quad \frac{\partial \phi}{\partial y} = bx, \quad \frac{\partial \phi}{\partial x} = -v_w \\ &\chi = 1 \qquad \text{PSC case} \\ &-D(c_w - c_\infty)\frac{\partial g}{\partial y} = Ex^s \qquad \text{PMF case.} \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\text{As} \quad y \to \infty, \quad \frac{\partial \phi}{\partial y} \to 0, \quad \frac{\partial^2 \phi}{\partial y^2} \to 0 \\ &\chi \to 0 \qquad \text{PSC case} \\ &g \to 0 \qquad \text{PMF case.} \end{aligned} \tag{3.6}$$

3.1. Scaling group of transformations

The simplified form of Lie-group transformations, namely the scaling group of transformations as given by Mukhopadhyay *et al.* [8] are given by

$$\begin{aligned}&\Gamma : x^* = xe^{\varepsilon\alpha_1}, y^* = ye^{\varepsilon\alpha_2}, \phi^* = \phi e^{\varepsilon\alpha_3}, u^* = ue^{\varepsilon\alpha_4},\\ &v^* = ve^{\varepsilon\alpha_5}, \chi^* = \chi e^{\varepsilon\alpha_6}, g^* = ge^{\varepsilon\alpha_7}\end{aligned} \tag{3.7}$$

where α_1, α_2, α_3, α_4, α_5, α_6 and α_7 are transformation parameters.

Equation (3.7) is considered as a point-transformation transforming co-ordinates $(x, y, \phi, u, v, \chi, g)$ into the co-ordinates $(x^*, y^*, \phi^*, u^*, v^*, \chi^*, g^*)$.

Substituting (3.7) in (3.2), (3.3) and (3.4)

$$\begin{aligned}&e^{\varepsilon(\alpha_1+2\alpha_2-2\alpha_3)}\left[\frac{\partial\phi^*}{\partial y^*}\frac{\partial^2\phi}{\partial x^*\partial y^*} - \frac{\partial\phi^*}{\partial x^*}\frac{\partial^2\phi}{\partial y^2}\right]\\ &= \nu e^{\varepsilon(3\alpha_2-\alpha_3)}\frac{\partial^3\phi^*}{\partial y^{*3}} - k_0 e^{\varepsilon(\alpha_1+4\alpha_2-2\alpha_3)}\left[\frac{\partial\phi^*}{\partial y^*}\frac{\partial^4\phi^*}{\partial x^*\partial y^{*3}} - \frac{\partial\phi^*}{\partial x^*}\frac{\partial^4\phi^*}{\partial y^{*4}}\right.\\ &\left. - \frac{\partial^2\phi^*}{\partial y^{*2}}\frac{\partial^3\phi^*}{\partial x^*\partial y^{*2}} + \frac{\partial^2\phi^*}{\partial x^*\partial y^*}\frac{\partial^3\phi^*}{\partial y^{*3}}\right] - \frac{\sigma B_0^2}{\rho}e^{\varepsilon(\alpha_2-\alpha_3)}\frac{\partial\phi^*}{\partial y^*},\end{aligned} \tag{3.8}$$

$$e^{\varepsilon(\alpha_1+\alpha_2-\alpha_3-\alpha_6)}\left[\frac{r}{x^*}\chi^*\frac{\partial\phi^*}{\partial y^*} + \frac{\partial\phi^*}{\partial y^*}\frac{\partial\chi^*}{\partial x^*} - \frac{\partial\phi^*}{\partial x^*}\frac{\partial\chi^*}{\partial y^*}\right] = De^{\varepsilon(2\alpha_2-\alpha_6)}\frac{\partial^2\chi^*}{\partial y^{*2}}, \tag{3.9}$$

$$e^{\varepsilon(\alpha_1+\alpha_2-\alpha_3-\alpha_7)}\left[\frac{\partial\phi^*}{\partial y^*}\frac{\partial g^*}{\partial x^*} - \frac{\partial\phi^*}{\partial x^*}\frac{\partial g^*}{\partial y^*}\right] = De^{\varepsilon(2\alpha_2-\alpha_7)}\frac{\partial^2 g^*}{\partial y^{*2}}. \tag{3.10}$$

Since the system remains invariant under the group of transformations Γ, the following relations among the parameters are deduced.

$$\begin{aligned}\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 = \alpha_1 + 4\alpha_2 - 2\alpha_3 = \alpha_2 - \alpha_3,\\ \alpha_1 + \alpha_2 - \alpha_3 - \alpha_6 = 2\alpha_2 - \alpha_6,\\ \alpha_1 + \alpha_2 - \alpha_3 - \alpha_7 = 2\alpha_2 - \alpha_7.\end{aligned}$$

The relation $3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3$ gives the value $\alpha_2 = 0$. Hence, $\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3$ gives $\alpha_1 = \alpha_3$. Thus, the boundary conditions become

$$\begin{aligned}&\text{At}\quad y^* = 0, \quad \frac{\partial\phi^*}{\partial y^*} = bx, \quad \frac{\partial\phi^*}{\partial x^*} = -v_w\\ &\chi^* = 1 \qquad \text{PSC case}\\ &-D(c_w - c_\infty)\frac{\partial g^*}{\partial y^*} = Ex^{*s} \qquad \text{PMF case.}\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\text{As}\quad y^* \to \infty, \quad \frac{\partial\phi^*}{\partial y^*} \to 0, \quad \frac{\partial^2\phi^*}{\partial y^{*2}} \to 0,\\ &\chi^* \to 0 \qquad \text{PSC case}\\ &g^* \to 0 \qquad \text{PMF case}\end{aligned} \tag{3.12}$$

with the conditions $\alpha_1 = \alpha_3 = \alpha_4$, $\alpha_2 = \alpha_5 = \alpha_6 = 0$, $\alpha_7 = s\alpha_1$. Thus, the set Γ reduce to a one parameter group of transformations as

$$x^* = xe^{\varepsilon\alpha_1}, y^* = y, \phi^* = \phi e^{\varepsilon\alpha_1}, u^* = ue^{\varepsilon\alpha_1}, v^* = v, \chi^* = \chi, g^* = ge^{\varepsilon s\alpha_1}. \tag{3.13}$$

Expanding by Taylor's method in powers of ε and keeping terms up to the order ε,

$$x^* - x = x\varepsilon\alpha_1,\ y^* - y = 0,\ \phi^* - \phi = \phi\varepsilon\alpha_1,\ u^* - u = u\varepsilon\alpha_1,$$

$$v^* - v = 0,\ \chi^* - \chi = 0,\ g^* - g = g\varepsilon s\alpha_1.$$

In terms of differentials, this yield,

$$\frac{dx}{x\alpha_1} = \frac{dy}{0} = \frac{d\phi}{\phi\alpha_1} = \frac{du}{u\alpha_1} = \frac{dv}{0} = \frac{d\chi}{0} = \frac{dg}{gs\alpha_1}.$$

Solving, one gets,

$$y^* = \eta, \quad x^* F(\eta) = \phi^*, \quad \chi^* = \chi, \quad g^* = x^{*s} G(\eta). \tag{3.14}$$

Substituting (3.14) in (3.8), (3.9), (3.10),

$$F'^2 - FF'' = \nu F''' - k_0 \left(2F'F''' - FF^{iv} - F''^2\right) - \frac{\sigma B_0^2}{\rho} F', \tag{3.15}$$

$$r\chi F' - F\chi' = D\chi'' \qquad \text{PSC case}, \tag{3.16}$$

$$sGF' - FG' = DG'' \qquad \text{PMF case}. \tag{3.17}$$

The boundary conditions are transformed to

$$\begin{aligned} &\text{At} \quad \eta = 0, \quad F' = 1, \quad F = -\frac{v_w}{\sqrt{b\nu}} \\ &\chi = 1 \qquad \text{PSC case} \\ &G' = -1 \quad \text{PMF case.} \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\text{As} \quad \eta \to \infty, \quad F' \to 0, \quad F'' \to 0 \\ &\chi \to 0 \qquad \text{PSC case} \\ &G \to 0 \qquad \text{PMF case.} \end{aligned} \tag{3.19}$$

The following transformations for η, F, χ and G are introduced in equations (3.15), (3.16) and (3.17).

$$\begin{aligned} \eta &= \nu^{\alpha} b^{\beta} \eta^* \\ F &= \nu^{\alpha^1} b^{\beta^1} F^* \\ \chi &= \nu^{\alpha^{11}} b^{\beta^{11}} \chi^* \\ G &= \nu^{\alpha^{111}} b^{\beta^{111}} G^* \end{aligned} \tag{3.20}$$

where α, α^1, α^{11}, α^{111}, β, β^1, β^{11} and β^{111} are transformation parameters.

Taking $F^* = f$, $\chi^* = \chi$, $G^* = g$ in the final equations, the following relations are obtained

$$f'^2 - ff'' = f''' - k_1 \left(2f'f''' - ff^{iv} - f''^2\right) - Mnf', \tag{3.21}$$

$$\chi'' - rScf'\chi + Sc\chi' f = 0 \qquad \text{PSC case,} \tag{3.22}$$

$$g^{''} - sScf'g + Scg'f = 0 \qquad \text{PMF case} \tag{3.23}$$

where $k_1 = k_0 b/\nu$ is the dimensionless visco-elastic parameter, $Mn = \sigma B_0/(\rho b)$ is the dimensionless magnetic parameter and $Sc = \nu/D$ is the Schmidt number.

The modified boundary conditions are

$$\begin{aligned} &\text{At} \quad \eta^* = 0, \quad f' = 1, \quad f = -\frac{v_w}{\sqrt{b\nu}} \\ &\chi = 1 \qquad \text{PSC case} \\ &g' = -1 \quad \text{PMF case} \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\text{As} \quad \eta^* \to \infty, \quad f' \to 0, \quad f'' \to 0 \\ &\chi \to 0 \qquad \text{PSC case} \\ &g \to 0 \qquad \text{PMF case.} \end{aligned} \tag{3.25}$$

In the absence of magnetic field, equations (3.21), (3.22), (3.23) along with the boundary conditions (3.24) and (3.25) are found to match with those derived by Sonth *et al.* [13] who used similarity transformations for the particular case when $r = s = 2$.

4. Solution of the momentum equation

The exact solution of the differential equation (3.21) satisfying the boundary conditions (3.24) and (3.25) is derived in this section.

New variables are introduced as

$$z = \alpha\eta, \quad S(z) = \alpha f(\eta). \tag{4.1}$$

Equation (4.1) transforms equation (3.21) and boundary conditions (3.24) and (3.25) to the following,

$$S'^2 - SS'' = \alpha^2 S''' - k_1\alpha^2[2S'S^{'''} - SS^{iv} - S''^2] - MS' \tag{4.2}$$

and

$$\begin{aligned} &S' = 1, \quad S = -\frac{v_w\alpha}{\sqrt{b\nu}} \quad \text{at } z = 0 \\ &S' \to 0, \quad S'' \to 0 \quad \text{as} \quad z \to \infty. \end{aligned} \tag{4.3}$$

The following approximation satisfies the boundary conditions as $z \to \infty$,

$$S' = \exp(-z). \tag{4.4}$$

On integrating equation (4.4) and using equation (4.3),

$$S(z) = 1 - e^{-z} - \frac{v_w\alpha}{\sqrt{b\nu}}. \tag{4.5}$$

To obtain an expression for α, equation (4.5) is substituted in equation (4.2) which gives the following cubic equation,

$$\alpha^3 + \frac{1-k_1}{k_1\left(v_w/\sqrt{b\nu}\right)}\alpha^2 + \frac{1}{k_1}\alpha - \frac{1+M}{k_1\left(v_w/\sqrt{b\nu}\right)} = 0. \tag{4.6}$$

Thus, the exact solutions for $f(\eta)$ and $f'(\eta)$ are given by

$$\begin{aligned} f(\eta) &= \frac{1-e^{-\alpha\eta}}{\alpha} - \frac{v_w}{\sqrt{b\nu}} \\ f'(\eta) &= e^{-\alpha\eta} \end{aligned} \tag{4.7}$$

where α is the real positive root of the cubic algebraic equation (4.6).

5. Solution of the mass transfer equation

The non-dimensional form of concentration equations for the two cases (PSC and PMF) are derived by the Lie group method and are given by equations (3.22) and (3.23) along with boundary conditions (3.24) and (3.25). These cases are now considered separately and solved analytically.

5.1. Case A: Prescribed Surface Concentration (PSC)

Substituting (4.7) in (3.22), we get

$$\chi'' - rSce^{-m\eta}\chi + Sc\chi'\left[\frac{1-e^{-m\eta}}{m} - \frac{v_w}{\sqrt{b\nu}}\right] = 0. \tag{5.1}$$

Introducing the following transformation

$$\psi = -\frac{Sc}{m^2}e^{-m\eta} \tag{5.2}$$

and using (5.2) in (5.1), the governing non-dimensional concentration equation is obtained in the form

$$\psi\frac{d^2\chi}{d\psi^2} + \frac{d\chi}{d\psi}\left(1-\psi-b_0\right) + r\chi = 0 \tag{5.3}$$

where

$$b_0 = \frac{Sc}{m^2} - \frac{v_w Sc}{m\sqrt{b\nu}}. \tag{5.4}$$

The corresponding boundary conditions are

$$\chi\left(\frac{-Sc}{m^2}\right) = 1, \quad \chi(0) = 0. \tag{5.5}$$

The solution of equation (5.3) satisfying boundary conditions given in (5.5) is derived in terms of confluent hyper geometric function

$$\chi(\psi) = \left(\frac{-m^2}{Sc}\psi\right)^{b_0}\frac{M\left[b_0-r, b_0+1, \psi\right]}{M\left[b_0-r, b_0+1, -Sc/m^2\right]} \tag{5.6}$$

where M is the Kummer's function [14] defined by

$$M(a_0, b_0, z) = 1 + \sum_{n=1}^{\infty} \frac{(a_0)_n \, z^n}{(b_0)_n \, n!}$$

$$(a_0)_n = a_0(a_0+1)(a_0+2)\ldots\ldots(a_0+n-1) \tag{5.7}$$

$$(b_0)_n = b_0(b_0+1)(b_0+2)\ldots\ldots(b_0+n-1)\,.$$

Substituting (5.2) in (5.6), the solution becomes

$$\chi(\eta) = e^{-mb_0\eta} \frac{M\left[b_0 - r, b_0 + 1, -Sc\; e^{-m\eta}/m^2\right]}{M\left[b_0 - r, b_0 + 1, -Sc/m^2\right]}. \tag{5.8}$$

The dimensionless concentration gradient $\chi'(0)$ at the sheet level is obtained as

$$\chi'(0) = -mb_0 + \left(\frac{Sc}{m}\right)\left(\frac{b_0 - r}{b_0 + 1}\right) \frac{M\left[b_0 - r + 1, b_0 + 2, -Sc/m^2\right]}{M\left[b_0 - r, b_0 + 1, -Sc/m^2\right]}. \tag{5.9}$$

5.2. Case B: Prescribed Mass Flux (PMF)

Substituting (4.7) in (3.23),

$$g'' - rSce^{-m\eta}g + Scg'\left[\frac{1 - e^{-m\eta}}{m} - \frac{v_w}{\sqrt{b\nu}}\right] = 0. \tag{5.10}$$

Introducing the following transformation

$$\xi = -\frac{Sc}{m^2}e^{-m\eta} \tag{5.11}$$

and using (5.11) in (5.10), the governing non-dimensional equation of concentration is obtained in the form

$$\xi \frac{d^2g}{d\xi^2} + \frac{dg}{d\xi}(1 - \xi - b_0) + sg = 0 \tag{5.12}$$

where

$$b_0 = \frac{Sc}{m^2} - \frac{v_w Sc}{m\sqrt{b\nu}}. \tag{5.13}$$

The corresponding boundary conditions are

$$g'\left(\frac{-Sc}{m^2}\right) = -\frac{m}{Sc}, \quad g(0) = 0. \tag{5.14}$$

The solution of equation (5.12) satisfying boundary conditions (5.14) is derived in terms of confluent hyper geometric function.

$$g(\xi) = \frac{-m\xi^{b_0} M\left[b_0 - s, b_0 + 1, \xi\right]}{\left(-Sc/m^2\right)^{b_0}\left\{ -b_0m^2M\left[b_0 - s, b_0 + 1, -Sc/m^2\right] + Sc\left(\frac{b_0 - s}{b_0 + 1}\right) M\left[b_0 - s + 1, b_0 + 2, -Sc/m^2\right] \right\}} \tag{5.15}$$

where M is the Kummer's function defined in (5.7). Substituting (5.11) in (5.15), the solution becomes

$$g(\eta) = \frac{-me^{-mb_0\eta} M\left[b_0 - s, b_0 + 1, -Sc\, e^{-m\eta}/m^2\right]}{\left\{ -b_0 m^2 M\left[b_0 - s, b_0 + 1, -Sc/m^2\right] + Sc\left(\frac{b_0 - s}{b_0 + 1}\right) M\left[b_0 - s + 1, b_0 + 2, -Sc/m^2\right] \right\}}. \tag{5.16}$$

The expression for dimensionless concentration parameter at sheet level $g(0)$ is obtained as

$$g(0) = \frac{-mM\left[b_0 - s, b_0 + 1, -Sc/m^2\right]}{\left\{ -b_0 m^2 M\left[b_0 - s, b_0 + 1, -Sc/m^2\right] + Sc\left(\frac{b_0 - s}{b_0 + 1}\right) M\left[b_0 - s + 1, b_0 + 2, -Sc/m^2\right] \right\}}. \tag{5.17}$$

In the absence of the magnetic field, results (5.8), (5.9) for the PSC case and (5.16) and (5.17) for the PMF case, are seen to match with the results of Sonth *et al.* [13] for the particular case when $r = s = 2$.

6. Discussion of the results

A boundary layer problem for momentum and mass transfer in a visco-elastic fluid flow over a stretching porous sheet in the presence of a transverse magnetic field is discussed in this paper. A simpler method of Lie group analysis, called the scaling group of transformations is used to reduce the highly non-linear boundary layer partial differential equations into a set of non-linear ordinary differential equations. Analytic solutions are obtained in terms of confluent hyper geometric functions (Kummer's function). Here two cases of mass transfer are considered:

(1) Prescribed Sheet Concentration (PSC) and
(2) Prescribed Mass Flux (PMF).

Analytical expressions have been derived for each case separately. The results of the momentum equation are not discussed in this paper as these have been discussed by the authors in their earlier paper [3]. The main aim in this paper is to study the effects of the flow parameters on the concentration profiles with the help of graphs and tables assigning the values $b = 1$, $\nu = 0.04$.

6.1. Concentration distribution

The concentration distribution of the flow field is shown in Figures 1–4 for both the PSC and PMF cases under prescribed values for the visco-elastic parameter (k_1), magnetic parameter (Mn), the Schmidt number (Sc), suction/blowing parameter (v_w), concentration parameter (r) for the PSC case and the mass flux parameter (s) for the PMF case.

6.1.1 *Effect of the visco-elastic parameter* (k_1). Figure 1 show the concentration distribution in PSC and PMF cases respectively for chosen values of the visco-elastic parameter (k_1) and the suction/blowing parameter (v_w). These figures reveal that an increase in the visco-elastic parameter leads to a corresponding increase in the concentration profiles in both the PSC and PMF cases. This is consistent with

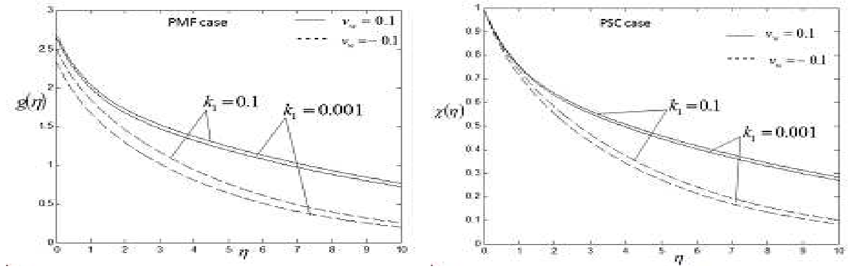

Figure 1. Dimensionless concentration profiles for different values of k_1 when $Mn = 0.5$, $Sc = 0.2$, $r = s = 2$.

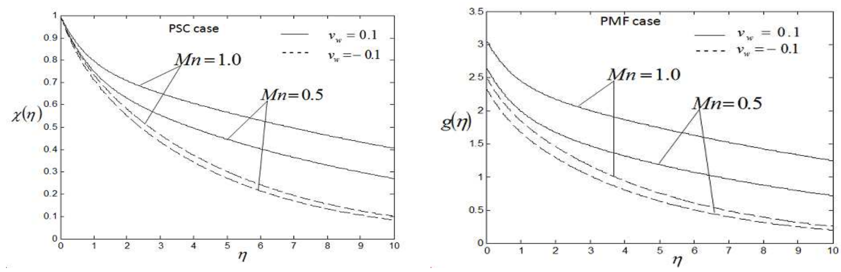

Figure 2. Dimensionless concentration profiles for different values of Mn when $k_1 = 0.001$, $Sc = 0.2$, $r = s = 2$.

the fact that the thickness of the concentration boundary layer occurs due to the presence of non-Newtonian visco-elastic normal stress. It is further observed that the concentration boundary layer is thinner in the case of suction as compared to blowing. This is evident from the flattening tendency of the curve for the latter as compared to the former. Further, the concentration level at the sheet is greater than unity in the PMF case while it is unity in the PSC case.

6.1.2. *Effect of the magnetic parameter* (Mn). Figures 2 are plotted to show the concentration profiles across the flow field in the PSC and PMF cases respectively for two chosen values each for the magnetic parameter (Mn) and the suction/blowing parameter (v_w). It is observed from these graphs that the effect of increasing the magnetic parameter is to increase the concentration in the flow field for both the cases. Further for the suction case, and for a particular value of η, the concentration is higher in the PMF case in comparison with the PSC case. Further, the magnetic field is seen to have significant effect on the concentration at the sheet level in the PMF case as compared to the PSC case.

6.1.3. *Effect of Schmidt number* (Sc). The concentration distribution is vastly affected by the presence of foreign species in the flow field. Figures 3 are plotted to show the concentration profiles across the flow field in the PSC and PMF cases respectively for different values of the Schmidt number (Sc) and the suction/blowing parameter (v_w). A comparative study of the curves reveals that the concentration

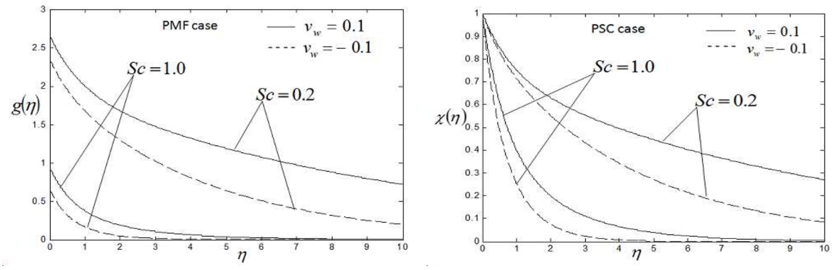

Figure 3. Dimensionless concentration profiles for different values of Sc when $k_1 = 0.001$, $Mn = 0.5$, $r = s = 2$.

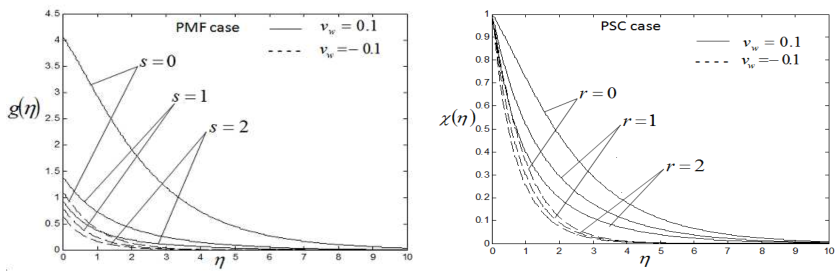

Figure 4. Dimensionless concentration profiles for (a) PSC case ($r = 0, 1, 2$) (b) PMF case ($s = 0, 1, 2$), when $k_1 = 0.001$, $Mn = 0.5$, $Sc = 1.0$.

boundary layer is thinner for values of Sc close to unity than for smaller values. This is more significant in the PMF case. It is also observed that for a given value of Sc, concentration is smaller for suction than for blowing. Physically, this implies that suction reduces the thickness of the concentration boundary layer while blowing increases the thickness both in the PSC and PMF cases.

6.1.4. *Effect of the concentration parameter* (r) *and mass flux parameter* (s). Figures 4 show the effects of the concentration parameter (r for PSC case) and mass flux parameter (s for PMF case) on the concentration distribution in the flow field considering the cases of suction and blowing in each case. It is observed that as r(or s) increases, the concentration boundary layer decreases. This decrease is more pronounced in the case of suction for both the PSC and PMF cases.

6.2. Surface concentration gradient $(\chi'(0))$ and surface concentration $g(0)$

The values of the surface concentration gradient in the PSC case are recorded in Tables 1 and 2 and surface concentrations in the PMF case are recorded in Tables 3 and 4 for various values of the parameters considered.

6.2.1. *Surface concentration gradient* $(\chi'(0))$. Table 1 shows the values of $\chi'(0)$ for the PSC case for various values of r. It is noticed that for increasing values of r, the surface concentration gradient is negative and its magnitude increases as r increases. Also, for a particular value of the suction/blowing parameter (v_w), the larger is the concentration parameter r, larger is the magnitude of the concentration gradient at the surface of the sheet. The surface concentration gradient is the least for the case of suction and maximum for the case of blowing for any particular value of r.

Table 1. Surfaces concentration gradient $\chi'(0)$ as a function of r at $Sc = 1.0$, $Mn = 0.5$, $k_1 = 0.001$.

r	$\chi'(0)$	$\chi'(0)$	$\chi'(0)$
	$v_w = -0.1$	$v_w = -0.0$	$v_w = 0.1$
0	-0.9008	-0.5366	-0.2463
1	-1.2369	-0.9407	-0.7137
2	-1.5282	-1.2695	-1.0665

Table 2 gives the values of $\chi'(0)$ for the PSC case for different values of the non-dimensional parameters k_1, Mn, Sc and for fixed value of $r = 2$. The effect of increasing Schmidt number is to decrease the concentration gradient whereas the effects of the visco-elastic and magnetic parameters is to increase the concentration gradient at the surface. The effect of suction parameter is to increase the numerical value of $\chi'(0)$ and that of the blowing parameter is to decrease the same for the same set of values of the parameters considered.

Table 2. Surfaces concentration gradient $\chi'(0)$ for the PSC case when $r = 2$.

k_1	Mn	Sc	$\chi'(0)$	$\chi'(0)$	$\chi'(0)$
			$v_w = -0.1$	$v_w = -0.0$	$v_w = 0.1$
0.001	0.5	0.2	-0.4258	-0.3945	-0.3757
0.1			-0.3978	-0.3812	-0.3702
0.001	1.0		-0.3975	-0.3578	-0.3250
0.1			-0.3711	-0.3446	-0.3236
0.001	0.5	1.0	-1.5282	-1.2695	-1.0665
0.1			-1.4831	-1.2508	-1.0602
0.001	1.0		-1.4825	-1.2156	-1.0028
0.1			-1.4348	-1.1941	-1.0008

6.2.2. *Surface concentration $g(0)$.* The values of $g(0)$ for the PMF case for different values of s are recorded in Table 3. It is observed that as s increases, $g(0)$ decreases for any particular value of the suction/blowing parameter v_w. For any particular value of s, the surface concentration for the case of suction is the least and is greater for the case of blowing. Table 4 gives the values of $g(0)$ for the PMF case for different values of the parameters considered.

The effect of the Schmidt number is to decrease the surface concentration whereas the effects of the visco-elastic and magnetic parameters is to increase the concentration at the surface. The effect of suction parameter is to decrease the value of $g(0)$ and that of the blowing parameter is to increase the same for the same set of values of the parameters considered.

Table 3. Surfaces concentration gradient $g(0)$ as a function of s at $Sc = 1.0$, $Mn = 0.5$, $k_1 = 0.001$.

s	$g(0)$	$g(0)$	$g(0)$
	$v_w = -0.1$	$v_w = -0.0$	$v_w = 0.1$
0	1.1102	1.8634	4.0602
1	0.8084	1.0630	1.4012
2	0.6544	0.7877	0.9377

Table 4. Surface concentration $g(0)$ for the PMF case when $s = 2$.

k_1	Mn	Sc	$g(0)$	$g(0)$	$g(0)$
			$v_w = -0.1$	$v_w = -0.0$	$v_w = 0.1$
0.001	0.5	0.2	2.3487	2.5350	2.6619
0.1			2.5139	2.6235	2.7010
0.001	1.0		2.5159	2.7947	3.0765
0.1			2.6948	2.9017	3.0902
0.001	0.5	1.0	0.6544	0.7877	0.9377
0.1			0.6743	0.7995	0.9432
0.001	1.0		0.6745	0.8227	0.9973
0.1			0.6970	0.8374	0.9992

7. Conclusions

The following inferences are derived from the above study on the concentration distribution of the flow field.

(1) The thickness of the concentration boundary layer occurs due to the presence of non-Newtonian visco-elastic normal stress. For small values of the Schmidt number (around 0.2), the concentration at the surface is greater in the PMF case when compared to the PSC case.

(2) The effect of transverse magnetic field is to increase concentration in the flow field. In the presence of a magnetic field, greater concentration occurs when the sheet is subjected to blowing.

(3) As the diffusing foreign species present in the flow field become heavier, there is a decrease in the concentration at all points in the flow field and suction reduces the concentration in the flow field while blowing increases the same both in the PSC and PMF cases.

(4) To obtain a reduced concentration boundary layer, an ideal combination of values for the parameters considered, would be to choose small values for the visco-elastic (around 0.001) and magnetic parameters (around 0.5) and a large value for the Schmidt number (around 1.0) while considering the case of suction of the fluid through the porous stretching sheet in the PMF case.

(5) Higher the value of the concentration parameter (PSC case) or the mass flux parameter (PMF case), the lower is the concentration in the flow field.

(6) The effect of the suction parameter is to decrease the surface concentration gradient for the PSC case and surface concentration for the PMF case while

that of the blowing parameter is to increase the same for the same set of values for the parameters considered.

(7) In the absence of magnetic field and when $r = s = 2$, the results of this paper are in excellent agreement with the results of Sonth *et al.* [13].

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