

Biharmonic Curves in Lorentzian Para-Sasakian Manifolds

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Abstract. In this paper we give necessary and sufficient conditions for space-like and timelike curves in a conformally flat, quasi conformally flat and conformally symmetric 4-dimensional Lorentzian Para-Sasakian (LP -Sasakian) manifold to be proper biharmonic. Also, we investigate proper biharmonic curves in the Lorentzian sphere S_1^4 .

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1. Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by Almansi, Levi-Civita and Nicolescu. Recently, biharmonic functions on Riemannian manifolds were studied by Caddeo and Vanhecke [4, 5], Sario, Nakai, Wang and Chung [36].

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations (see [11, 24, 39, 45, 46]), because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

Let $C^\infty(M, N)$ denote the space of smooth maps $\Psi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds. A map $\Psi \in C^\infty(M, N)$ is called *harmonic* if it is a critical point of the *energy* functional

$$E : C^\infty(M, N) \rightarrow \mathbb{R}, E(\Psi) = \frac{1}{2} \int_M |d\Psi|^2 v_g$$

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and is characterized by the vanishing of the tension field $\tau(\Psi) = \text{trace } \nabla d\Psi$ where ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^Ψ . As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by Eells and Sampson in [16]. *Biharmonic maps* between Riemannian manifolds $\Psi : (M, g) \rightarrow (N, h)$ are the critical points of the *bienergy functional*

$$E_2(\Psi) = \frac{1}{2} \int_M |\tau(\Psi)|^2 v_g.$$

The first variation formula for the bienergy which is derived in [22, 23] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - \text{trace } R^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where $\Delta = -\text{trace}(\nabla^\Psi \nabla^\Psi - \nabla_{\frac{\Psi}{\Psi}})$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature operator on N . From the expression of the bitension field τ_2 , it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

In a different setting, Chen [12] defined biharmonic submanifolds $M \subset E^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian, and stated the following

- Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

If the definition of biharmonic maps is applied to Riemannian immersions into Euclidean space, the notion of Chen's biharmonic submanifold is obtained, so the two definitions agree.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

- Generalized Chen's conjecture: Biharmonic submanifolds of a manifold N with $Riem^N \leq 0$ are minimal,

encouraged the study of proper biharmonic submanifolds, that is submanifolds such that the inclusion map is a biharmonic map, in spheres or another non-negatively curved spaces (see [6, 8, 17, 20, 30, 33]).

Of course, the first and easiest examples can be found by looking at differentiable curves in a Riemannian manifold. Obviously geodesics are biharmonic. Non-geodesic biharmonic curves are called proper biharmonic curves. Chen and Ishikawa [13] showed non-existence of proper biharmonic curves in Euclidean 3-space E^3 . Moreover they classified all proper biharmonic curves in Minkowski 3-space E_1^3 (see also [19]). Caddeo, Montaldo and Piu showed that on a surface with non-positive Gaussian curvature, any biharmonic curve is a geodesic of the surface [7]. So they gave a positive answer to generalized Chen's conjecture. Caddeo *et al.* in [6] studied biharmonic curves in the unit 3-sphere. More precisely, they showed that proper biharmonic curves in S^3 are circles of geodesic curvature 1 or helices which are geodesics in the Clifford minimal torus. Then the same authors studied the biharmonic submanifolds of unit n -sphere [8].

On the other hand, there are a few results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves in the Heisenberg group H_3 are investigated in [9] by Caddeo *et al.* In [17] Fetcu studied biharmonic curves in the generalized Heisenberg group and obtained two families of proper biharmonic curves. Also, the explicit parametric equations for the biharmonic curves on Berger spheres S_ε^3 are obtained by Balmuş in [3].

In contact geometry, there is a well-known analog of real space form, namely a Sasakian space form. In particular, a simply connected three-dimensional Sasakian space form of constant holomorphic sectional curvature 1 is isometric to S^3 . So in this context Inoguchi classified in [20] the proper biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form and in [18] the explicit parametric equations were obtained. Sasahara [37], analyzed the proper biharmonic Legendre surfaces in Sasakian space forms and in the case when the ambient space is the unit 5-dimensional sphere S^5 he obtained their explicit representations. Also, Özgür and Tripathi [35] proved that a Legendre curve in an α -Sasakian manifold is biharmonic if and only if its curvature is zero.

Other results on biharmonic Legendre curves and biharmonic anti-invariant surfaces in Sasakian space forms and (κ, μ) -manifolds are given in [1, 2]. In [31] it was proved that all invariant submanifolds of non-Sasakian (κ, μ) -manifolds are always totally geodesic. Thus it is obvious that biharmonic invariant submanifolds of (κ, μ) -manifolds are not proper.

It is known that the solution to a problem first formulated in Euclidean spaces may sometimes look considerably different when considered in pseudo-Euclidean spaces. Although no examples of proper biharmonic submanifolds in Euclidean spaces are known, in the pseudo-Euclidean spaces E_t^4 , ($t = 1, 2$), many examples of proper biharmonic spacelike surfaces with constant mean curvature were given by Chen and Ishikawa in [14]. They also investigated biharmonic surfaces with lightlike mean curvature vector and biharmonic surfaces with constant Gauss curvature in pseudo-Euclidean space. In [21] W -surfaces in a 4-dimensional pseudo-Euclidean space E_t^4 , ($t = 1, 2$), is defined and a classification theorem for biharmonic W -surfaces with flat normal connection in E_t^4 is obtained.

Despite the existence of proper biharmonic submanifolds in semi-Euclidean spaces, biharmonicity may still imply minimality in some specific cases. The authors in [14] showed that any biharmonic surface in E_t^3 , ($t = 1, 2$), is also minimal. In [15] it is proved that a nondegenerate biharmonic hypersurface of 4-dimensional pseudo-Euclidean space with diagonalizable shape operator must be minimal.

Pseudo-Riemannian spaces especially the constant curvature ones, namely de Sitter, Minkowski, anti de Sitter space, play important roles in the general relativity. Ouyang [34] and Sun [40] studied the spacelike biharmonic submanifolds in the Pseudo-Riemannian spaces. In [47] Zhang constructed examples of proper biharmonic hypersurfaces in the anti de Sitter space.

In this paper we give some necessary and sufficient conditions for a spacelike and a timelike curve lying in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric LP -Sasakian manifold to be proper biharmonic.

The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto in 1989 [26]. Also he introduced the notion of LP -Sasakian manifold. Mihai

and Rosca [28] defined the same notion independently and thereafter many authors [25, 29, 42, 43] studied LP -Sasakian manifolds and their submanifolds.

2. Preliminaries

2.1. Biharmonic maps between Riemannian manifolds

Let (M, g) and (N, h) be Riemannian manifolds and $\Psi : (M, g) \rightarrow (N, h)$ be a smooth map. The tension field of Ψ is given by $\tau(\Psi) = \text{trace } \nabla d\Psi$, where $\nabla d\Psi$ is the second fundamental form of Ψ defined by $\nabla d\Psi(X, Y) = \nabla_X^\Psi d\Psi(Y) - d\Psi(\nabla_X^M Y)$, $X, Y \in \Gamma(TM)$. For any compact domain $\Omega \subseteq M$, the bienergy is defined by

$$E_2(\Psi) = \frac{1}{2} \int_{\Omega} |\tau(\Psi)|^2 v_g.$$

Then a smooth map Ψ is called biharmonic map if it is a critical point of the bienergy functional for any compact domain $\Omega \subseteq M$. We have for the bienergy the following first variation formula:

$$(2.1) \quad \frac{d}{dt} E_2(\Psi_t; \Omega)|_{t=0} = \int_{\Omega} \langle \tau_2(\Psi), w \rangle v_g$$

where v_g is the volume element, w is the variational vector field associated to the variation $\{\Psi_t\}$ of Ψ and

$$\tau_2(\Psi) = -J(\tau_2(\Psi)) = -\Delta^\Psi \tau(\Psi) - \text{trace } R^N(d\Psi, \tau(\Psi))d\Psi.$$

$\tau_2(\Psi)$ is called bitension field of Ψ . Here Δ^Ψ is the rough Laplacian on the sections of the pull-back bundle $\Psi^{-1}TN$ which is defined by

$$\Delta^\Psi V = - \sum_{i=1}^m \{ \nabla_{e_i}^\Psi \nabla_{e_i}^\Psi V - \nabla_{\nabla_{e_i}^M e_i}^\Psi V \}, \quad V \in \Gamma(\Psi^{-1}TN),$$

where ∇^Ψ is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame on M . When the target manifold is semi-Riemannian manifold, the bienergy and bitension field can be defined in the same way.

Let M be a semi-Riemannian manifold and $\gamma : I \rightarrow M$ be a non-null curve parametrized by arclength. By using the definition of the tension field we have

$$\tau(\gamma) = \nabla_{\frac{\partial}{\partial s}}^\gamma d\gamma\left(\frac{\partial}{\partial s}\right) = \nabla_T T,$$

where $T = \gamma'$. Consider a smooth variation of γ , that is a smooth map $\beta : I \times (-\delta, \delta) \rightarrow M$, $\beta(s, t) = \gamma_t(s)$, such that $\gamma_0 = \gamma$ (see [32]). Then from (2.1) we can write the first variation formula for the bienergy functional of γ

$$\begin{aligned} \frac{d}{dt} E_2(\gamma_t; I)|_{t=0} &= \int_I \langle \nabla_{\frac{\partial}{\partial s}}^\beta \nabla_{\frac{\partial}{\partial s}}^\beta d\beta\left(\frac{\partial}{\partial t}\right) - \nabla_{\nabla_{\frac{\partial}{\partial s}}^I \frac{\partial}{\partial s}}^\beta \frac{\partial \beta}{\partial t}, \tau(\gamma_t) \rangle |_{t=0} ds \\ &+ \int_I \langle R^M\left(\frac{\partial \beta}{\partial t}, d\beta\left(\frac{\partial}{\partial s}\right)\right) d\beta\left(\frac{\partial}{\partial s}\right), \tau(\gamma_t) \rangle |_{t=0} ds, \end{aligned}$$

where ∇^I denotes the connection on I . Since $\nabla^I_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0$ and the Laplace operator is self-adjoint then we have

$$\frac{d}{dt} E_2(\gamma_t; D)|_{t=0} = \int_I \langle \nabla_T^3 T - R^M(T, \nabla_T T)T, w \rangle ds.$$

Here w is the variation vector field of γ and ∇ denotes the connection on M . In this case biharmonic equation for the curve γ reduces to

$$(2.2) \quad \nabla_T^3 T - R(T, \nabla_T T)T = 0,$$

that is, γ is called a biharmonic curve if it is a solution of the equation (2.2) (see also [30]).

2.2. Lorentzian almost paracontact manifolds

Let M be an n -dimensional smooth connected paracompact Hausdorff manifold with a Lorentzian metric g , i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow R$ is a non-degenerate inner product of signature $(-, +, \dots, +)$, where $T_p M$ is the tangent space of M at the point p . Then (M, g) is known to be a Lorentzian manifold. A non-zero vector $X_p \in T_p M$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$ ($X_p \neq 0$) or $g_p(X_p, X_p) < 0$, respectively.

Let M be an n -dimensional differentiable manifold equipped with a triple (ϕ, ξ, η) , where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form on M such that [26]

$$(2.3) \quad \eta(\xi) = -1,$$

$$(2.4) \quad \phi^2 = I + \eta \otimes \xi,$$

where I denotes the identity map of $T_p M$ and \otimes is the tensor product. The equations (2.3) and (2.4) imply that

$$\begin{aligned} \eta \circ \phi &= 0, \\ \phi \xi &= 0, \\ \text{rank}(\phi) &= n - 1. \end{aligned}$$

Then M admits a Lorentzian metric g , such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

and M is said to admit a Lorentzian almost paracontact structure (ϕ, ξ, η, g) . Then we get

$$\begin{aligned} g(X, \xi) &= \eta(X), \\ \Phi(X, Y) &= g(X, \phi Y) = g(\phi X, Y) = \Phi(Y, X), \\ (\nabla_X \Phi)(Y, Z) &= g(Y, (\nabla_X \phi)Z) = (\nabla_X \Phi)(Z, Y), \end{aligned}$$

where ∇ is the covariant differentiation with respect to g . It is clear that Lorentzian metric g makes ξ a timelike unit vector field, i.e, $g(\xi, \xi) = -1$. The manifold M equipped with a Lorentzian almost paracontact structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold (for short LAP -manifold) [26, 27].

In equations (2.3) and (2.4) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by Satō [38].

A Lorentzian almost paracontact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian paracontact manifold (for short LP -manifold) [26] if

$$\Phi(X, Y) = \frac{1}{2}((\nabla_X \eta)Y + (\nabla_Y \eta)X).$$

A Lorentzian almost paracontact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian para Sasakian manifold (for short LP -Sasakian) [26] if

$$(\nabla_X \phi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi.$$

In a LP -Sasakian manifold the 1-form η is closed.

Also Matsomoto in [26] showed that if an n -dimensional Lorentzian manifold (M, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then (M, g) admits an LP -Sasakian structure.

The conformal curvature tensor C is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} \{g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y\} \\ + \frac{r}{(n-1)(n-2)} \{g(Y, Z)X - g(X, Z)Y\},$$

where $S(X, Y) = g(QX, Y)$. If $C = 0$ then the LP -Sasakian manifold is called conformally flat.

The quasi-conformal curvature tensor \tilde{C} is given by

$$\tilde{C}(X, Y)Z = aR(X, Y)Z + b\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ - g(X, Z)QY\} - \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) \{g(Y, Z)X - g(X, Z)Y\},$$

where a, b constants such that $ab \neq 0$ and $S(Y, Z) = g(QY, Z)$. If $\tilde{C} = 0$ then the LP -Sasakian manifold is called quasi conformally flat. In [41] it was proved that a conformally flat and a quasi conformally flat LP -Sasakian manifold is of constant curvature and the value of this constant is $+1$. Also the same authors showed in [41] that if in an LP -Sasakian manifold M^n ($n > 3$) the relation $R(X, Y).C = 0$ holds, then it is locally isometric to a Lorentzian unit sphere.

For a conformally symmetric Riemannian manifold [10], we have $\nabla C = 0$. Hence for such a manifold $R(X, Y).C = 0$ holds. Thus a conformally symmetric LP -Sasakian manifold M^n ($n > 3$) is locally isometric to a Lorentzian unit sphere [41].

For a conformally flat, quasi conformally flat and conformally symmetric LP -Sasakian manifold M^n , we have [41]

$$(2.5) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in \Gamma(TM).$$

An arbitrary curve $\gamma : I \rightarrow M$, $\gamma = \gamma(s)$, in a LP -Sasakian manifold is called spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike). If $\gamma(s)$ is a spacelike or timelike curve, we can reparametrize it such that $g(\gamma'(s), \gamma'(s)) = \varepsilon$ where $\varepsilon = 1$ if γ is spacelike and $\varepsilon = -1$ if γ is timelike, respectively. In this case $\gamma(s)$ is said to be unit speed or arclength parametrization.

Let M be a 4-dimensional LP -Sasakian manifold. Denote by $\{T, N, B_1, B_2\}$ the moving Frenet frame along the curve γ in M . Then T, N, B_1, B_2 are respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields.

Let $\gamma(s)$ be a curve in a 4-dimensional LP -Sasakian manifold parametrized by arclength function s . Then for the curve γ the following Frenet equations are given in [44]:

Case I. γ is a spacelike curve: Then T is a spacelike vector, so depending on the casual character of the principal normal vector N and the first binormal vector B_1 , we have the following Frenet formulas:

Case I.1. N and B_1 are spacelike;

$$(2.6) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T, T) = g(N, N) = g(B_1, B_1) = 1, \quad g(B_2, B_2) = -1.$$

Proof. Let M be a 4-dimensional LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve. Let $\{T, N, B_1, B_2\}$ be an orthonormal frame field tangent to M along the curve γ where $T = \dot{\gamma}$ unit vector field tangent to γ , N is the unit spacelike principal normal vector field in the direction $\nabla_T T$, that is $k_1 = |\nabla_T T|$, B_1 is a unit spacelike vector field and B_2 is a unit timelike vector field. Then we can write

$$\begin{aligned} \nabla_T T &= k_1 N, \\ \nabla_T N &= a_{21} T + a_{22} N + a_{23} B_1 + a_{24} B_2, \\ \nabla_T B_1 &= a_{31} T + a_{32} N + a_{33} B_1 + a_{34} B_2, \\ \nabla_T B_2 &= a_{41} T + a_{42} N + a_{43} B_1 + a_{44} B_2. \end{aligned}$$

By straightforward calculation we have

$$\begin{aligned} a_{21} &= g(\nabla_T N, T) = -g(\nabla_T T, N) = -k_1, & a_{22} &= g(\nabla_T N, N) = 0, \\ a_{23} &= g(\nabla_T N, B_1) = k_2, & a_{24} &= -g(\nabla_T N, B_2), \\ a_{31} &= g(\nabla_T B_1, T) = -g(\nabla_T T, B_1) = 0, \\ a_{32} &= g(\nabla_T B_1, N) = -g(\nabla_T N, B_1) = -k_2, \\ a_{33} &= g(\nabla_T B_1, B_1) = 0, & a_{34} &= -g(\nabla_T B_1, B_2) = k_3, \\ a_{41} &= g(\nabla_T B_2, T) = -g(\nabla_T T, B_2) = 0, & a_{42} &= g(\nabla_T B_2, N), \\ a_{43} &= g(\nabla_T B_2, B_1) = -g(\nabla_T B_1, B_2) = k_3, & a_{44} &= -g(\nabla_T B_2, B_2) = 0. \end{aligned}$$

Furthermore

$$a_{24} = -g(\nabla_T N, B_2) = g(\nabla_T B_2, N) = a_{42}.$$

Since

$$\nabla_T N = -k_1 T + k_2 B_1 + a_{42} B_2,$$

$$\nabla_T B_2 = a_{42}N + k_3 B_1$$

then

$$\begin{aligned} a_{42} &= g(\nabla_T B_2, N) \\ &= -g(\nabla_T N, B_2) \\ &= g(-k_1 T + k_2 B_1 + a_{42} B_2, B_2) \\ &= -a_{42} \end{aligned}$$

which gives $a_{42} = a_{24} = 0$. Therefore we obtain (2.6). \blacksquare

By the similar way following in the proof of Case I.1, we can deduce the Frenet equations for all cases.

Case I.2. N is spacelike, B_1 is timelike;

$$(2.7) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T, T) = g(N, N) = g(B_2, B_2) = 1, \quad g(B_1, B_1) = -1.$$

Case I.3. N is spacelike, B_1 is null;

$$(2.8) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy the equations

$$\begin{aligned} g(T, T) = g(N, N) = 1, \quad g(B_1, B_1) = g(B_2, B_2) = 0, \\ g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, \quad g(B_1, B_2) = 1. \end{aligned}$$

Case I.4. N is timelike, B_1 is spacelike;

$$(2.9) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(N, N) = -1.$$

Case I.5. N is null, B_1 is spacelike;

$$(2.10) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where the first curvature k_1 takes only two values: 0 when γ is a geodesic or 1 in all other cases. In this case, T, N, B_1, B_2 satisfy the equations

$$g(T, T) = g(B_1, B_1) = 1, \quad g(N, N) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, \quad g(N, B_2) = 1.$$

Remark 2.1. For a spacelike curve with null principal normal N , one can change the role of first binormal B_1 and the second binormal B_2 in the Case I.5. In this case we have

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & k_2 \\ -k_1 & 0 & 0 & k_3 \\ 0 & -k_3 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 satisfy the equations

$$g(T, T) = g(B_2, B_2) = 1, \quad g(N, N) = g(B_1, B_1) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_2) = g(B_1, B_2) = 0, \quad g(N, B_1) = 1.$$

Case II. γ is a timelike curve: In this case T is a timelike vector, so the Frenet formulae have the form

$$(2.11) \quad \begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(N, N) = g(B_1, B_1) = g(B_2, B_2) = 1, \quad g(T, T) = -1.$$

3. Biharmonic curves in LP -Sasakian manifolds

In this section we characterize the spacelike and timelike proper biharmonic curves in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric LP -Sasakian manifold.

Theorem 3.1. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(N, N) = g(B_1, B_1) = 1$ and $g(B_2, B_2) = -1$. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 + k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that is its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_1 is a unit spacelike and B_2 is a unit timelike

vector. The tension field of γ is $\tau(\gamma) = \nabla_T T$. Then by using the Frenet formulas (2.6) and the equation (2.5) we obtain the Euler-Lagrange equation of the bienergy:

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \nabla_T^3 T - R(T, k_1 N)T \\ &= (-3k_1 k_1')T + (k_1'' - k_1^3 - k_1 k_2^2)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\ &= (-3k_1 k_1')T + (k_1'' - k_1^3 - k_1 k_2^2 + k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\ &= 0. \end{aligned}$$

where k_1, k_2 and k_3 are respectively the first, the second and the third curvature of the curve $\gamma(s)$.

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' - k_1(k_1^2 + k_2^2 - 1) &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned} k_1 &= \text{constant} \neq 0, \\ k_2 &= \text{constant}, \\ k_1^2 + k_2^2 &= 1, \\ k_2 k_3 &= 0. \end{aligned}$$

This completes the proof. ■

Theorem 3.2. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(N, N) = g(B_2, B_2) = 1$ and $g(B_1, B_1) = -1$. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 - k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that is its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_2 is a unit spacelike and B_1 is a unit timelike vector. Since the tension field of γ is $\tau(\gamma) = \nabla_T T$ then by using the Frenet formulas given in (2.7) and the equation (2.5), we obtain the biharmonic equation for γ :

$$\tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T$$

$$\begin{aligned}
 &= \nabla_T^3 T - R(T, k_1 N)T \\
 &= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1 k_2^2)N \\
 &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\
 &= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1 k_2^2 + k_1)N \\
 &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\
 &= 0.
 \end{aligned}$$

where k_1, k_2 and k_3 are respectively the first, the second and the third curvature of curve $\gamma(s)$.

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned}
 k_1 k_1' &= 0, \\
 k_1'' - k_1(k_1^2 - k_2^2 - 1) &= 0, \\
 2k_1' k_2 + k_1 k_2' &= 0, \\
 k_1 k_2 k_3 &= 0.
 \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned}
 k_1 &= \text{constant} \neq 0, \\
 k_2 &= \text{constant}, \\
 k_1^2 - k_2^2 &= 1, \\
 k_2 k_3 &= 0.
 \end{aligned}$$

This completes the proof. ■

Theorem 3.3. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame such that N is a spacelike and B_1 is a null vector. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if $k_1 = 1$ and either $k_2 = 0$, or $\ln k_2(s) = -\int k_3(s) ds$.*

Proof. Let $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength on a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold M . Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame such that

$$g(T, T) = g(N, N) = 1, \quad g(B_1, B_1) = g(B_2, B_2) = 0,$$

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, \quad g(B_1, B_2) = 1.$$

Then by using the Frenet equations given by (2.8), we have

$$\begin{aligned}
 \tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\
 &= \nabla_T^3 T - R(T, k_1 N)T \\
 &= (-3k_1 k_1')T + (k_1'' - k_1^3 + k_1)N + (2k_1' k_2 + k_1 k_2' + k_1 k_2 k_3)B_1
 \end{aligned}$$

where k_1, k_2 and k_3 are respectively the first, the second and the third curvature of curve $\gamma(s)$. From the biharmonic equation of γ above, we can say γ is a biharmonic

curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' - k_1^3 + k_1 &= 0, \\ 2k_1' k_2 + k_1 k_2' + k_1 k_2 k_3 &= 0. \end{aligned}$$

For biharmonic curves with $k_1 \neq 0$ that is if we investigate the nongeodesic solutions, we obtain

$$\begin{aligned} k_1 &= 1, \\ k_2' + k_2 k_3 &= 0. \end{aligned}$$

Thus we have the assertion of the theorem. ■

Theorem 3.4. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ such that $g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1$ and $g(N, N) = -1$. Then $\gamma : I \rightarrow M$ is a biharmonic curve if and only it is a geodesic of M .*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP-Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a spacelike curve that is its velocity vector $T = \gamma'(s)$ is spacelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit timelike vector field in the direction $\nabla_T T$, B_1 and B_2 are unit spacelike vectors. The tension field of γ is $\tau(\gamma) = \nabla_T T$. Then by using the tension field of γ , Frenet formulas in (2.9) and the equation (2.5) we obtain the Euler-Lagrange equation of the bienergy:

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \nabla_T^3 T - R(T, k_1 N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 + k_1 k_2^2)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 + k_1 k_2^2 + k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 \\ &= 0. \end{aligned}$$

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' + k_1(k_1^2 + k_2^2 + 1) &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$k_1 = \text{constant} \neq 0,$$

$$\begin{aligned} k_2 &= \text{constant}, \\ k_1^2 + k_2^2 &= -1, \\ k_2 k_3 &= 0. \end{aligned}$$

This shows that we have no nongeodesic solution for the biharmonic equation for the curve γ . ■

Theorem 3.5. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength. Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame along γ such that N is a null vector. Then $\gamma : I \rightarrow M$ is a biharmonic curve if and only if γ is a geodesic of M .*

Proof. Let $\gamma : I \rightarrow M$ be a spacelike curve parametrized by arclength on a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold M . Suppose that $\{T, N, B_1, B_2\}$ be a moving Frenet frame along the curve γ such that

$$\begin{aligned} g(T, T) &= g(B_1, B_1) = 1, & g(N, N) &= g(B_2, B_2) = 0, \\ g(T, N) &= g(T, B_1) = g(T, B_2) = g(N, B_1) = g(B_1, B_2) = 0, & g(N, B_2) &= 1. \end{aligned}$$

If we consider the Frenet formulas given in (2.10), we obtain the biharmonic equation for the curve γ :

$$0 = \tau_2(\gamma) = (k_1'' + k_1 k_2 k_3 + k_1)N + (2k_1' k_2 + k_1 k_2')B_1 + (-k_1 k_2^2)B_2$$

Then γ is a biharmonic curve if and only if

$$\begin{aligned} k_1'' + k_1 k_2 k_3 + k_1 &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2^2 &= 0. \end{aligned}$$

Since γ is a spacelike curve with a null normal vector, k_1 can take only two values: 0 and 1. If we look for nongeodesic solutions, we get $k_2 = 0$. But from the first equation above, we have a contradiction such that $k_2 k_3 + 1 = 0$. So the only biharmonic spacelike curves on M with a null normal vector are the geodesics of M . ■

Remark 3.1. Let $\gamma : I \rightarrow M$ is a spacelike curve parametrized by arclength in a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold with a null principal normal N . If we change the role of B_1 and B_2 we can easily see by the similiar way following in the proof of Theorem 3.5 that $\gamma : I \rightarrow M$ is a biharmonic curve if and only if γ is a geodesic of M .

Now let us investigate the biharmonicity of a timelike curve in a 4-dimensional conformally flat, quasi conformally flat and conformally symmetric LP -Sasakian manifold. We have,

Theorem 3.6. *Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold and $\gamma : I \rightarrow M$ be a timelike curve parametrized by arclength. Then $\gamma : I \rightarrow M$ is a proper biharmonic curve if and only if either γ is a circle with $k_1 = 1$, or γ is a helix with $k_1^2 - k_2^2 = 1$.*

Proof. Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold endowed with the structure (ϕ, ξ, η, g) and $\gamma : I \rightarrow M$ be a curve parametrized by arclength. Suppose that γ is a timelike curve that is its velocity vector $T = \gamma'(s)$ is timelike. Let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field tangent to M along γ , where N is the unit spacelike vector field in the direction $\nabla_T T$, B_1 and B_2 are unit spacelike vectors. Then by using the Frenet formulas (2.11), we have:

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T - R(T, \nabla_T T)T \\ &= \nabla_T^3 T - R(T, k_1 N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 - k_1 k_2^2)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2 - k_1 R(T, N)T \\ &= (3k_1 k_1')T + (k_1'' + k_1^3 - k_1 k_2^2 - k_1)N \\ &\quad + (2k_1' k_2 + k_1 k_2')B_1 + (k_1 k_2 k_3)B_2. \end{aligned}$$

It follows that γ is a biharmonic curve if and only if

$$\begin{aligned} k_1 k_1' &= 0, \\ k_1'' + k_1(k_1^2 - k_2^2 - 1) &= 0, \\ 2k_1' k_2 + k_1 k_2' &= 0, \\ k_1 k_2 k_3 &= 0. \end{aligned}$$

If we look for nongeodesic solutions, that is for biharmonic curves with $k_1 \neq 0$, we obtain

$$\begin{aligned} k_1 &= \text{constant} \neq 0, \\ k_2 &= \text{constant}, \\ k_1^2 - k_2^2 &= 1, \\ k_2 k_3 &= 0. \end{aligned} \quad \blacksquare$$

4. Biharmonic curves in S_1^4

Let M be a 4-dimensional conformally flat, quasi conformally flat or conformally symmetric LP -Sasakian manifold. Since M is locally isometric to a Lorentzian unit sphere S_1^4 , by using the theorems stated in the Section 3 we shall give some characterizations for nongeodesic biharmonic curves in a Lorentzian unit sphere S_1^4 . The Lorentzian unit sphere of radius 1 can be seen as the hyperquadric

$$S_1^4 = \{p \in R_1^5 : \langle p, p \rangle = 1\}$$

in a Minkowski space R_1^5 with the metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2.$$

Let $\gamma : I \rightarrow S_1^4$ be a non-null curve parametrized by arclength. Denote by ∇ the covariant derivative along γ in S_1^4 . Then for any vector field X along γ we have

$$\nabla_T X = X' + \langle T, X \rangle \gamma.$$

It's also known that S_1^4 is a Lorentzian space form of the scalar curvature 1. Then for all vector fields X, Y, Z in the tangent bundle of S_1^4 we have

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$$

where R is the Riemannian curvature tensor of S_1^4 .

Proposition 4.1. *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N and first binormal vector B_1 are spacelike. Then*

$$(4.1) \quad \gamma^{(IV)} + 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

Proof. From the Frenet formulas (2.6), by taking the covariant derivative of $\nabla_T N$ with respect to T , we have

$$\begin{aligned} \nabla_T^2 N &= -k_1 \nabla_T T + k_2 \nabla_T B_1 \\ &= -k_1^2 N + k_2(-k_2 N + k_3 B_2) \\ &= -(k_1^2 + k_2^2)N + k_2 k_3 B_2 \\ &= -N. \end{aligned}$$

If we use the Gauss equation of $S_1^4 \subset R_1^5$, that for any vector field X along γ is

$$\nabla_T X = X' + \langle T, X \rangle \gamma,$$

we get

$$\begin{aligned} \nabla_T^2 N &= \nabla_T [N' + \langle T, N \rangle \gamma] \\ &= \nabla_T N' \\ &= N'' + \langle T, N' \rangle \gamma \\ &= N'' + \langle T, \nabla_T N - \langle N, T \rangle \gamma \rangle \gamma \\ &= N'' + \langle T, \nabla_T N \rangle \gamma \\ &= N'' - k_1 \gamma \end{aligned}$$

and

$$N = \frac{1}{k_1}(\gamma'' + \gamma).$$

By substituting the above expressions of $\nabla_T^2 N$ and N in the equation $\nabla_T^2 N + N = 0$, we obtain the differential equation (4.1). ■

From Proposition 4.1, it is obvious that to find nongeodesic biharmonic curves in S_1^4 we must investigate the solutions of (4.1). By integrating the differential equation (4.1), we have:

Theorem 4.1. *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N and first binormal vector B_1 are spacelike. Then we have two cases:*

- γ is a circle of radius $1/\sqrt{2}$;
-

$$\gamma(s) = \left(0, \frac{\cos(as)}{\sqrt{2}}, \frac{\sin(as)}{\sqrt{2}}, \frac{\cos(bs)}{\sqrt{2}}, \frac{\sin(bs)}{\sqrt{2}}\right).$$

Proof. If $k_1 = 1$, then the general solution of (4.1) is

$$\gamma(s) = c_1 + c_2s + c_3 \cos(\sqrt{2}s) + c_4 \sin(\sqrt{2}s).$$

Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, we have $c_2 = 0$, while c_1, c_3, c_4 are constant vectors orthogonal to each other with $|c_1|^2 = |c_3|^2 = |c_4|^2 = 1/2$. Then the solution becomes

$$\gamma(s) = \left(d_1, \frac{\cos(\sqrt{2}s)}{\sqrt{2}}, \frac{\sin(\sqrt{2}s)}{\sqrt{2}}, d_2, d_3 \right),$$

with $-d_1^2 + d_2^2 + d_3^2 = 1/2$. It is obvious that γ is a circle of radius $1/\sqrt{2}$. If $0 < k_1 < 1$, then the general solution of (4.1) is

$$\gamma(s) = c_1 \cos(as) + c_2 \sin(as) + c_3 \cos(bs) + c_4 \sin(bs)$$

where $a = \sqrt{1 - k_1}$ and $b = \sqrt{1 + k_1}$. Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, we obtain that the vectors $c_i, i = 1, 2, 3, 4$, are orthogonal to each other and $|c_1|^2 = |c_2|^2 = |c_3|^2 = |c_4|^2 = 1/2$. Then the curve γ becomes

$$\gamma(s) = \left(0, \frac{\cos(as)}{\sqrt{2}}, \frac{\sin(as)}{\sqrt{2}}, \frac{\cos(bs)}{\sqrt{2}}, \frac{\sin(bs)}{\sqrt{2}} \right). \quad \blacksquare$$

Proposition 4.2. *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that the principal normal vector N is spacelike and first binormal vector B_1 is timelike. Then*

$$(4.2) \quad \gamma^{(IV)} + 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

If $k_1 = 1$, then it is obvious that the general solution of (4.2) is a circle of radius $1/\sqrt{2}$. If $k_1 > 1$, then the general solution of (4.2) is

$$\gamma(s) = c_1 e^{as} + c_2 e^{-as} + c_3 \cos(bs) + c_4 \sin(bs)$$

with $a = \sqrt{k_1 - 1}$ and $b = \sqrt{k_1 + 1}$. Here $c_i, i = 1, 2, 3, 4$ are constant vectors. Since $|\gamma|^2 = 1$ and $|\gamma'|^2 = 1$, by choosing

$$c_1 = (1, 0, 0, 0, 1), \quad c_2 = \left(-1, \frac{\sqrt{7}}{4}, 0, 0, -\frac{3}{4} \right),$$

$$c_3 = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0 \right), \quad c_4 = \left(-\frac{\sqrt{7}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, -\frac{\sqrt{7}}{\sqrt{2}} \right),$$

such that

$$\begin{aligned} \langle c_1, c_1 \rangle &= \langle c_2, c_2 \rangle = 0, \\ \langle c_3, c_3 \rangle &= \langle c_4, c_4 \rangle = \frac{3}{b^2}, \\ \langle c_1, c_2 \rangle &= \frac{1}{a^2}, \\ \langle c_1, c_3 \rangle &= \langle c_1, c_4 \rangle = 0, \\ \langle c_2, c_3 \rangle &= \langle c_2, c_4 \rangle = 0, \\ \langle c_3, c_4 \rangle &= 0, \end{aligned}$$

with $a = 2$, $b = \sqrt{6}$, we obtain following special solution of differential equation (4.2)

$$\gamma(s) = \left(e^{2s} - e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{6}s), \frac{\sqrt{7}}{4} e^{-2s} + \frac{1}{\sqrt{2}} \sin(\sqrt{6}s), \right. \\ \left. \frac{1}{2} \cos(\sqrt{6}s), \frac{1}{2} \cos(\sqrt{6}s), e^{2s} - \frac{3}{4} e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{6}s) \right),$$

which is a helix with $k_1 = 5$ and $k_2 = 2\sqrt{6}$.

Proposition 4.3. *Let $\gamma : I \rightarrow S_1^4$ be a spacelike nongeodesic biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a moving Frenet frame along γ such that the principal normal vector N is spacelike and first binormal vector B_1 is null. Then*

$$(4.3) \quad \gamma^{(IV)} + 2\gamma'' = 0.$$

It can be easily seen that the general solution of differential equation (4.3) is a circle of radius $1/\sqrt{2}$.

Proposition 4.4. *Let $\gamma : I \rightarrow S_1^4$ be a timelike nongeodesic biharmonic curve parametrized by arclength. Then*

$$(4.4) \quad \gamma^{(IV)} - 2\gamma'' + (1 - k_1^2)\gamma = 0.$$

If $k_1 = 1$, then the general solution of (4.4) is

$$\gamma(s) = c_1 + c_2s + c_3e^{-\sqrt{2}s} + c_4e^{\sqrt{2}s}$$

Here $c_i, i = 1, 2, 3, 4$, are constant vectors. Since $\langle \gamma(s), \gamma(s) \rangle = 1$ and $\langle \gamma'(s), \gamma'(s) \rangle = -1$, by choosing

$$c_1 = \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 1 \right), \quad c_2 = (0, 0, 0, 0, 0), \\ c_3 = \left(-1, \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}} \right), \quad c_4 = \left(1, -\frac{\sqrt{2}}{4}, \frac{1}{2\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right),$$

such that

$$\langle c_1, c_1 \rangle = \frac{1}{2} \\ \langle c_2, c_2 \rangle = \langle c_3, c_3 \rangle = \langle c_4, c_4 \rangle = 0, \\ \langle c_1, c_2 \rangle = \langle c_1, c_3 \rangle = \langle c_1, c_4 \rangle = 0, \\ \langle c_2, c_3 \rangle = \langle c_2, c_4 \rangle = 0, \\ \langle c_3, c_4 \rangle = \frac{1}{4},$$

we obtain following special solution of differential equation (4.4)

$$\gamma(s) = \left(\frac{1}{\sqrt{2}} - e^{-\sqrt{2}s} + e^{\sqrt{2}s}, \frac{e^{-\sqrt{2}s}}{\sqrt{2}} - \frac{e^{\sqrt{2}s}}{2\sqrt{2}}, \frac{e^{\sqrt{2}s}}{2\sqrt{2}}, \frac{e^{\sqrt{2}s}}{2}, 1 - \frac{e^{-\sqrt{2}s}}{\sqrt{2}} + \frac{e^{\sqrt{2}s}}{\sqrt{2}} \right),$$

which is a circle. If $k_1 > 1$, then the general solution of (4.4) is

$$\gamma(s) = c_1 e^{as} + c_2 e^{-as} + c_3 \cos(bs) + c_4 \sin(bs)$$

with $a = \sqrt{k_1 + 1}$ and $b = \sqrt{k_1 - 1}$. Here c_i , $i = 1, 2, 3, 4$, are constant vectors. Since again $\langle \gamma(s), \gamma(s) \rangle = 1$ and $\langle \gamma'(s), \gamma'(s) \rangle = -1$, by choosing

$$c_1 = (1, 0, 0, 0, 1), \quad c_2 = \left(-1, \frac{\sqrt{7}}{4}, 0, 0, -\frac{3}{4} \right),$$

$$c_3 = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0 \right), \quad c_4 = \left(-\frac{\sqrt{7}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, -\frac{\sqrt{7}}{\sqrt{2}} \right),$$

such that c_1 and c_2 are null vectors and

$$\langle c_3, c_3 \rangle = \langle c_4, c_4 \rangle = \frac{1}{b^2},$$

$$\langle c_1, c_2 \rangle = \frac{1}{a^2},$$

$$\langle c_1, c_3 \rangle = \langle c_1, c_4 \rangle = 0,$$

$$\langle c_2, c_3 \rangle = \langle c_2, c_4 \rangle = 0,$$

$$\langle c_3, c_4 \rangle = 0,$$

with $a = 2$, $b = \sqrt{2}$, we obtain following special solution of differential equation (4.4)

$$\gamma(s) = \left(e^{2s} - e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{2}s), \frac{\sqrt{7}}{4} e^{-2s} + \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \right.$$

$$\left. \frac{1}{2} \cos(\sqrt{2}s), \frac{1}{2} \cos(\sqrt{2}s), e^{2s} - \frac{3}{4} e^{-2s} - \frac{\sqrt{7}}{\sqrt{2}} \sin(\sqrt{2}s) \right),$$

which is a helix with $k_1 = 3$ and $k_2 = 2\sqrt{2}$.

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References

- [1] K. Arslan, R. Ezentas, C. Murathan and T. Sasahara, Biharmonic submanifolds in 3-dimensional (κ, μ) -manifolds, *Int. J. Math. Math. Sci.* **2005**, no. 22, 3575–3586.
- [2] K. Arslan, R. Ezentas, C. Murathan and T. Sasahara, Biharmonic anti-invariant submanifolds in Sasakian space forms, *Beiträge Algebra Geom.* **48** (2007), no. 1, 191–207.
- [3] A. Balmuş, On the biharmonic curves of the Euclidian and Berger 3-dimensional spheres, *Sci. Ann. Univ. Agric. Sci. Vet. Med.* **47** (2004), no. 2, 87–96.
- [4] R. Caddeo, Riemannian manifolds on which the distance function is biharmonic, *Rend. Sem. Mat. Univ. Politec. Torino* **40** (1982), no. 1, 93–101.
- [5] R. Caddeo and L. Vanhecke, Does “ $\Delta^2 d^{2-n} = 0$ on a Riemannian manifold” imply flatness?, *Period. Math. Hungar.* **17** (1986), no. 2, 109–117.
- [6] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds of S^3 , *Internat. J. Math.* **12** (2001), no. 8, 867–876.
- [7] R. Caddeo, S. Montaldo and P. Piu, Biharmonic curves on a surface, *Rend. Mat. Appl. (7)* **21** (2001), no. 1–4, 143–157.

- [8] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds in spheres, *Israel J. Math.* **130** (2002), 109–123.
- [9] R. Caddeo, C. Oniciuc and P. Piu, Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group, *Rend. Sem. Mat. Univ. Politec. Torino* **62** (2004), no. 3, 265–277.
- [10] M. C. Chaki and B. Gupta, On conformally symmetric spaces, *Indian J. Math.* **5** (1963), 113–122.
- [11] S.-Y. A. Chang, L. Wang and P. C. Yang, A regularity theory of biharmonic maps, *Comm. Pure Appl. Math.* **52** (1999), no. 9, 1113–1137.
- [12] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.* **17** (1991), no. 2, 169–188.
- [13] B.-Y. Chen and S. Ishikawa, Biharmonic surfaces in pseudo-Euclidean spaces, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **45** (1991), no. 2, 323–347.
- [14] B.-Y. Chen and S. Ishikawa, Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces, *Kyushu J. Math.* **52** (1998), no. 1, 167–185.
- [15] F. Defever, G. Kaimakamis and V. Papantoniou, Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space \mathbb{E}_s^4 , *J. Math. Anal. Appl.* **315** (2006), no. 1, 276–286.
- [16] J. Eells, Jr. and J. H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.
- [17] D. Fetcu, Biharmonic curves in the generalized Heisenberg group, *Beiträge Algebra Geom.* **46** (2005), no. 2, 513–521.
- [18] D. Fetcu and C. Oniciuc, Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres, *Abh. Math. Sem. Univ. Hamburg* **77** (2007), 179–190.
- [19] J.-I. Inoguchi, Biharmonic curves in Minkowski 3-space, *Int. J. Math. Math. Sci.* **2003**, no. 21, 1365–1368.
- [20] J.-I. Inoguchi, Submanifolds with harmonic mean curvature vector field in contact 3-manifolds, *Colloq. Math.* **100** (2004), no. 2, 163–179.
- [21] S. Ishikawa, Biharmonic W -surfaces in 4-dimensional pseudo-Euclidean space, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **46** (1992), no. 2, 269–286.
- [22] G. Y. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser. A* **7** (1986), no. 2, 130–144.
- [23] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, *Chinese Ann. Math. Ser. A* **7** (1986), no. 4, 389–402.
- [24] T. Lamm, Biharmonic map heat flow into manifolds of nonpositive curvature, *Calc. Var. Partial Differential Equations* **22** (2005), no. 4, 421–445.
- [25] K. Matsumoto and I. Mihai, On a certain transformation in a Lorentzian para-Sasakian manifold, *Tensor (N.S.)* **47** (1988), no. 2, 189–197.
- [26] K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. Yamagata Univ. Natur. Sci.* **12** (1989), no. 2, 151–156.
- [27] K. Matsumoto, I. Mihai and R. Roşca, ξ -null geodesic gradient vector fields on a Lorentzian para-Sasakian manifold, *J. Korean Math. Soc.* **32** (1995), no. 1, 17–31.
- [28] I. Mihai and R. Roşca, On Lorentzian P -Sasakian manifolds, in *Classical Analysis (Kazimierz Dolny, 1991)*, 155–169, World Sci. Publ., River Edge, NJ.
- [29] I. Mihai, A. A. Shaikh and U. C. De, On Lorentzian para-Sasakian manifolds, *Rendicontidel Seminario Matematico di Messina, Serie II*, 1999.
- [30] S. Montaldo and C. Oniciuc, A short survey on biharmonic maps between Riemannian manifolds, *Rev. Un. Mat. Argentina* **47** (2006), no. 2, 1–22 (2007).
- [31] B. C. Montano, L. Di Terlizzi and M. M. Tripathi, Invariant submanifolds of contact (κ, μ) -manifolds, *Glasg. Math. J.* **50** (2008), no. 3, 499–507.
- [32] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
- [33] Y.-L. Ou, p -harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps, *J. Geom. Phys.* **56** (2006), no. 3, 358–374.
- [34] C. Z. Ouyang, 2-harmonic space-like submanifolds of a pseudo-Riemannian space form, *Chinese Ann. Math. Ser. A* **21** (2000), no. 6, 649–654.
- [35] C. Özgür and M. M. Tripathi, On Legendre curves in α -Sasakian manifolds, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 1, 91–96.

- [36] L. Sario, M. Nakai, C. Wang and L. Chung, *Classification Theory of Riemannian Manifolds*, Springer, Berlin, 1977.
- [37] T. Sasahara, Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors, *Publ. Math. Debrecen* **67** (2005), no. 3-4, 285–303.
- [38] I. Satō, On a structure similar to the almost contact structure, *Tensor (N.S.)* **30** (1976), no. 3, 219–224.
- [39] P. Strzelecki, On biharmonic maps and their generalizations, *Calc. Var. Partial Differential Equations* **18** (2003), no. 4, 401–432.
- [40] H. Sun, 2-harmonic spacelike submanifolds in De Sitter space, *Journal of Southern Institute of Metallurgy*, **21** (2000), no. 2, 138–142.
- [41] M. Tarafdar and A. Bhattacharyya, On Lorentzian para-Sasakian manifolds, in *Steps in Differential Geometry (Debrecen, 2000)*, 343–348, Inst. Math. Inform., Debrecen.
- [42] M. M. Tripathi and U. C. De, Lorentzian almost paracontact manifolds and their submanifolds, *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **8** (2001), no. 2, 101–125.
- [43] M. M. Tripathi and S. S. Shukla, On submanifolds of Lorentzian almost paracontact manifolds, *Publ. Math. Debrecen* **59** (2001), no. 3-4, 327–338.
- [44] J. Walrave, Curves and surfaces in Minkowski Space, Doctoral Thesis, K. U. Leuven, Fac. of Science, 1995.
- [45] C. Wang, Biharmonic maps from \mathbf{R}^4 into a Riemannian manifold, *Math. Z.* **247** (2004), no. 1, 65–87.
- [46] C. Wang, Remarks on biharmonic maps into spheres, *Calc. Var. Partial Differential Equations* **21** (2004), no. 3, 221–242.
- [47] W. Zhang, *Biharmonic space-like hypersurfaces in pseudo-Riemannian space*, arxiv: math/0808.1346v1.