

Simplicity of 2-Graph Algebras Associated to Dynamical Systems

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Abstract. We give a combinatorial description of a family of 2-graphs which subsumes those described by Pask, Raeburn and Weaver. Each 2-graph Λ we consider has an associated C^* -algebra, denoted $C^*(\Lambda)$, which is simple and purely infinite when Λ is aperiodic. We give new, straightforward conditions which ensure that Λ is aperiodic. These conditions are highly tractable as we only need to consider the finite set of vertices of Λ in order to identify aperiodicity. In addition, the path space of each 2-graph can be realised as a two-dimensional dynamical system which we show must have zero entropy.

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1. Introduction

Higher-rank graphs, or k -graphs, were introduced by Kumjian and Pask [4] as a graph based model for the higher-rank Cuntz-Krieger algebras studied by Robertson and Steger [15]. The C^* -algebras associated to higher-rank graphs have been generating a lot of interest [2, 3, 17, 18]. In this paper we show how to obtain a class of 2-graphs from a small set of parameters, which we will call “basic data”. Examples generated in this way include all those considered by Pask, Raeburn and Weaver in [10]. Our examples will therefore add to the emerging connections between 2-graphs, classification theory and two-dimensional shift spaces [5, 9, 20, 19].

This paper has three aims: The first aim is to model a family of 2-graphs which subsumes those studied in [10]. The second aim is to identify when the associated C^* -algebra is simple and purely infinite using the new results of [7, 14]. The third aim is to establish connections between the 2-graphs we generate and two-dimensional shift spaces, as defined in [1, 8, 12]. In doing this, we generalise certain results of Pask, Raeburn and Weaver in [10].

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For our first aim, we write down a set of basic data, BD which determines a finite set of labelled tiles. In Theorem 3.1 we show that BD completely specifies a 2-graph, denoted Λ_{BD} : The vertices of Λ_{BD} are a set of labelled tiles, with an edge between two vertices if their labels overlap in a particular way. The 2-graph Λ_{BD} has a visual representation, called the skeleton of Λ_{BD} . We prove several basic properties about the skeleton of Λ_{BD} in Proposition 3.3 and Proposition 3.4. In particular we show that Λ_{BD} is strongly connected, row-finite and has no sources.

For our second aim, we investigate when the C^* -algebra associated to Λ_{BD} is purely infinite and simple. The results of [4, 7, 13, 17] lead to a characterisation of simplicity of k -graph C^* -algebras in terms of two conditions on the underlying k -graph; namely aperiodicity and cofinality. Since all our examples are cofinal, we focus our attention on identifying when Λ_{BD} is aperiodic. Theorem 4.1 gives a condition on the set of vertices of Λ_{BD} which ensures that Λ_{BD} is aperiodic. As the set of vertices is finite, checking this condition is straightforward. Our aperiodicity result then allows us to prove the primary result of this paper, Theorem 5.1, which gives necessary conditions for $C^*(\Lambda_{BD})$ to be simple and purely infinite.

For our third aim, we note that the shift map σ induces a \mathbb{Z}^2 -action on the two-sided infinite path space Λ_{BD}^Δ of Λ_{BD} and so Λ_{BD}^Δ is a shift space. In Theorem 6.1 we show that the shift space Λ_{BD}^Δ is conjugate to a two-dimensional shift of finite type, as studied by Quas and Trow [12]. Unlike the 2-graphs studied in [10] the path space Λ_{BD}^Δ has no underlying algebraic properties.

We first introduce some background in Section 2. In Section 3, we describe the basic data from which we generate a 2-graph Λ_{BD} . We go on to prove some general properties about the skeletons of Λ_{BD} . In Section 4, we detail a condition on the vertices of Λ_{BD} which will ensure that Λ_{BD} is aperiodic. We use this condition to establish when $C^*(\Lambda_{BD})$ is unital, nuclear, purely infinite and simple. This result is proved in Section 5. Finally we connect our work to the literature on two-dimensional shift spaces in Section 6.

2. Background information

2.1. Higher-rank graphs

We regard \mathbb{N}^k as a semigroup under addition, with basis elements e_i . For $m, n \in \mathbb{N}^k$ we write m_i, n_i for the i th coordinates of m and n , $m \vee n$ for their coordinatewise maximum and $m \wedge n$ for their coordinatewise minimum. We let \leq denote the usual coordinatewise partial order on \mathbb{N}^k .

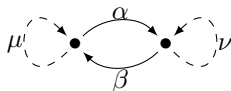
A k -graph is a pair (Λ, d) consisting of a countable category Λ , together with a functor $d : \Lambda \rightarrow \mathbb{N}^k$ which satisfies the *factorisation property*: For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfying $d(\lambda) = m + n$ there exists unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We refer to d as the *degree map* and think of it as a generalised length function for elements of Λ . We think of objects in the category Λ as vertices, and morphisms in Λ as paths. We refer to morphisms of degree e_i as edges. For $n \in \mathbb{N}^k$ let $\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}$ be the set of all paths of degree n . We frequently omit the functor d when referring to a k -graph.

For each $\lambda \in \Lambda^n$ the factorisation property implies there exist unique elements $r(\lambda), s(\lambda) \in \Lambda^0$ such that $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$. Hence we can identify the objects in

a k -graph with the paths of degree 0. For $X \subseteq \Lambda$ and $v \in \Lambda^0$ let $vX = \{\lambda \in X : r(\lambda) = v\}$ and $Xv = \{\lambda \in X : s(\lambda) = v\}$. For $u, v \in \Lambda^0$ set $uXv = uX \cap Xv$.

In this paper we will be working exclusively with the case when $k = 2$. All 2-graphs in this paper are row-finite and free of sinks and sources: For every $v \in \Lambda^0$ and every $n \in \mathbb{N}^2$ we have $0 < |v\Lambda^n| < \infty$ and $0 < |\Lambda^n v|$. (See Proposition 3.3).

To visualise a 2-graph we draw its *skeleton*. The skeleton of a 2-graph Λ is a bicoloured directed graph with vertices Λ^0 and edges $\Lambda^{e_1} \cup \Lambda^{e_2}$ with range and source maps inherited from Λ . We call edges of degree e_1 blue edges, and draw them using solid lines. We call edges of degree e_2 red edges, and draw them using dashed lines. The following picture is an example of a skeleton of a 2-graph Λ .



The skeleton itself does not encode all the information in Λ , we also need to record the factorisation of paths of degree $e_1 + e_2$ in Λ . In the skeleton given above, there are only two such factorisations, namely $\alpha\mu = \nu\alpha$ and $\beta\mu = \nu\beta$. Since there is at most one edge of each colour between any two vertices, and for every blue-red path between two vertices there is exactly one red-blue path between the same vertices, the skeleton itself determines the factorisations of paths of degree $e_1 + e_2$, and so we need not record them. The 2-graphs we study in this paper will also have this property (see Proposition 3.3 (4)). A concise overview of skeletons can be found in [13, Section 2].

Let Λ be a row-finite 2-graph with no sources. A *Cuntz-Krieger Λ -family* is a set of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ satisfying:

- (CK1) $\{S_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
- (CK2) $S_\mu S_\nu = S_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
- (CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$ and;
- (CK4) $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^2$.

The C^* -algebra $C^*(\Lambda)$ is then the universal C^* -algebra generated by a Cuntz-Krieger Λ -family $\{s_\lambda : \lambda \in \Lambda\}$: Given any other Cuntz-Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ in a C^* -algebra B there exists a unique homomorphism $\pi_S : C^*(\Lambda) \rightarrow B$ such that $\pi_S(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$. By [4, Proposition 2.11] there is a Cuntz-Krieger Λ -family where each s_λ is non-zero. Further background on higher-rank graph C^* -algebras can be found in [13, Section 3].

2.2. Two-dimensional shift spaces

Let A be a finite set of symbols and denote by $A^{\mathbb{Z}^2}$ the set of all functions $\mathbb{Z}^2 \rightarrow A$. We think of elements in $A^{\mathbb{Z}^2}$ as being an infinite two-dimensional array of symbols from A . For $x \in A^{\mathbb{Z}^2}$ and $S \subset \mathbb{Z}^2$, let $x|_S$ denote the restriction of x to S . A map $E : S \rightarrow A$ is called a *configuration*. For a subset X of $A^{\mathbb{Z}^2}$ a configuration E is said to be *allowed* for X if there exists $x \in X$ such that $x|_S = E$. In this case we say that E occurs in x .

For every $b \in \mathbb{Z}^2$, define $\sigma_b : A^{\mathbb{Z}^2} \rightarrow A^{\mathbb{Z}^2}$ by $\sigma_b(x)(m) = x(m + b)$ for all $x \in A^{\mathbb{Z}^2}$ and $m \in \mathbb{Z}^2$. Since σ_b is a homeomorphism and $\sigma_a \sigma_b = \sigma_{a+b}$ for all $a, b \in \mathbb{Z}^2$ we see that \mathbb{Z}^2 acts on $A^{\mathbb{Z}^2}$. A nonempty subset X of $A^{\mathbb{Z}^2}$ which is invariant under σ_b for all $b \in \mathbb{Z}^2$ is called a *shift space*. A shift space X is called a *shift of finite type* if there exists a finite set $S \subset \mathbb{Z}^2$ and a nonempty subset $Q \subset A^S$ such that

$$X = \{x \in A^{\mathbb{Z}^2} : x|_{S+b} \in Q \text{ for every } b \in \mathbb{Z}^2\}.$$

We think of Q as a finite set of allowed configurations for X .

For a shift space X and a fixed $d \in \mathbb{N}$, let $\mathcal{B}_d(X)$ denote the set of all $d \times d$ square configurations which occur in X . We define the *topological entropy* of a shift space X to be $\lim_{d \rightarrow \infty} (1/2^d) \log |\mathcal{B}_d(X)|$ and denote it by $h(X)$. Further background information about the shift spaces used in this paper, and their entropy can be found in [12].

3. The 2-graph construction

We begin by introducing some notation for the data which we use to determine a 2-graph.

A subset $T \subset \mathbb{N}^2$ is *hereditary* if $n \in T$ and $0 \leq m \leq n$ implies $m \in T$. Define a *tile* T to be a hereditary subset of \mathbb{N}^2 with finite cardinality $|T|$ and let $(c_1, c_2) := \bigvee \{i : i \in T\}$. Then, using the visualisation convention of [10, Section 1] for tiles, T consists of $|T|$ squares with $c_1 + 1$ squares in the first row and $c_2 + 1$ squares in the first column. If T is a tile with $|T| \geq 2$ then we say that T is *nondegenerate*.

Example 3.1. The set $T = \{0, e_1, 2e_1, e_1 + e_2, e_2\}$ is a nondegenerate tile with $c_1 = 2$ and $c_2 = 1$. We visualise T as



Note that the first row has $c_1 + 1 = 3$ squares and the first column has $c_2 + 1 = 2$ squares.

If T is a nondegenerate tile we define $P = T \setminus \{c_1 e_1, c_2 e_2\}$. If $c_1, c_2 \geq 1$ then we visualise P as T with the top left and bottom right squares removed. If $c_1 = 0, c_2 \geq 1$ (resp. $c_2 = 0, c_1 \geq 1$) then we visualise P as T with the squares corresponding to the origin 0 and $c_2 e_2$ (resp. $c_1 e_1$) removed. Note that $P = \emptyset$ if and only if $|T| = 2$.

Definition 3.1. A set of basic data is a triple $BD := (T, A, \{f_p : p \in A^P\})$, where T is a nondegenerate tile, A is a nonempty set (which we call the alphabet), and for each $p \in A^P$ the map $f_p : A \rightarrow A$ is a bijection.

Remark 3.1.

- (1) Recall that for a nonempty set A we have $A^\emptyset = \{\emptyset\}$.
- (2) If T is degenerate, that is $T = \{0\}$, then we define the basic data to be the triple $(T, A, \{f_a\})$ where for $a \in A$, f_a is the function $f_a : T \rightarrow A$ given by $f_a(0) = a$. Following this case through the following text would obfuscate the main results, we comment separately on this case in Remarks 3.3, 3.4 and 6.1.

The main result of this section is Theorem 3.1 which describes the unique 2-graph Λ_{BD} associated to the basic data BD . In order to state this theorem we require some notation. For each $p \in A^P$ and $a \in A$ define $F_{p,a} : T \rightarrow A$ by

$$(3.1) \quad F_{p,a}(t) = \begin{cases} a & \text{when } t = c_2e_2, \\ f_p(a) & \text{when } t = c_1e_1, \\ p(t) & \text{when } t \in P. \end{cases}$$

If $P = \emptyset$ then either $T = \{0, e_1\}$ in which case $F_{\emptyset,a}(0) = a$, $F_{\emptyset,a}(e_1) = f_\emptyset(a)$ for all $a \in A$, or $T = \{0, e_2\}$ in which case $F_{\emptyset,a}(e_2) = a$, $F_{\emptyset,a}(0) = f_\emptyset(a)$ for all $a \in A$.

We denote the collection $\bigcup_{p \in A^P, a \in A} F_{p,a}$ by Λ_{BD}^0 .

Definition 3.2. For $S \subset \mathbb{N}^2$ and $n \in \mathbb{N}^2$ let $S + n = \{m + n : m \in S\}$ denote the translate of S by n and let $T(n) := \bigcup_{0 \leq m \leq n} T + m$ denote the union of translates of T by $m \leq n$.

Example 3.2. For the tile $T = \{0, e_1, 2e_1, e_1 + e_2, e_2\}$ defined in Example 3.1 we have

$$T(2, 1) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

If $f : S \rightarrow A$ is a function defined on a subset $S \subset \mathbb{N}^2$ containing $T + n$, then we define $f|_{T+n} : T \rightarrow A$ by $f|_{T+n}(i) = f(i + n)$ for $i \in T$. (In [10] $f|_{T+n}$ was denoted $f|_{T+n}$). Note that $f|_T = f|_T$.

Definition 3.3. For $n \in \mathbb{N}^2$ a path of degree n is a function $\lambda : T(n) \rightarrow A$ such that $\lambda|_{T+m} \in \Lambda_{BD}^0$ for $0 \leq m \leq n$, with source $s(\lambda) = \lambda|_T$ and range $r(\lambda) = \lambda|_{T+n}$. We denote the set of all paths of degree n by Λ_{BD}^n and define $\Lambda_{BD}^* = \bigcup_{n \in \mathbb{N}^2} \Lambda_{BD}^n$.

For $\lambda \in \Lambda_{BD}^\ell$ and $0 \leq m \leq n \leq \ell$ the factorisation $\lambda(m, n)$ is a path of degree $n - m$ defined by

$$\lambda(m, n)(i) = \lambda(m + i) \text{ for } i \in T(n - m).$$

Observe in particular that $\lambda(m, m) = \lambda|_{T+m} \in \Lambda_{BD}^0$. This is a deviation from notation used elsewhere in the literature [3, 7, 13, 14] and so it is worth highlighting the fact that $\lambda(i) \neq \lambda(i, i)$. This is because $\lambda(i) \in A$ is a value of λ at a specific coordinate while $\lambda(i, i) \in A^T$ is a vertex. Thus we continue the convention of [10].

Now we have all the notation required to state the main theorem of this section.

Theorem 3.1. Given basic data BD , the quadruple $\Lambda_{BD} := (\Lambda_{BD}^0, \Lambda_{BD}^*, r, s)$ is a category. The function $d : \Lambda_{BD} \rightarrow \mathbb{N}^2$ given by $d(\lambda) = n$ for $\lambda \in \Lambda_{BD}^n$ is a functor satisfying the factorisation property; hence (Λ_{BD}, d) is a 2-graph.

The following two key examples were used to motivate Theorem 3.1.

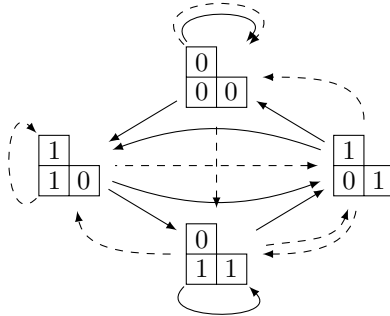
Example 3.3.

- (1) Let $A = \{0, 1\}$ and $T = \{0, e_1, e_2\}$ be a nondegenerate tile. Then $P = \{0\}$ and so $A^P = \{p[0], p[1]\}$ where for $i \in A$ we define $p[i] : P \rightarrow A$ by $p[i](0) = i$. To complete our basic data BD we define bijections $f_{p[0]}, f_{p[1]} : A \rightarrow A$ by

$f_{p[0]}(a) = a$ and $f_{p[1]}(a) = a + 1 \pmod{2}$ for all $a \in A$. Then by Theorem 3.1 the 2-graph Λ_{BD} associated to BD has vertices

$$F_{p[0],0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, F_{p[0],1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F_{p[1],0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F_{p[1],1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The above blocks form the Ledrappier system, named after the two-dimensional shift space studied in [6]. In [10] it was shown that the associated 2-graph has skeleton:



- (2) Let $T = \{0, e_1, e_2, e_1 + e_2\}$ be a nondegenerate tile and $A = \{0, 1\}$. Then $P = \{0, e_1 + e_2\}$ and so $A^P = \{p[0, 0], p[0, 1], p[1, 0], p[1, 1]\}$ where for $a, b \in A$ we define $p[a, b] : P \rightarrow A$ by $p[a, b](0) = a$ and $p[a, b](e_1 + e_2) = b$. To obtain a set of basic data BD we define a set of bijections as follows: Let $f_{p[0,0]}(a) = a$ for every $a \in A$ and $f_p(a) = a + 1 \pmod{2}$ for every $a \in A$ and every $p \in A^P \setminus \{p[0, 0]\}$. Then by Theorem 3.1 the 2-graph Λ_{BD} associated to this basic data has vertices

$$(3.2) \quad \begin{aligned} F_{p[0,0],0} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & F_{p[0,0],1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & F_{p[0,1],0} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, & F_{p[0,1],1} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \\ F_{p[1,0],0} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, & F_{p[1,0],1} &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, & F_{p[1,1],0} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, & F_{p[1,1],1} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Remark 3.2. In [10, Theorem 3.4], Pask, Raeburn and Weaver use four parameters to generate a 2-graph: A tile T , an alphabet $\mathbb{Z}/q\mathbb{Z}$ where $q \geq 2$, a trace $t \in \mathbb{Z}/q\mathbb{Z}$ and a rule $w : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that $w(c_1e_1)$ and $w(c_2e_2)$ are invertible in the ring $\mathbb{Z}/q\mathbb{Z}$. The vertices of the 2-graph $\Lambda(T, q, t, w)$ associated to the parameters (T, q, t, w) consist of functions $v : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that $\sum_{i \in T} v(i)w(i) = t \pmod{q}$. The parameters $T = \{0, e_1, e_2, e_1 + e_2\}$, $q = 2$, $t = 0$ and $w \equiv 0$ are the only ones which yield the eight vertices shown in (3.2). However $\Lambda(T, 2, 0, 0)$ has sixteen vertices, hence the 2-graph Λ_{BD} of Example 3.3 (2) is **not** in the family of 2-graphs described in [10].

It turns out that all 2-graphs described in [10] can be replicated using the basic data given in this paper.

Proposition 3.1. *Let T be a nondegenerate tile, $q \geq 2$ an integer, $t \in \mathbb{Z}/q\mathbb{Z}$, and $w : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ a function with $w(c_1e_1), w(c_2e_2)$ invertible in $\mathbb{Z}/q\mathbb{Z}$. Let $\Lambda(T, q, t, w)$ be the associated 2-graph (see [10]). For each $p \in (\mathbb{Z}/q\mathbb{Z})^P$ there is a bijection $f_p : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$ such that $\Lambda(T, q, t, w) \cong \Lambda_{BD}$ where $BD = (T, \mathbb{Z}/q\mathbb{Z}, \{f_p : p \in (\mathbb{Z}/q\mathbb{Z})^P\})$.*

Proof. For $p \in (\mathbb{Z}/q\mathbb{Z})^P$ and $a \in \mathbb{Z}/q\mathbb{Z}$ we may define $f_p : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$ by

$$(3.3) \quad f_p(a) = (t - \sum_{i \in P} w(i)p(i) - w(c_2e_2)a)w(c_1e_1)^{-1}$$

since $w(c_1e_1)$ is invertible in $\mathbb{Z}/q\mathbb{Z}$. Observe that

$$(3.4) \quad \sum_{i \in P} w(i)p(i) + w(c_1e_1)f_p(a) + w(c_2e_2)a = t \text{ for all } a \in \mathbb{Z}/q\mathbb{Z}.$$

To see that $f_p : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$ is a bijection we solve (3.4) for a , and then define

$$f_p^{-1}(a) = (t - \sum_{i \in P} w(i)p(i) - w(c_1e_1)f_p(a))w(c_2e_2)^{-1}.$$

Which makes sense since $w(c_2e_2)$ is invertible in $\mathbb{Z}/q\mathbb{Z}$.

Set $BD = (T, \mathbb{Z}/q\mathbb{Z}, \{f_p : p \in (\mathbb{Z}/q\mathbb{Z})^P\})$ and let $v \in \Lambda(T, q, t, w)^0$. We define $p(v) \in (\mathbb{Z}/q\mathbb{Z})^P$ by $p(v) = v|_P$. Define the map $\phi : \Lambda(T, q, t, w)^0 \rightarrow \Lambda_{BD}^0$ by $\phi(v) = F_{p(v), v(c_2e_2)}$ where $p(v) : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$ is the bijection defined in (3.3). For $F_{p,a} \in \Lambda_{BD}^0$ define $v_{p,a} : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_{p,a}|_P = p$, $v_{p,a}(c_2e_2) = a$ and $v_{p,a}(c_1e_1) = f_p(a)$. By (3.4) we see that $v_{p,a} \in \Lambda(T, q, t, w)^0$. Hence $F_{p,a} \mapsto v_{p,a}$ defines a map $\psi : \Lambda_{BD}^0 \rightarrow \Lambda(T, q, t, w)^0$ such that $\phi \circ \psi(F_{p,a}) = F_{p,a}$ for all $p \in (\mathbb{Z}/q\mathbb{Z})^P$ and $a \in \mathbb{Z}/q\mathbb{Z}$. It follows that $\Lambda(T, q, t, w)^0 = \Lambda_{BD}^0$ as functions from T to $\mathbb{Z}/q\mathbb{Z}$. Similarly, we have $\Lambda(T, q, t, w)^n = \Lambda_{BD}^n$ as maps from $T(n)$ to $\mathbb{Z}/q\mathbb{Z}$ for all $n \in \mathbb{N}^2$. Hence $\Lambda(T, q, t, w)^* = \Lambda_{BD}^*$ and since the structure maps and the degree map are defined in the same manner it follows that the 2-graphs $\Lambda(T, q, t, w)$ and Λ_{BD} are isomorphic, as required. ■

Remark 3.3. If $T = \{0\}$ is a degenerate tile, then one checks that $\Lambda(T, q, t, w) \cong \Lambda_{BD}$ where $BD = (T, \mathbb{Z}/q\mathbb{Z}, \{f_{tw(0)^{-1}}\})$.

We now turn to the proof of Theorem 3.1. Our goal is for Λ_{BD}^* to be the morphisms in a category so that from a set of basic data we can generate a 2-graph. To this end we need to say how to compose elements $\mu, \nu \in \Lambda_{BD}^*$. Our method is motivated by [10].

Proposition 3.2. *Suppose that $\mu, \nu \in \Lambda_{BD}^*$ satisfy $s(\mu) = r(\nu)$. Then there exists a unique path $\lambda \in \Lambda_{BD}^{d(\mu)+d(\nu)}$ such that*

$$(3.5) \quad \lambda(0, d(\mu)) = \mu \quad \text{and} \quad \lambda(d(\mu), d(\mu) + d(\nu)) = \nu.$$

Notice that λ is already explicitly defined on $T(d(\mu)) \cup (T(d(\nu)) + d(\mu))$. Since $\mu\nu \in \Lambda_{BD}^*$ it remains to show that there is a unique function λ' which extends λ to $T(d(\mu) + d(\nu))$ in such a way that $\lambda' \downarrow_{T+k} \in \Lambda_{BD}^0$ for every $k \in T(d(\mu) + d(\nu)) \setminus (T(d(\mu)) \cup (T(d(\nu)) + d(\mu)))$.

In [10, Lemma 3.3] we are given an algorithm for extending λ from $T(d(\mu)) \cup (T(d(\nu)) + d(\mu))$ to $T(d(\mu) + d(\nu))$. This involves filling in the missing regions

one square at a time so that at each stage there is only one unique choice for the additional value we wish to add. The next two lemmas are an analogue of this.

Lemma 3.1. *Let T be a nondegenerate tile. Fix $n \in \mathbb{N}^2$ and S a subset of \mathbb{N}^2 containing $T + n + e_1$ and $T + n - e_2$. Suppose that $\lambda : S \rightarrow A$ is a function such that $\lambda|_{T+k}$ belongs to Λ_{BD}^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S$. Then there is a unique function*

$$\lambda' : S' := S \cup \{n + (c_1 + 1)e_1 - e_2\} \rightarrow A$$

such that $\lambda'|_S = \lambda$ and $\lambda'|_{T+k}$ belongs to Λ_{BD}^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S'$.

Proof. The proof here closely follows the proof of [10, Lemma 3.3]. Fix $n \in \mathbb{N}^2$ and $S \subset \mathbb{N}^2$ as in the statement of the Lemma. If $n + (c_1 + 1)e_1 - e_2 \in S$ then there is nothing to do. So suppose $n + (c_1 + 1)e_1 - e_2 \notin S$. Let $i \in T \setminus \{c_1 e_1\}$. Then either $i_2 = 0$ and $i_1 < c_1$ or $i_2 > 0$. If $i_2 = 0$ and $i_1 < c_1$ we have $i + e_1 \in T$ and hence $n + e_1 - e_2 + i \in T + n - e_2$. Alternatively if $i_2 > 0$, we have $i - e_2 \in T$ and it follows that $n + e_1 - e_2 + i \in T + n + e_1$. Hence $(T \setminus \{c_1 e_1\}) + n + e_1 - e_2 \subseteq S$ so the corner $n + (c_1 + 1)e_1 - e_2$ is the only element of $T + n + e_1 - e_2$ which is not in S . To say what that point is we now deviate from [10, Proof of Lemma 3.3].

Let $T' = (T \setminus \{c_1 e_1\}) + n + e_1 - e_2 \subseteq S$, $p = \lambda|_{T' \setminus \{c_2 e_2\}}$, and $t := (n + e_1 + (c_2 - 1)e_2) \in T'$. Then t is the top left hand corner of T' . We now define $\lambda'|_S = \lambda$ and

$$(3.6) \quad \lambda'(n + (c_1 + 1)e_1 - e_2) := F_{p, \lambda(t)}(c_1 e_1) = f_p(\lambda(t)).$$

To see that this is the required function fix $k \in \mathbb{N}^2$ such that $T \setminus \{c_1 e_1\} \subset S'$. If $T + k \subset S$, we have $\lambda'|_{T+k} = \lambda|_{T+k} \in \Lambda_{BD}^0$. On the other hand, if $T + k \not\subset S$ we must have $k = n + e_1 - e_2$. Then $\lambda'|_{T \setminus \{c_1 e_1\} + k} = \lambda|_{T \setminus \{c_1 e_1\} + k}$ and (3.6) gives the unique value $\lambda'(n + e_1 - e_2 + c_1 e_1)$ such that $\lambda'|_{T+k} \in \Lambda_{BD}^0$ as required. \blacksquare

Lemma 3.2. *Let T be a nondegenerate tile. Fix $n \in \mathbb{N}^2$ and S a subset of \mathbb{N}^2 containing $T + n + e_1$ and $T + n - e_2$. Suppose that $\lambda : S \rightarrow A$ is a function such that $\lambda|_{T+k}$ belongs to Λ_{BD}^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S$. Then there is a unique function*

$$\lambda' : S' := S \cup \{n + c_2 e_2\} \rightarrow A$$

such that $\lambda'|_S = \lambda$ and $\lambda'|_{T+k}$ belongs to Λ_{BD}^0 for every $k \in \mathbb{N}^2$ such that $T + k \subset S'$.

Remark. The proof of Lemma 3.2 is very similar to that of Lemma 3.1 and so we omit it. The key point of difference is that for Lemma 3.2 we are filling in the upper left hand corner $\lambda'(n + c_2 e_2)$ while in Lemma 3.1 we filled in the lower right hand corner $\lambda'(n + (c_1 + 1)e_1 - e_2)$. A mirror argument of the proof of Lemma 3.1 using the fact that f_p is a bijection for each $p \in A^P$ will prove Lemma 3.2.

Proof of Proposition 3.2. To obtain $\lambda \in \Lambda_{BD}^{d(\mu)+d(\nu)}$ we start with the region $T(d(\mu)) \cup (T(d(\nu)) + d(\mu))$. We must fill in the two remaining rectangular regions given by

$$BR := \{j \in \mathbb{N}^2 : c_1 + d(\mu)_1 < j_1 \leq c_1 + d(\mu)_1 + d(\nu)_1, j_2 < d(\mu)_2\}, \text{ and}$$

$$UL := \{j \in \mathbb{N}^2 : j_1 < d(\mu)_1, c_2 + d(\mu)_2 < j_2 \leq c_2 + d(\mu)_2 + d(\nu)_2\}.$$

We consider each region separately. First consider the bottom right rectangle BR . We order BR lexicographically by starting at the top of the $(c_1 + d(\mu)_1 + 1)$ th column and going down. We then repeat with the $(c_1 + d(\mu)_1 + 2)$ th column and so on. Now

we proceed by induction. For our base case consider $d(\mu) + (c_1 + 1)e_1 - e_2 \in BR$. We apply Lemma 3.1 with $n = d(\mu)$ to give $\lambda(d(\mu) + (c_1 + 1)e_1 - e_2) = f_p(\lambda(d(\mu) + c_1e_1 - e_2))$ for $p = \lambda|_{P+d(\mu)-e_2}$.

Fix $k \in BR$. Suppose $\lambda'|_{T+j} \in \Lambda_{BD}^0$ for every j such that $d(\mu) + (c_1 + 1)e_1 - e_2 \leq j \leq k$ in the lexicographic order described above. For the inductive step there are two cases to consider. Firstly, if $k_2 \neq 0$, let $n = k - (c_1 + 1)e_1$. Then $\lambda'(n + e_1)$ and $\lambda'(n - e_2)$ are given by the inductive hypothesis. Then Lemma 3.1 with this n gives the unique value for $\lambda'(k - e_2)$. On the other hand, if $k_2 = 0$ then the next value we need to determine is $\lambda(k + e_1 + (d(\mu)_2 - 1)e_2)$, that is, the top value of the next column in BR . Then $\lambda(T + k + d(\mu)_2e_2) = \nu(T + (k_1 - d(\mu)_1)e_1)$ and $T(k - e_1 + (d(\mu)_2 - 1)e_2)$ is given by the inductive hypothesis. Thus Lemma 3.1 with $n = k - c_1e_1 + d(\mu)_2e_2$ gives the unique value $\lambda(k + e_1 + (d(\mu)_2 - 1)e_2)$.

It remains to show how to fill in the upper left rectangle UL . We again order elements lexicographically. However this time we start at the right hand end of the $(c_2 + d(\mu)_2 + 1)$ th row and count left. At the end of that row we then continue from the right hand end of the $(c_2 + d(\mu)_2 + 2)$ th row, and so on. For a base case for an induction argument we consider $d(\mu) - e_1 + e_2$. A mirror argument of the argument in the preceding two paragraphs, replacing the use of Lemma 3.1 with Lemma 3.2 will complete the proof. \blacksquare

We now know how to compose elements $\mu, \nu \in \Lambda_{BD}^*$ and can hence prove Theorem 3.1.

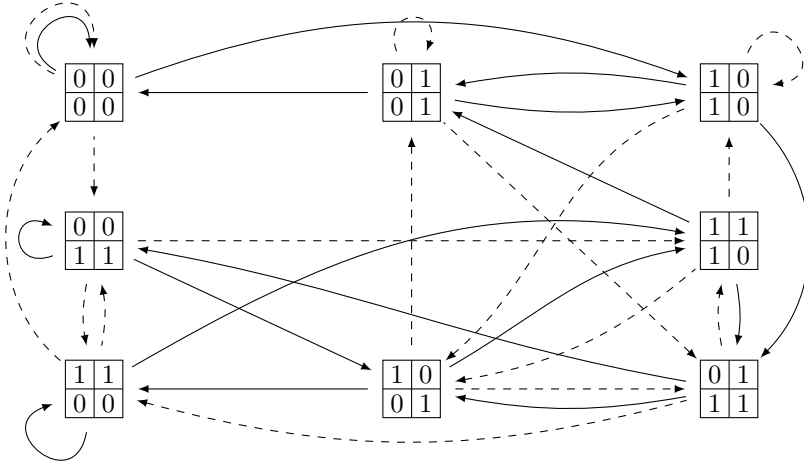
Proof of Theorem 3.1. Proposition 3.2 ensures that composition in $(\Lambda_{BD}^0, \Lambda_{BD}^*, r, s)$ is well defined. We now show that composition in Λ_{BD} is associative. Fix $\mu, \nu, \rho \in \Lambda_{BD}^*$ such that $s(\mu) = r(\nu)$ and $s(\nu) = r(\rho)$. Then Proposition 3.2 implies that $\mu\nu$ is the unique path in $\Lambda_{BD}^{d(\mu)+d(\nu)}$ such that $(\mu\nu)(0, d(\mu)) = \mu$ and $(\mu\nu)(d(\mu), d(\mu) + d(\nu)) = \nu$. Proposition 3.2 also implies that $\nu\rho$ is the unique path in $\Lambda_{BD}^{d(\nu)+d(\rho)}$ such that $(\nu\rho)(0, d(\nu)) = \nu$ and $(\nu\rho)(d(\nu), d(\nu) + d(\rho)) = \rho$. Then

$$\begin{aligned} (\mu\nu)\rho &= ((\mu\nu)(0, d(\mu))) ((\mu\nu)(d(\mu), d(\mu) + d(\nu))) \rho \\ &= \mu\nu\rho \\ &= \mu((\nu\rho)(0, d(\nu))) ((\nu\rho)(d(\nu), d(\nu) + d(\rho))) \\ &= \mu(\nu\rho). \end{aligned}$$

Hence Λ_{BD} is a category, which is countable as Lemma 3.1 and Lemma 3.2 imply that $|\Lambda_{BD}^{e_1} \cup \Lambda_{BD}^{e_2}| < \infty$. The map $d : \Lambda_{BD} \rightarrow \mathbb{N}^2$ satisfies $d(\mu\nu) = d(\mu) + d(\nu)$ and so it is a functor. To see that Λ_{BD} is a 2-graph it remains to show that d satisfies the factorisation property. Fix $m, n \in \mathbb{N}^2$ and $\lambda \in \Lambda_{BD}^{m+n}$. Then (3.5) ensures $\mu := \lambda(0, m)$ and $\nu := \lambda(m, m + n)$ are the two unique paths with $\lambda = \mu\nu$. \blacksquare

Remark 3.4. If $|A| = 1$ then Λ_{BD} is the 2-graph consisting of just one vertex with one blue loop and one red loop. If $T = \{0\}$ then every vertex in Λ_{BD} connects to every other vertex in Λ_{BD} with exactly one blue and one red edge.

Example 3.4. The 2-graph associated to the basic data given in Example 3.3 (2) has the skeleton given below.



Next we prove some general facts about the skeleton of a basic data 2-graphs.

Proposition 3.3. Let Λ_{BD} be the 2-graph associated to the basic data BD , then

- (1) $|\Lambda_{BD}^0| = |A|^{|P|+1}$;
 - (2) for any $u, v \in \Lambda_{BD}^0$ and $i \in \{1, 2\}$ we have $v\Lambda_{BD}^{e_i}u \neq \emptyset$ if and only if
- (3.7) $v(m) = u(m - e_i)$ for every $m \in T \cap (T + e_i)$,
- in which case $|v\Lambda_{BD}^{e_i}u| = 1$; and
- (3) $|v\Lambda_{BD}^{e_1}| = |A|^{c_2} = |\Lambda_{BD}^{e_1}v|$ and $|v\Lambda_{BD}^{e_2}| = |A|^{c_1} = |\Lambda_{BD}^{e_2}v|$ for every $v \in \Lambda_{BD}^0$.
 - (4) $|v\Lambda_{BD}^{e_1+e_2}u| = 1$ for all $u, v \in \Lambda_{BD}^0$.

Remark 3.5. Proposition 3.3 (3) shows that Λ_{BD} is row-finite and has no sinks or sources.

Proof of Proposition 3.3.

- (1) Recall that $\Lambda_{BD}^0 = \{F_{p,a} : p \in A^P, a \in A\}$. For each $p \in A^P$ and each $a \in A$ the functions $F_{p,a}$ are distinct; hence we have that $|\Lambda_{BD}^0| = |A^P||A| = |A|^{|P|+1}$.
- (2) Let $u, v \in \Lambda_{BD}^0$ and $i \in \{1, 2\}$. Suppose $v\Lambda_{BD}^{e_i}u \neq \emptyset$ and fix $m \in T \cap (T + e_i)$. Then there exists $\lambda \in v\Lambda_{BD}^{e_i}u$. To show (3.7) holds we calculate

$$v(m) = \lambda|_T(m) = \lambda|_T(m) = \lambda|_{T+e_i}(m - e_i) = u(m - e_i).$$

For the reverse implication suppose u, v and i satisfy (3.7). Define $\lambda : T(e_i) \rightarrow A$ by

$$\lambda(m) = \begin{cases} v(m) & \text{for } m \in T \\ u(m) & \text{for } m \in (T + e_i) \setminus T. \end{cases}$$

Then $r(\lambda) = \lambda \downarrow_T = v$ and (3.7) ensures that for every $m \in T \cap (T + e_i)$ we have $\lambda(m) = v(m) = u(m - e_i)$, which forces $s(\lambda) = \lambda \downarrow_{T+e_i} = u$. Hence $\lambda \in v\Lambda_{BD}^{e_i}u$. To see that $|v\Lambda_{BD}^{e_i}u| = 1$ suppose $\lambda' \in v\Lambda_{BD}^{e_i}u$. Then $s(\lambda') = u$, $r(\lambda') = v$ and (3.7) forces $\lambda' = \lambda$.

- (3) Fix $v \in \Lambda_{BD}^0$ and let $S = \{u \in \Lambda_{BD}^{e_1} : v\Lambda_{BD}^{e_1}u \neq \emptyset\}$. Then Proposition 3.3 (2) implies $|v\Lambda_{BD}^{e_1}| = |v\Lambda_{BD}^{e_1}S| = |S|$. We claim $|S| = |A|^{c_2}$. For this we construct $u \in \Lambda_{BD}^0$ such that $u \in S$. Let $u|_{T \cap (T - e_1)} = v|_{T \cap (T + e_1)}$. Then there remains $|T| - |T \cap (T - e_1)| = c_2 + 1$ values left to be determined, one of which must be the corner $u(c_1e_1)$. We may arbitrarily define values for $u|_{(T \cap (T - e_1)) \setminus \{c_1e_1\}}$, which may be done in $|A|^{c_2}$ ways. Then $u(c_1e_1) = f_{u|_P, u(c_2e_2)}(c_1e_1) = f_{u|_P}(c_2e_2)$. Hence $|v\Lambda_{BD}^{e_1}| = |U| = |A|^{c_2}$.

To see that $|\Lambda_{BD}^{e_1}v| = |A|^{c_2}$ let $R = \{u \in \Lambda_{BD}^0 : u\Lambda_{BD}^{e_1}v \neq \emptyset\}$. Then a similar argument to that given above shows that $|\Lambda_{BD}^{e_1}v| = |R| = |A|^{c_2}$. The mirror results for edges of degree e_2 follow by symmetry.

- (4) Follows by applying Lemma 3.1 and Lemma 3.2 to $S = T \cup (T + e_1 + e_2)$. ■

Definition 3.4. A k -graph Λ is strongly connected if for every $u, v \in \Lambda^0$ there exists $\lambda \in \Lambda$ such that $r(\lambda) = v$ and $s(\lambda) = u$.

It turns out that Λ_{BD} is always strongly connected.

Proposition 3.4. Let Λ_{BD} be the 2-graph associated to the basic data BD . Suppose $k \in \mathbb{N}$ satisfies $(k - 1)(e_1 + e_2) \in T$ and $k(e_1 + e_2) \notin T$. Then for every $u, v \in \Lambda_{BD}^0$ there exists $\lambda \in \Lambda_{BD}^{k(e_1+e_2)}$ such that $r(\lambda) = v$ and $s(\lambda) = u$. Hence Λ_{BD} is strongly connected.

Proof. We will use the following notation: For $n \in \mathbb{N}$ let $T_n := T + n(e_1 + e_2)$, $P_n := P + n(e_1 + e_2)$ and $\mu^n \in \Lambda_{BD}^{n(e_1+e_2)}$. Fix $u, v \in \Lambda^0$ and $\mu^m \in v\Lambda^{m(e_1+e_2)}$. Then

$$(3.8) \quad s(\mu^m)(i) = u(i - (k - m)(e_1 + e_2)) \text{ for every } i \in T \cap T_{k-m}.$$

We show that μ^m exists for $m \leq k$ so that μ^k will be the required path. We do this by an induction argument. For a base case let $\mu^0 = v$. Now suppose for $0 \leq m < k$ that $s(\mu^m)$ satisfies (3.8). To show that $s(\mu^{m+1})$ satisfies (3.8) it suffices to show there exists $\lambda \in \Lambda_{BD}^{e_1+e_2}$ such that $r(\lambda) = s(\mu^m)$ and

$$(3.9) \quad s(\lambda)(i) = u(i - (k - (m + 1))(e_1 + e_2)) \text{ for } i \in T \cap T_{k-(m+1)}$$

so that Proposition 3.2 implies $\mu^m\lambda = \mu^{m+1}$.

Let $S = T_m \cup T_{m+1}$. We will define a function $\lambda' : S \rightarrow A$. Let $\lambda' \downarrow_{T_m} = s(\mu^m)$ and for $i \in T_k \cap T_{k-(m+1)}$ let $\lambda'(i) = u(i - k(e_1 + e_2))$. This fixes those values of λ' whose domain coincides with the domain of $s(\mu^m)$ or the domain of u . We may now define the values of $\lambda'|_{P_{m+1}}$ and $j = \lambda'(m(e_1 + e_2) + c_2e_2)$ arbitrarily. Then $\lambda'(m(e_1 + e_2) + c_1e_1) = f_{\lambda'|_{P_{m+1}}}(j)$. This completes the definition of λ' . Now Lemma 3.1 and Lemma 3.2 with these S and λ' give $\lambda \in \Lambda_{BD}^{e_1+e_2}$ such that (3.9) holds and $r(\lambda) = \lambda' \downarrow_{T_m} = s(\mu^m)$. The final remark now follows from the definition. ■

4. Aperiodicity

The results of [4, 7, 13, 17] show that a condition, called aperiodicity, plays an important role in simplicity results for the C^* -algebras associated to k -graphs. In this section, we prove Theorem 4.1 which gives conditions on the basic data BD which ensure that the associated 2-graph Λ_{BD} is aperiodic. Theorem 4.1 generalises [10, Theorem 5.2] as it allows us to prove aperiodicity for a larger class of 2-graphs.

There have been many formulations of aperiodicity, see [3, 4, 7, 13, 14, 17]. The new formulation we propose is inspired by condition (4) given by Robertson and Sims in [14, Lemma 3.3].

Definition 4.1. *Let Λ be a row-finite k -graph with no sources then Λ is aperiodic if for every $v \in \Lambda^0$ and for every distinct $m, n \in \mathbb{N}^k$ with $m \wedge n = 0$ there exists a path $\lambda \in \Lambda$ such that $r(\lambda) = v$, $d(\lambda) \geq m \vee n$ and*

$$(4.1) \quad \lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

The following lemma shows that our definition of aperiodicity is equivalent to condition (4) of [14, Lemma 3.3]. Briefly, condition (4) of [14, Lemma 3.3], which we shall refer to as the *Robertson-Sims condition* amounts to demanding that (4.1) holds for all $m, n \in \mathbb{N}^2$.

Lemma 4.1. *Let Λ be a row-finite 2-graph with no sources. Then Λ is aperiodic if and only if it satisfies the Robertson-Sims condition.*

Proof. Suppose that Λ is aperiodic. Fix $v \in \Lambda^0$ and suppose that $m, n \in \mathbb{N}^2$ are such that $m \wedge n \neq 0$. Put $m' = m - (m \wedge n)$ and $n' = n - (m \wedge n)$. Then $m' \wedge n' = 0$ and $m' \vee n' = (m \vee n) - (m \wedge n)$. Let w be such that $v\Lambda^{m \wedge n}w \neq \emptyset$. Then by hypothesis there exists $\lambda' \in w\Lambda$ with $d(\lambda') \geq m' \vee n'$ and

$$\lambda'(m', m' + d(\lambda') - (m' \vee n')) \neq \lambda'(n', n' + d(\lambda') - (m' \vee n')).$$

Let $\mu \in v\Lambda^{m \wedge n}w$. Then $\lambda = \mu\lambda'$ is such that

$$\begin{aligned} \lambda(m, m + d(\lambda) - (m \vee n)) &= \lambda'(m - (m \wedge n), m - (m \wedge n) + d(\lambda) - (m \vee n)) \\ &= \lambda'(m', m' + d(\lambda') + (m \wedge n) - (m \vee n)) \\ &= \lambda'(m', m' + d(\lambda') - (m' \vee n')) \\ &\neq \lambda'(n', n' + d(\lambda') - (m' \vee n')) \\ &= \lambda'(n', n' + d(\lambda') + (m \wedge n) - (m \vee n)) \\ &= \lambda'(n - (m \wedge n), n - (m \wedge n) + d(\lambda) - (m \vee n)) \\ &= \lambda(n, n + d(\lambda) - (m \vee n)), \end{aligned}$$

and so Λ satisfies the Robertson-Sims condition. If $m \wedge n = 0$ then the Robertson-Sims condition follows immediately from the aperiodicity of Λ .

If Λ satisfies the Robertson-Sims condition then it is clear that it is aperiodic; which completes the proof. ■

We now give a condition on the basic data BD which, as we shall see in Theorem 4.1, ensures that Λ_{BD} is aperiodic.

Definition 4.2. Let Λ_{BD} be the 2-graph associated to the basic data BD . For $a \in A$ we say that BD has a blue a -breaking cycle if there exists $k > 1$ and distinct $v_1, \dots, v_k \in \Lambda_{BD}^0$ such that for every $i \in \{1, \dots, k\}$ we have $v_i|_{T \cap (T+e_2)} \equiv a$ and either

- (1) $|v_i \Lambda^{e_1} v_{i+1}| = 1$ for every $i \in \{1, \dots, k\}$ where $v_{k+1} = v_1$; or
- (2) $|v_i \Lambda^{e_1} v_i| = 1$ for every $i \in \{1, \dots, k\}$.

We may define what it means for BD to have a red a -breaking cycle by reversing the roles of the coordinates. We say that BD has an a -breaking cycle if it has either a blue a -breaking cycle or a red a -breaking cycle.

Example 4.1.

- (1) The basic data given in Example 3.3 has a blue 0-breaking cycle as it satisfies part (2) of Definition 4.2 where $v_1 = F_{p[0],0}$ and $v_2 = F_{p[1],0}$. Moreover, it has a blue 1-breaking cycle as it satisfies part (1) of Definition 4.2 where $v_1 = F_{p[0],1}$ and $v_2 = F_{p[1],1}$. One also checks that there are a red 1-breaking cycle and a red 0-breaking cycle.
- (2) The basic data given in Example 3.3 (2) has a blue 0-breaking cycle as it satisfies part (2) of Definition 4.2 where $v_1 = F_{p[0,0],0}$ and $v_2 = F_{p[1,0],1}$. One also checks that there is a red 0-breaking cycle, but no red 1-breaking cycle.
- (3) Let $A = \{0, 1\}$, and $T = \{0, e_1, 2e_1, e_2\}$, so that $P = \{0, e_1\}$. For $a, b \in \{0, 1\}$ let $p[a, b] : \{0, e_1\} \rightarrow A$ be given by $p[a, b](0) = a$ and $p[a, b](e_1) = b$, so that $A^P = \{p[0,0], p[0,1], p[1,0], p[1,1]\}$. Define a bijection $f_{p[0,0]} : A \rightarrow A$ by $f_{p[0,0]}(a) = a$ for all $a \in A$, and bijections $f_{p[1,0]}, f_{p[1,1]}, f_{p[0,1]} : A \rightarrow A$ by $f_{p[1,0]}(a) = f_{p[0,1]}(a) = f_{p[1,1]}(a) = a + 1 \pmod{2}$ for all $a \in A$ which gives us basic data BD . The basic data BD has a blue 0-breaking cycle as it satisfies part (2) of Definition 4.2 with $v_1 = F_{p[0,0],0}$ and $v_2 = F_{p[1,1],1}$. It also has a blue 1-breaking cycle as it satisfies part (1) of Definition 4.2 with $v_1 = F_{p[0,0],1}$, $v_2 = F_{p[0,1],0}$ and $v_3 = F_{p[1,0],0}$. One checks that there is a red 1-breaking cycle but not a red 0-breaking cycle.

We now give the main result of this paper.

Theorem 4.1. Let Λ_{BD} be the 2-graph associated to the basic data BD . If BD has an a -breaking cycle for some $a \in A$ then Λ_{BD} is aperiodic.

Proof. Fix $v \in \Lambda_{BD}^0$ and distinct $m, n \in \mathbb{N}^2$. We also fix $\alpha \in v\Lambda_{BD}^{m \vee n}$. If $\alpha|_{T+m} \neq \alpha|_{T+n}$ then $\lambda := \alpha$ will do. So suppose otherwise. Recall from Lemma 4.1 that it is enough show that (4.1) is satisfied for every distinct $m, n \in \mathbb{N}^2$ such that $m \wedge n = 0$.

Without loss of generality we may suppose $m_1 > n_1 = 0$ and $n_2 > m_2 = 0$. The proof proceeds as follows: In the case where BD has a blue a -breaking cycle we first show there exists $\nu \in \Lambda_{BD}^{m \vee n}$ whose bottom c_2 rows are all a 's. We then attach an edge μ to the head of ν so that $\rho = (\mu\nu)(0, m \vee n)$ satisfies $\rho(m, m) \neq \rho(n, n)$. We then use the fact that Λ_{BD} is strongly connected to prove there exist $\beta \in s(\alpha)\Lambda_{BD}r(\rho)$ and then show that $\alpha\beta\rho$ satisfies (4.1).

Let $T^- \subset \mathbb{N}^2$ denote the bottom c_2 rows of T . We claim that for all $l \geq 0$ there exists $\nu' \in \Lambda_{BD}^{le_1}$ such that $\nu'|_{T-(le_1)} \equiv a$. We shall prove our claim by induction: For a base case, $l = 0$, fix $p \in A^P$ such that $p \equiv a$. Then $F_{p, f_p^{-1}(a)} \in \Lambda_{BD}^0$ and

$F_{p, f_p^{-1}(a)}|_{T^-} \equiv a$. Now suppose that for $k \geq 1$ there is $\phi \in \Lambda_{BD}^{ke_1}$ which satisfies $\phi|_{T^-(ke_1)} \equiv a$. Define $q \in A^P$ by

$$q(j) = \begin{cases} \phi(j - e_1) & \text{for } j \in P \cap (P + e_1), \\ a & \text{for } j \notin P \cap (P + e_1). \end{cases}$$

Then $F_{q, f_q^{-1}(a)} \in \Lambda_{BD}^0$ and $F_{q, f_q^{-1}(a)}|_{T^-} \equiv a$. Then Proposition 3.3 (2) implies there exists $\theta \in F_{q, f_q^{-1}(a)} \Lambda_{BD}^{e_1} r(\phi)$. Then $\nu' = \theta\phi \in \Lambda_{BD}^{(k+1)e_1}$ satisfies $\nu'|_{T^-((k+1)e_1)} \equiv a$ as required. Hence by the principle of mathematical induction our claim follows.

Now we may prove the existence of a $\nu \in \Lambda_{BD}^{m \vee n}$ whose bottom c_2 rows are all a 's. By the claim established in the above paragraph there is $\nu' \in \Lambda_{BD}^{m_1 e_1}$ such that $\nu'|_{T^-(m_1 e_1)} \equiv a$. Since Λ has no sources there exists $\nu \in \Lambda_{BD}^{m \vee n}$ such that $\nu(0, m) = \nu'$ and hence $\nu|_{T^-(m)} = \nu'|_{T^-(m)} \equiv a$. In other words, the bottom c_2 rows of ν are all a 's.

By Proposition 3.3 (2) there exist unique $\mu_1 \in v_1 \Lambda_{BD}^{e_2} r(\nu)$ and $\mu_2 \in v_2 \Lambda_{BD}^{e_2} r(\nu)$. We claim $(\mu_1 \nu)(m, m) \neq (\mu_2 \nu)(m, m)$. Suppose $\{v_k\}$ satisfies condition (1) of Definition 4.2. Then for every $l \leq m$ let $i = 1 + l_1 \pmod k$. Then Lemma 3.1 implies $(\mu_1 \nu)(l, l) = v_i \neq v_{i+1} = (\mu_2 \nu)(l, l)$. On the other hand, if $\{v_k\}$ satisfies condition (2) of Definition 4.2 then Lemma 3.1 implies $(\mu_1 \nu)(l, l) = v_1 \neq v_2 = (\mu_2 \nu)(l, l)$ for every $l \leq m$. In either case taking $l = m$ proves the claim.

If $(\mu_1 \nu)(m, m) \neq (\mu_1 \nu)(n, n)$ we let $\rho = (\mu_1 \nu)(0, m \vee n) \in \Lambda_{BD}^{m \vee n}$. Otherwise let $\rho = (\mu_2 \nu)(0, m \vee n) \in \Lambda_{BD}^{m \vee n}$. Then $\rho(m, m) \neq \rho(n, n)$. Now because Λ_{BD} is strongly connected there exist $\beta \in s(\alpha) \Lambda_{BD} r(\rho)$. To finalise the proof we show that $\lambda = \alpha\beta\rho$ satisfies (4.1). Since

$$\begin{aligned} s(\lambda(m, m + d(\lambda) - (m \vee n))) &= s((\alpha\beta\rho)(m, m + d(\alpha\beta\rho) - (m \vee n))) \\ &= s((\alpha\beta\rho)(m, m + d(\alpha\beta))) \\ &= \rho(m, m) \\ &\neq \rho(n, n) = s(\lambda(n, n + d(\lambda) - (m \vee n))) \end{aligned}$$

we see that $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$.

If BD instead has a red a -breaking cycle then a similar argument can be constructed by reversing the roles of the coordinates. ■

Remark. In Examples 4.1 (1) we showed that the basic data BD from Example 3.3 (2) has a blue 0-breaking cycle and so satisfies the hypotheses of Theorem 4.1. Hence we are able to prove aperiodicity for 2-graphs which do not arise from the parameters of [10].

The aperiodicity result given in [10, Theorem 5.2] employs a condition, called three invertible corners. It implies that all 2-graphs which are defined using nondegenerate tiles T with $c_1 = 0$ or $c_2 = 0$ cannot be aperiodic (see [11]). Using Weaver's argument from [21, Lemma 6.3.1], we may show that the same is true here:

Lemma 4.2. *Let BD be basic data for a nondegenerate tile with $c_1 = 0$ or $c_2 = 0$, then BD does not have an a -breaking cycle for all $a \in A$. Moreover, Λ_{BD} is not aperiodic and so $C^*(\Lambda_{BD})$ is not simple.*

Proof. Suppose $c_2 = 0$ and BD is basic data for the nondegenerate tile $T = \{0, e_1, \dots, c_1 e_1\}$. Since $T \cap (T + e_2) = \emptyset$, there is no blue a -breaking cycle for any $a \in A$.

Suppose $v \in \Lambda_{BD}^0$ is such that $v|_{T \cap (T + e_1)} \equiv a$ for some $a \in A$. Since $T \cap (T + e_1) = \{0, e_1, \dots, c_1 e_1\}$ this implies that $v(je_1) = a$ for all $1 \leq j \leq c_1$. By (3.1) $v(0) = v(c_2 e_2)$ is completely determined by $v(c_1 e_1)$, so there is only one vertex $v \in \Lambda_{BD}^0$ with $v|_{T \cap (T + e_1)} \equiv a$ for some $a \in A$. Hence by Definition 4.2 there is no red a -breaking cycle for any $a \in A$.

Since $c_2 = 0$ it follows by Proposition 3.3 (3) that $|v\Lambda_{BD}^{e_1}| = 1$ for all $v \in \Lambda_{BD}^0$; in particular there is only one blue edge coming into each $v \in \Lambda_{BD}^0$. Since Λ_{BD}^0 is finite, it follows that every sufficiently long blue path in Λ_{BD} must pass through the same vertex twice. Hence the blue subgraph of the skeleton of Λ_{BD} consists of a finite number of disjoint cycles. Let n be the lowest common multiple of the lengths of these cycles. Hence every path $\lambda \in \Lambda_{BD}$ has a horizontal repeating pattern of a block of width n . In particular for $\lambda \in \Lambda_{BD}$ and $l \in \mathbb{Z}^2$ we have $\lambda|_{T+l} = \lambda|_{T+l+ne_1}$ whenever $T+l$ and $T+l+ne_1$ are in the domain of λ . Fix $v \in \Lambda_{BD}^0$, $m, p \in \mathbb{N}^2$ satisfying $p_1 - m_1 = n$ and $p_2 - m_2 = 0$. Then every $\lambda \in v\Lambda_{BD}$ with $d(\lambda) \geq p$ has

$$\lambda(m, m + d(\lambda) - p) = \lambda(p, d(\lambda)).$$

Hence (4.1) cannot be satisfied which implies that Λ_{BD} is not aperiodic by Lemma 4.1. Thus [14, Theorem 3.1] implies that $C^*(\Lambda_{BD})$ is not simple.

Similar arguments apply for nondegenerate tiles with $c_1 = 0$. ■

For a general nondegenerate tile, we do not know if the converse of Theorem 4.1 holds. Recall from Proposition 3.1 that given parameters $(T, q, 0, w)$ from [10] with $w(c_1 e_1)$ and $w(c_2 e_2)$ invertible in $\mathbb{Z}/q\mathbb{Z}$ then there is basic data BD such that $\Lambda_{BD} \cong \Lambda(T, q, 0, w)$. We now show that if $w(0)$ is invertible in $\mathbb{Z}/q\mathbb{Z}$ (and so $\Lambda(T, q, 0, w)$ is aperiodic by [10, Theorem 5.2]) then Λ_{BD} is aperiodic.

Proposition 4.1. *Suppose $(T, q, 0, w)$ is a set of parameters and BD is a set of basic data such that $\Lambda_{BD} \cong \Lambda(T, q, 0, w)$. If $c_1, c_2 \geq 1$ and $w(0)$ is invertible in $\mathbb{Z}/q\mathbb{Z}$ then BD has an a -breaking cycle for some $a \in \mathbb{Z}/q\mathbb{Z}$ and so Λ_{BD} aperiodic.*

Proof. We claim that the basic data BD has a blue 1-breaking cycle. We do this by finding distinct vertices $v_1, \dots, v_k \in \Lambda_{BD}^0$ with $v_i|_{T \cap (T + e_2)} \equiv 1$ which satisfy either (1) or (2) of Definition 4.2. Let $n = \sum_{i \in T \cap (T + e_2)} w(i)$ and $n' \in \mathbb{Z}/q\mathbb{Z}$ be the unique number such that $n + n' = 0$. Since vertices $v \in \Lambda(T, q, 0, w)^0$ must satisfy $\sum_{i \in T} v(i)w(i) = 0$, the vertices we produce must therefore have a weighted sum of n' along the bottom row.

If $n' = 0$, then since $w(0), w(c_1 e_1)$ are invertible in $\mathbb{Z}/q\mathbb{Z}$ we may find nonzero elements $a, b \in \mathbb{Z}/q\mathbb{Z}$ such that $aw(0) + bw(c_1 e_1) = 0$. Define $v_1 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_1|_{T \cap (T + e_2)} \equiv 1$, with $v_1(0) = a$, $v_1(c_1 e_1) = b$ and if $c_1 \geq 2$ we set $v_1(je_1) = 0$ for $1 \leq j \leq c_1 - 1$. Since the weighted sum along the bottom row is 0 we have $v_1 \in \Lambda(T, q, 0, w)^0 = \Lambda_{BD}^0$.

If $n' \neq 0$. Define $v_1 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_1|_{T \cap (T + e_2)} \equiv 1$, with $v_1(je_1) = 0$ for $1 \leq j \leq c_1$ and $v_1(0) = n'w(0)^{-1}$. Since the weighted sum along the bottom row is n' we have $v_1 \in \Lambda(T, q, 0, w)^0 = \Lambda_{BD}^0$.

Define $v_2 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_2|_{T \cap (T+e_2)} \equiv 1$, with $v_2(je_1) = v_1((j+1)e_1)$ for $0 \leq j \leq c_1 - 1$ and

$$v_2(c_1e_1) = w(c_1e_1)^{-1} \left(n' - \sum_{j=0}^{c_1-1} w(je_1)v_2(je_1) \right).$$

Since the weighted sum along the bottom row is n' we have $v_2 \in \Lambda(T, q, 0, w)^0 = \Lambda_{BD}^0$ and $|v_1\Lambda^{e_1}v_2| = 1$.

If $n' \neq 0$ then one checks that $v_1 \neq v_2$. If $n' = 0$ then $v_1 = v_2$ only if $c_1 = 1$ and $a = b$; in this case we define $v_2 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_2|_{T \cap (T+e_2)} \equiv 1$, with $v_2(je_1) = 0$ for $0 \leq j \leq c_1$. Since the weighted sum along the bottom row is 0 we have $v_2 \in \Lambda(T, q, 0, w)^0 = \Lambda_{BD}^0$ and $|v_2\Lambda_{BD}^{e_1}v_2| = 1$. Hence the vertices v_1, v_2 satisfy condition (2) of Definition 4.2. Hence we may assume that $v_1 \neq v_2$ from here on.

Now define $v_3 : T \rightarrow \mathbb{Z}/q\mathbb{Z}$ by $v_3|_{T \cap (T+e_2)} \equiv 1$ by $v_3(je_1) = v_2((j+1)e_1)$ for $0 \leq j \leq c_1 - 1$, and

$$v_3(c_1e_1) = w(c_1e_1)^{-1} \left(n' - \sum_{j=0}^{c_1-1} w(je_1)v_3(je_1) \right).$$

Since the weighted sum along the bottom row is n' we have $v_3 \in \Lambda(T, q, 0, w)^0$ and $|v_2\Lambda_{BD}^0v_3| = 1$.

If $v_3 = v_1$ then $\{v_1, v_2\}$ satisfy condition (1) of Definition 4.2. Suppose that $v_2 = v_3$ then entries of the bottom row of v_2 must all be equal. But $v_2(0) = 0$ and

$$v_2(c_1e_1) = w(c_1e_1)^{-1} \left(n' - \sum_{j=0}^{c_1-1} w(je_1)v_2(je_1) \right) = w(c_1e_1)^{-1}n' \neq 0$$

which is a contradiction. Hence $v_2 \neq v_3$.

Continuing in this way, we produce $k \geq 2$ vertices v_1, v_2, \dots, v_k in $\Lambda(T, q, 0, w)^0 = \Lambda_{BD}^0$ with $|v_i\Lambda_{BD}v_{i+1}| = 1$ for $1 \leq i \leq k - 1$. Since Λ_{BD}^0 is finite we must have $v_i = v_j$ for some $i < j$. Hence the vertices v_i, \dots, v_j satisfy condition (1) of Definition 4.2. The final statement now follows from Theorem 4.1. ■

The proof of Proposition 4.1 may be modified to show that there are also red and blue a -breaking cycles for all $a \in \mathbb{Z}/q\mathbb{Z}$.

Remark. It follows from Proposition 4.1 that our aperiodicity result, Theorem 4.1, subsumes [10, Theorem 5.2]. Furthermore, Theorem 4.1 allows us to deduce aperiodicity for a wider class of 2-graphs described using the parameters of [10]. In Examples 4.1 (3) we gave basic data BD which has a blue 0-breaking cycle, and so by Theorem 4.1 we conclude that the associated 2-graph is aperiodic. However the basic data BD corresponds to the parameters $(T, 2, 0, w)$ of [10] where the rule w is given by $w(0) = 0$ and $w(e_1) = w(e_2) = w(2e_1) = 1$. Since $w(0)$ is not invertible in $\mathbb{Z}/2\mathbb{Z}$, we cannot apply [10, Theorem 5.2] to deduce aperiodicity.

5. Simplicity of basic data 2-graph C^* -algebras.

We can now use the properties of our basic data 2-graphs to determine the structure of the corresponding higher rank graph C^* -algebras. Recall from [7, Remark A.3]

that for a row-finite 2-graph Λ with no sources, we say Λ is *cofinal* if for every $v, w \in \Lambda^0$ there exist $n \in \mathbb{N}^2$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^n$.

Theorem 5.1. *Let Λ_{BD} be the 2-graph associated to the basic data BD which has an a -breaking cycle. Then $C^*(\Lambda_{BD})$ is unital, nuclear, simple, purely infinite and belongs to the bootstrap class \mathcal{N} .*

Proof. Since Λ_{BD}^0 is finite the sum $\sum_{v \in \Lambda_{BD}^0} s_v$ is an identity for $C^*(\Lambda_{BD})$. As Λ_{BD} is a 2-graph, [4, Theorem 5.5] implies $C^*(\Lambda_{BD})$ belongs to the bootstrap class \mathcal{N} for which the UCT holds, and so $C^*(\Lambda_{BD})$ is nuclear. Since Λ_{BD} has an a -breaking cycle, Λ_{BD} is aperiodic by Theorem 4.1, and hence [14, Lemma 3.2] implies Λ_{BD} has no local periodicity. Since Λ_{BD} is strongly connected it is clearly cofinal. Then [7, Remark A.3 and Theorem 5.1] show that the definition of cofinality used here is equivalent to the definition of cofinality used in [14]. Then [14, Theorem 3.1] implies that $C^*(\Lambda_{BD})$ is simple.

We use the results of [18] to show that $C^*(\Lambda_{BD})$ is purely infinite: We must show that every $v \in \Lambda_{BD}^0$ can be reached from a loop with an entrance and that Λ_{BD} satisfies the technical condition (C) of [18, Section 7]. The argument that every $v \in \Lambda_{BD}^0$ can be reached from a loop with an entrance is the same as in the proof of [10, Theorem 6.1]. It remains to show that Λ_{BD} satisfies (1) and (2) of condition (C). Since Λ_{BD} is aperiodic we may use [7, Proposition 3.6] to conclude that Λ_{BD} satisfies (1) of condition (C). By Proposition 3.3, Λ_{BD} is row-finite with no sinks or sources, so each finite exhaustive set is a union of sets of the form Λ_{BD}^n and hence trivially satisfies (2) of condition (C). The result now follows from [18, Proposition 8.8]. \blacksquare

6. Two-dimensional shift spaces of basic data 2-graphs.

Here we connect our work to the literature on dynamical systems. In our construction of the 2-graph Λ_{BD} from the basic data BD we have not used any algebraic properties of A in our definitions. For this reason we align our work to that of Quas and Trow [12] rather than the work of Schmidt [16]. Given basic data BD and associated 2-graph Λ_{BD} , the two-sided infinite path space Λ_{BD}^Δ may be given the structure of a shift space (see [5]). In Theorem 6.1 we show that the shift space associated to a 2-graph Λ_{BD} is conjugate to a two-dimensional shift of finite type Y_{BD} , as described in Section 2.2.

Fix a set of basic data BD and define Y_{BD} to be the subset

$$\{y \in A^{\mathbb{Z}^2} : y \downarrow_{T+n} \in \Lambda_{BD}^0 \text{ for every } n \in \mathbb{Z}^2\} \subset A^{\mathbb{Z}^2}.$$

Then it is routine to show that Y_{BD} is a shift space with shift map σ_b given by $(\sigma_b y)(i) = y(i+b)$ for $b \in \mathbb{Z}^2$. In fact Y_{BD} is a shift of finite type since $Q = \Lambda_{BD}^0$ is a set of allowed finite configurations for $S = T \subset \mathbb{Z}^2$.

Kumjian and Pask [5] introduced the idea of a two-sided infinite path space for a k -graph Λ with no sinks or sources. We will use their construction with $k = 2$. Let

$$\Delta = \{(m, n) : m, n \in \mathbb{Z}^2, m \leq n\};$$

where $r(m, n) = m$, $s(m, n) = n$ and $d(m, n) = n - m$. Then (Δ, d) is a 2-graph with no sources or sinks. Then the two-sided infinite path space of Λ is

$$\Lambda^\Delta = \{x : \Delta \rightarrow \Lambda \text{ such that } x \text{ is a degree preserving functor}\}.$$

Kumjian and Pask show that Λ^Δ has a locally compact topology generated by the cylinder sets

$$Z(\lambda, n) = \{x \in \Lambda^\Delta : x(n, n + d(\lambda)) = \lambda\} \text{ where } \lambda \in \Lambda \text{ and } n \in \mathbb{Z}^2.$$

In [5, Section 3] Kumjian and Pask also demonstrated that there is a metric ρ which induces this topology and that Λ^Δ is compact if Λ^0 is finite. This fact is used in the proof of Theorem 6.1 and details of this construction are given in [5, Section 3] and [19, Section 1.1]. For every $b \in \mathbb{Z}^2$, define $\sigma^b : \Lambda^\Delta \rightarrow \Lambda^\Delta$ by $\sigma^b(x)(m, n) = x(m + b, n + b)$. A straightforward argument shows that σ^b is a homeomorphism and so Λ_{BD}^Δ is a shift space.

Theorem 6.1. *Suppose we have basic data BD with associated 2-graph Λ_{BD} , and Y_{BD} defined as above. Then there is a homeomorphism $h : \Lambda_{BD}^\Delta \rightarrow Y_{BD}$ such that $\sigma_b \circ h = h \circ \sigma^b$ for every $b \in \mathbb{Z}^2$, and so Λ_{BD}^Δ is conjugate to Y_{BD} .*

Proof. Recall for any $x \in \Lambda_{BD}^\Delta$ that $x(i, i) \in \Lambda_{BD}^0$ is a function from T to A , so in order to evaluate $x(i)$, we evaluate $x(i, i)$ at 0. Define $h : \Lambda_{BD}^\Delta \rightarrow A^{\mathbb{Z}^2}$ by $(h(x))(i) = x(i, i)(0)$ where $x \in \Lambda_{BD}^\Delta$ and $i \in \mathbb{Z}^2$. Fix $x \in \Lambda_{BD}^\Delta, n \in \mathbb{Z}^2$ and let $p = x|_{P+n}$. To show $h : \Lambda_{BD}^\Delta \rightarrow Y_{BD}$ we must show $h(x)|_{T+n} \in \Lambda_{BD}^0$. Since $p \in A^P$ it only remains to show $(h(x))(n + c_1e_1) = f_p(x(n + c_2e_2))$. Since $x(n, n) \in \Lambda_{BD}^0$ we have

$$(h(x))(n + c_1e_1) = x(n + c_1e_1, n + c_1e_1)(0) = x(n, n)(c_1e_1) = f_p(x(n + c_2e_2)).$$

Next we show h is a homeomorphism. Since Λ_{BD}^0 is finite it follows that Λ_{BD}^Δ is compact and so it suffices show that h is a continuous bijection. To this end for each $y \in Y_{BD}$ we define a functor $g(y) \in \Lambda_{BD}^\Delta$ by

$$(g(y))(m, n) = y|_{T(n-m)+m} \text{ for } m, n \in \mathbb{N}^2 \text{ such that } m \leq n.$$

Then the definition of Y_{BD} ensures $(g(y))(m, n)$ is a path in Λ_{BD} with degree $n - m$. So g is degree preserving. To see that $g(y)$ is a functor from Δ to Λ_{BD} , fix $m, M \in \mathbb{Z}^2$ such that $m \leq n \leq M$. Then Proposition 3.2 implies

$$(g(y))(m, M) = y|_{T(M-m)+m} = y|_{T(n-m)+m}y|_{T(M-n)+n} = (g(y))(m, n)(g(y))(n, M).$$

Since $g(y)$ is a degree preserving functor for every $y \in Y_{BD}$ we now know that $g : Y_{BD} \rightarrow \Lambda_{BD}^\Delta$ and we claim $g = h^{-1}$. Fix $y \in Y_{BD}$ and $i \in \mathbb{N}^2$. Then

$$(h(g(y)))(i) = (g(y))(i, i)(0) = y|_{T(0)+i}(0) = y(i)$$

To finalise the claim we must show $g(h(x)) = x$. Fix $m, n \in \mathbb{N}^2$ such that $m \leq n$ and let $i \in T(n - m)$. Then

$$\begin{aligned} ((g(h(x)))(m, n))(i) &= (h(x)|_{T(n-m)+m})(i) \\ &= h(x)(i + m) \\ &= x(i + m, i + m)(0) = x(i + m, i + n)(0) = x(m, n)(i). \end{aligned}$$

This proves the claim that $g = h^{-1}$. Thus to show that h is a homeomorphism it remains to show that h is continuous. Since Λ_{BD}^{Δ} and Y_{BD} are both metric spaces it suffices to show that $h(x_m)(i) \rightarrow h(x)(i)$ for every $i \in \mathbb{Z}^2$. Fix $i \in \mathbb{Z}^2$. Then since $\sigma^{-i}(x) = x$ it follows that $Z(x(i, i), 0)$ is an open neighbourhood of x . So for sufficiently large m we have $x_m \in Z(x(i, i), 0)$. That is, $\lim_{m \rightarrow \infty} x_m(0, 0) = x(i, i)$ and hence

$$\lim_{m \rightarrow \infty} h(x_m)(i) = \lim_{m \rightarrow \infty} x_m(i, i)(0) = x(i, i)(0) = h(x)(i).$$

To finalise the proof for $x \in \Lambda_{BD}^{\Delta}$ and $b \in \mathbb{Z}^2$ we compute

$$\sigma_b(h(x))(i) = (h(x))(i + b) = x(i + b, i + b)(0) = \sigma^b(x)(i, i)(0) = h(\sigma^b(x))(i). \quad \blacksquare$$

Remark 6.1. If $T = \{0\}$ then the 2-graph associated to the basic data $BD = (T, A, \{f_a\})$ has only one infinite path, namely the function which is identically a . One checks that this function is also the only element of Y_{BD} .

Suppose we have a set of basic data BD . We now compute the topological entropy of the shift space Λ_{BD}^{Δ} (and hence of the shift space Y_{BD}).

By definition $|\mathcal{B}_d(\Lambda_{BD}^{\Delta})| = |\Lambda_{BD}^{\Delta(d,d)}|$ for all $d \geq 1$. Since $|\mathcal{B}_d(\Lambda_{BD}^{\Delta})| \leq |A|^{d^2}$ for all $d \geq 1$ the topological entropy of Λ_{BD}^{Δ} is,

$$h(\Lambda_{BD}^{\Delta}) = \lim_{d \rightarrow \infty} \frac{1}{2^d} \log |\mathcal{B}_d(\Lambda_{BD}^{\Delta})| \leq \lim_{d \rightarrow \infty} \frac{1}{2^d} \log |A|^{d^2} = \lim_{d \rightarrow \infty} \frac{d^2}{2^d} \log |A| = 0.$$

This computation agrees with [19, Proposition 1.2] where a different definition of entropy, formulated using the notion of separating subsets, is used.

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