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Lexicographic Product of Extendable Graphs

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Abstract. Lexicographic product $G \circ H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. If every matching of G of size k can be extended to a perfect matching in G, then G is called *k-extendable*. In this paper, we study matching extendability in lexicographic product of graphs. The main result is that the lexicographic product of an m-extendable graph and an n-extendable graph is $(m+1)(n+1)$-extendable. In fact, we prove a slightly stronger result.

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1. Introduction

The graphs considered in this paper will be finite, undirected, simple and connected.

A *matching* in a graph G is a set of pairwise nonadjacent edges and a matching M is called a *perfect matching* if $V(M) = V(G)$. If every matching of size k can be extended to a perfect matching in G, then G is called *k-extendable*. To avoid triviality, we require that $|V(G)| \geq 2k+2$ for k-extendable graphs. In particular, 0-extendable means there exists a perfect matching in G.

A graph G is *k-factor-critical*, if it satisfies that $G - S$ has a perfect matching for any k-subset S of $V(G)$. Clearly, a $2k$-factor-critical graph is k-extendable, but the reverse is not true (e.g., complete bipartite graphs). Note that if G is $2k$-factor-critical, then all graphs obtained by adding any number of edges to G are still k-extendable. In fact, adding any number of edges to G being still k-extendable is a sufficient and necessary condition for G being $2k$-factor-critical.

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It is natural to study factor criticality and matching extendability of different types of graph products, as such products contain a large number of perfect matchings. Our motivation is from the study of Cayley graphs since graph products often form a 'skeleton' of Cayley graphs. Győri and Plummer [3] showed that the Cartesian product of an m-extendable graph and an n-extendable graph is $(m+n+1)$-extendable. Győri and Imrich [4] proved that the strong product of an m-extendable graph and an n-extendable graph is $[(m+1)(n+1)]_2$-factor-critical. Here, for a real number x, $[x]_2$ denotes the biggest *even* integer not greater than x. In the same paper, they also conjectured that the factor-criticality of strong product can be improved to $[(m+2)(n+2)]_2-2$. Liu and Yu [6] studied matching extension properties in Cartesian products and lexicographic products. In particular, they investigated the matching extension from a prescribed vertex set in lexicographic product of graphs. Readers can see [6] for more details. Wu, Yang and Yu [9] investigated factor-criticality of the Cartesian product of an m-factor-critical and an n-factor-critical graph. More research on graph products can be found in the book written by Imrich and Klavžar [5].

In this paper, we investigate the factor-criticality and extendability in the lexicographic product of an m-extendable graph and an n-extendable graph.

The *lexicographic product* $G \circ H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *strong product* $G \boxtimes H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. Note that $G \circ K_n = G \boxtimes K_n$ and $G \circ H \ncong H \circ G$ whenever $G \ncong H$ and neither of G and H is trivial. A lexicographic product of graphs may not be commutative, even when both factors are connected. For example, $K_2 \circ P_3 \ncong P_3 \circ K_2$.

Let $T \subseteq V(G)$ be a given subset with $|T|$ even. An edge set $F \subseteq E(G)$ is called a *T-join*, if

$$d_F(x) \equiv \begin{cases} 1 & \pmod 2, \text{ if } x \in T \\ 0 & \pmod 2, \text{ if } x \notin T, \end{cases}$$

where $d_F(x)$ denotes the number of edges incident with x in F.

For terminology and notation not defined in this paper, readers are referred to [7].

2. Main results and preliminaries

One of the main results of this paper is the following.

Theorem 2.1. *Let G_1 be m-extendable and G_2 be n-extendable. Then their lexicographic product $G_2 \circ G_1$ is $2(m+1)(n+1)$-factor-critical. In particular, it is $(m+1)(n+1)$-extendable.*

Remark 2.1. For $n \to \infty$, Theorem 2.1 is close to be sharp. To see this, take an arbitrary m-extendable graph G_1 with $|V(G_1)| = 2m+2$ containing a vertex x of degree $m+1$ (e.g. $K_{m+1,m+1}$) and an arbitrary n-extendable graph G_2 with a vertex y of degree $n+1$. Then the degree of the vertex (x, y) in the lexicographic product $G_2 \circ G_1$ is $2(n+1)(m+1) + (m+1)$. Clearly, if X contains all the neighbors of (x, y), then $(G_2 \circ G_1) - X$ obviously does not have a perfect matching. Since $\lim_{n\to\infty} d_{G_2\circ G_1}(x, y)/[2(n + 1)(m + 1)] = 1$, we are nearly able to choose a vertex set X to contain all neighbors of (x, y).

In the special case $G_1 = K_2$ or $G_2 = K_2$, a higher factor-critical number can be proved. With a similar discussion we see that the result is best possible.

Theorem 2.2. *If G is an m-extendable graph of order $2p$, then $G \circ K_2$ is $2(m+1)$-factor-critical and $K_2 \circ G$ is $2p$-factor-critical.*

From the above theorem, it seemed to suggest the following conjecture: If G_1 is m-extendable and G_2 is n-extendable $(m, n \geq 0)$, then their lexicographic product $G_2 \circ G_1$ is $(n + 1)|V(G_1)|$-factor-critical.

Favaron [2] and Yu [10] introduced the concept of k-factor-criticality, independently, and studied the basic properties of k-factor-critical graphs. Several of these properties will be used in our proofs, so we summarize them below.

Theorem 2.3. [2, 10] *Let G be a k-factor-critical graph with $k \geq 2$ and $|V(G)| > k$, then G is also $(k-2)$-factor-critical. Moreover, G is k-factor-critical if and only if $c_o(G - S) \leq |S| - k$ for any $S \subseteq V(G)$ with $|S| \geq k$, where $c_o(G - S)$ is the number of odd components in $G - S$.*

Plummer [8] proved fundamental properties of k-extendable graphs and we summarize them as follows.

Theorem 2.4. [8] *Let G be a k-extendable graph with $k \geq 1$ and $|V(G)| > 2k$. Then*

(a) *G is also $(k-1)$-extendable;*
(b) *G is $(k+1)$-connected;*
(c) *$\delta(G) \geq k + 1$.*

Győri and Imrich [4] considered the strong product of an m-extendable graph and an n-extendable graph, and they gave the following result.

Theorem 2.5. [4] *Let G be a k-extendable graph. Then $G \boxtimes K_2$ is $2(k+1)$-factor-critical.*

In some special cases, applying properties of Hamilton cycles can lead to a shorter proof, so we present a classical theorem of Dirac.

Theorem 2.6. [1] *Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle.*

Before giving the proofs of the main results, we need the following lemma which is used in our proofs. Let $G^{u,v}$ denote the subgraph of $G \circ H$ induced by vertex set $\{(x, u), (x, v) : x \in V(G)\}$ for any $uv \in E(H)$. Similarly, $H^{x,y}$ is defined for any $xy \in E(G)$. It is not difficult to see that $G^{x,y} \cong G \circ K_2$ and $H^{x,y} \cong K_2 \circ H$.

Lemma 2.1. *Let G be m-extendable and H be Hamiltonian with even order, and X be an arbitrary even subset of $V(G \circ H)$. If for any $uv \in E(H)$, $|X \cap V(G^{u,v})| \leq 2m + 1$, then $(G \circ H) - X$ has a perfect matching.*

Proof. Let $u_1u_2\dots u_{2t}$ be a Hamilton cycle of H. By assumption, $|X\cap V(G^{u_{2i-1},u_{2i}})| \le 2m+1$ for $1 \le i \le t$.

If $|X\cap V(G^{u_{2i-1},u_{2i}})|$ is even for $1 \le i \le t$, then by Theorems 2.2 and 2.3, there is a perfect matching of $(G\circ H)-X$. If $|X\cap V(G^{u_{2i-1},u_{2i}})|$ is odd (we call such pair 'odd') for some i, there must be another odd pair $\{u_{2j-1}, u_{2j}\}$. Choose such a j nearest to i along the cycle in 'clockwise' order, then we get a path P_{ij} on this cycle. Deal with other odd pairs in the same way. Thus, the Hamilton cycle $u_1u_2\dots u_{2t-1}u_{2t}$ can be viewed as an ordered components sequence, connected together in order. Each component is either an even path (we say a path p is even if $|V(P)|$ is even; otherwise, it is odd) from one odd pair to another or an edge $u_{2k-1}u_{2k}$ with $|X\cap V(G^{u_{2k-1},u_{2k}})|$ even and no more than $2m+1$. If we can find a perfect matching of $(G\circ P)-X$ for each such even path P, we obtain a perfect matching of $(G\circ H)-X$. Let $P = u_{2i_0-1}u_{2i_0}\dots u_{2j_0-1}u_{2j_0}$. We will construct a set M of independent edges such that $|(X\cup V(M))\cap V(G^{u_{2i-1},u_{2i}})|$ is even for all $u_{2i-1}u_{2i}\in E(P)$. Initially, let $M=\emptyset$. For each edge $e=xy$ (considering each edge once and only once) in P,

(a) if $e\ne u_{2i-1}u_{2i}$ for each i, $1\le i\le t$, then there exists an edge e' between G^x and G^y such that both end vertices of e' are not covered by X and M. Set $M := M\cup\{e'\}$; (If no such e' exists, $G^{x,y}-X$ is disconnected. Note that $\{(v,x),(v,y)\}$ occurs in pair in a component of $G^{x,y}-X$ unless either (v,x) or (v,y) lies in X for any $v\in V(G)$. However, as $|X\cup V(G^{u,v})|\le 2m+1$ for any $uv\in E(G)$, $V(G^{x,y})\cap X$ contains at most m pairs of vertices $\{(v,x),(v,y)\}$. It contradicts to fact that G is $(m+1)$-connected.)

(b) if $e=u_{2i-1}u_{2i}$ for some i, $1\le i\le t$, then set $M:=M$.

Thus, it is not too hard to verify that $|(X\cup V(M))\cap V(G^{u_{2i-1}u_{2i}})|$ is even and no more than $2m+2$ for each $u_{2i-1}u_{2i}\in E(P)$. By Theorems 2.2 and 2.3, $G^{u_{2i-1}u_{2i}} - (X\cup V(M))$ has a perfect matching. Therefore, the union of these perfect matchings together with M is a perfect matching of $(G\circ P)-X$ and thus we obtain a perfect matching of $(G\circ H)-X$. ∎

3. Proofs of the main results

Since Theorem 2.2 will be used in the proof of the main theorem several times, we ought to prove it first.

Proof of Theorem 2.2. The first assertion follows from Theorem 2.5 and the fact that $G\circ K_2 \cong G\boxtimes K_2$. Next, we prove the second part. Let $V(K_2)=\{v_1,v_2\}$.

To the contrary, suppose $K_2\circ G$ is not $2p$-factor-critical. Then by Theorem 2.3, there exists a set $S\subseteq V(K_2\circ G)$ with $|S|\ge 2p$ such that $c_o((K_2\circ G)-S) > |S|-2p$. By parity, $c_o((K_2\circ G)-S)\ge |S|-|V(G)|+2\ge 2$. Note that all components of $(K_2\circ G)-S$ must lie in the same 'layer' G^{v_i}, $i=1$ or 2, since it induces a complete bipartite graph between G^{v_1} and G^{v_2} in $K_2\circ G$, $G^{v_i}\subseteq S$ for some $v_i\in V(K_2)$, say G^{v_1}. Thus, there exists $S'\subseteq V(G^{v_2})$ such that $S'\subseteq S$ and $c_o(G^{v_2}-S')\ge |S'|+2\ge 2$, therefore, $G(\cong G^{v_2})$ has no perfect matching, a contradiction to that G is m-extendable. This completes the proof. ∎

Proof of Theorem 2.1. First, assume $|V(G_1)| \geq 2m+4$. We use induction on n. For the case $n = 0$, we show the following claim.

Claim 1. If G_1 is m-extendable and G_2 is connected with a perfect matching, then $G_2 \circ G_1$ is $2(m+1)$-factor-critical.

Proof. Fix a perfect matching $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$ in G_2. We show the claim by induction on t. The case $t = 1$ follows from Theorems 2.2 and 2.3. Assume that it holds for smaller t. Extend $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$ to a spanning tree of G_2 and contract the edges $v_1v_2, \ldots, v_{2t-1}v_{2t}$. Then a spanning tree in G_2 is transformed into a spanning tree of the contracted graph and the new tree contains a vertex of degree one. Without loss of generality, assume that the vertex obtained from the contraction of v_1v_2 has degree one. It implies that $G_2 - \{v_1, v_2\}$ is connected and has a perfect matching $\{v_3v_4, \ldots, v_{2t-1}v_{2t}\}$. In other words, it is 0-extendable. Since G_2 is connected, we may assume that v_1 has a neighbor in $\{v_3v_4, \ldots, v_{2t-1}v_{2t}\}$. Let X be an arbitrary vertex set in $G_2 \circ G_1$ with $|X| = 2(m+1)$. If $|X \cap V(G_1^{v_1,v_2})|$ is even, both $((G_2 - \{v_1, v_2\}) \circ G_1) - X$ and $G_1^{v_1,v_2} - X$ have a perfect matching M_1 and M_2, respectively. Then $M_1 \cup M_2$ is a perfect matching of $(G_2 \circ G_1) - X$. If $|X \cap V(G_1^{v_1,v_2})|$ is odd, we can pick an arbitrary vertex (u, v_1) in $G_1^{v_1} - X$. Since the vertex (u, v_1) has at least $|V(G_1)|$ neighbors in $G_2 \circ G_1 - V(G_1^{v_1,v_2})$ by the choice of v_1 and the definition of the lexicographic graph, there exists a vertex $(u', w) \in V(G_2 \circ G_1) - V(G_1^{v_1,v_2})$ such that $(u', w) \notin X$ and $(u, v_1)(u', w) \in E(G_2 \circ G_1)$ as $|X \cap V((G_2 - \{v_1, v_2\}) \circ G_1)| \leq 2m + 1 < |V(G_1)|$. Then, $G_1^{v_1,v_2} - (X \cup \{(u, v_1)\})$ has a perfect matching M_1 by Theorems 2.2 and 2.3, and $((G_2 - \{v_1, v_2\}) \circ G_1) - (X \cup \{(u', w)\})$ has a perfect matching M_2 by the induction hypothesis. Then $M_1 \cup M_2 \cup \{(u, v_1)(u', w)\}$ is a perfect matching in $G_2 \circ G_1 - X$. ∎

Now, assume $n \geq 1$. Let X be an arbitrary subset of $V(G_2 \circ G_1)$ with $|X| = 2(m+1)(n+1)$. We consider two cases based on $|X \cap V(G_1^{v,v'})|$.

Case 1. There exists an edge $v_1v_2 \in E(G_2)$ for which $|X \cap V(G_1^{v_1,v_2})| \geq 2(m+1)$.

Take $2m+2$ vertices, say $X_1 = \{x_1, \cdots, x_{2m+2}\}$, in $X \cap V(G_1^{v_1,v_2})$, then $G_1^{v_1,v_2} - X_1$ has a perfect matching M. Consider the edges $y_1z_1, \ldots, y_pz_p$ of M such that $z_i \in X - X_1$ and $y_i \notin X - X_1$. Note that every vertex y_i has at least $n|V(G_1)|$ ($\geq (2m+2)n$) neighbors in $(G_2 \circ G_1) - V(G_1^{v_1,v_2})$. Let $C_1, \ldots, C_k$ (note that $k > 1$ implies $n = 1$) denote the components of $G_2 - \{v_1, v_2\}$. Clearly, each C_j $(1 \leq j \leq k)$ has a perfect matching. Note that when $n = 1$, as G_2 is 1-extendable, both v_1 and v_2 are adjacent to vertices in C_j for all $1 \leq j \leq k$, and hence each y_i has at least $2m + 2$ neighbors in $C_j \circ G_1$ for all $1 \leq j \leq k$. Since $|G_1^{v_1,v_2} \cap X| \geq 2m + 2 + p$, then $|(V(G_2 \circ G_1) - V(G_1^{v_1,v_2})) \cap X| \leq (2m + 2)n - p$. Therefore, there exist vertices $w_1, \ldots, w_p \in V(G_2 \circ G_1) - V(G_1^{v_1,v_2}) - X$ such that $y_iw_i \in E(G_2 \circ G_1)$ for $i = 1, \ldots, p$, and $|(X \cup \{w_1, \ldots, w_p\}) \cap C_j|$ is even for all $1 \leq j \leq k$. By the induction hypothesis, there exists a perfect matching M'_j in $(C_j \circ G_1) - X \cup \{w_1, \ldots, w_p\}$. If M_0 denotes the set of edges of M with both end-vertices in X, then $\bigcup_{j=1}^{k} M'_j \cup (M - M_0) \cup \{y_1w_1, \ldots, y_pw_p\} - \{y_1z_1, \ldots, y_pz_p\}$ is a perfect matching of $(G_2 \circ G_1) - X$.

Case 2. For every edge $v_iv_j \in E(G_2)$, we have $|X \cap V(G_1^{v_i,v_j})| \leq 2m+1$.

Since G_2 is n-extendable, it has a perfect matching denoted by $\{v_1v_2, \ldots, v_{2t-1}v_{2t}\}$. Contracting each edge $v_{2i-1}v_{2i}$ of G_2 to a vertex w_i, we obtain a new graph G_2' with the vertex set $\{w_1, \ldots, w_t\}$. Let I_0 denote the set of indices i such that $|X \cap V(G_1^{v_{2i-1},v_{2i}})|$ is odd, $T = \{w_i \,|\, i \in I_0\}$. Without loss of generality, we assume $T \neq \emptyset$ and thus $|T|$ is even. Our proof relies on the existence of a T-join, which can be stated as the following claim.

Claim 2. There exists a T-join F of G_2' satisfying:

$$d_F(w_i) + d_X(w_i) \leq |V(G_1)| \text{ for all } 1 \leq i \leq t \tag{3.1}$$

where $d_F(w_i)$ denotes the degree of w_i in F and $d_X(w_i) = |X \cap V(G_1^{v_{2i-1},v_{2i}})|$ for $w_i \in V(G_2')$.

Proof. Starting with the empty forest, we construct a T-join F step by step, such that it always satisfies the property (3.1). Set $I := I_0$ at first.

Suppose that a forest F has been chosen already. Let A denote the set of vertices w_i in G_2' satisfying $d_F(w_i) + d_X(w_i) = |V(G_1)|$ already and $|A| = a$. If $I \neq \emptyset$, let $i, j \in I$. Suppose there exists a path P from w_i to w_j avoiding A. If there exists some vertex $w_k \in T \cap V(P)$ different from w_i, w_j satisfying $d_F(w_k) + d_X(w_k) = |V(G_1)| - 1$, let w_k be the vertex nearest to w_i in P. Clearly, $w_k \in I$. Let P' be the subgraph of P from w_i to w_k, and set $F := E(F) \triangle E(P')$, where $\triangle$ denotes symmetric difference, and $I := I \setminus \{i, k\}$. If there exists no such a vertex w_k, then set $F := E(F) \triangle E(P)$ and $I := I \setminus \{i, j\}$. If F contains an Eulerian subgraph, then delete its edges from F. Clearly, the new constructed subgraph F is a forest satisfying (3.1), and if w_i is an end vertex of P, $d_F(w_i) + d_X(w_i) \leq |V(G_1)| - 1 + 1 = |V(G_1)|$; if w_i is an interval vertex of P, then $d_F(w_i) + d_X(w_i) \leq |V(G_1)| - 2 + 2 = |V(G_1)|$, and nothing changes for the vertices in A. Repeating this process until $I = \emptyset$. By the construction of F, we know that (3.1) is satisfied and T-join is preserved.

The problem becomes to show the existence of P stated above. We consider two subcases based on $a = |A|$.

Subcase 2.1. $a \leq n$.

If $a < n$, then $G_2 - \cup_{w_i \in A}\{v_{2i-1}, v_{2i}\}$ is $(n-a)$-extendable and $(n-a+1)$-connected. In particular, $G_2' - A$ is connected. Thus, there is a path P_{ij} from w_i to w_j avoiding A.

When $a = n$, if $G_2 - \cup_{w_i \in A}\{v_{2i-1}, v_{2i}\}$ is connected, we are done; otherwise, by n-extendability of G_2, it is not hard to see that every vertex in $\{v_{2i-1}, v_{2i} : w_i \in A\}$ is adjacent to all components of $G_2 - \cup_{w_i \in A}\{v_{2i-1}, v_{2i}\}$. Substituting the initial matching of G_2 at the beginning of Case 2 by a new matching so that it covers $\{v_{2i-1}, v_{2i} : w_i \in A\}$ by at least $n + \lceil n/2 \rceil$ edges (note: this is possible because G_2 is n-extendable). So we obtain a new graph G_2' and a corresponding vertex set I_0 and thus can reconstruct a new T-join F together with a new set A step by step. Note that this time, $G_2' - A$ is connected even if $|A| = n$.

Subcase 2.2. $a \geq n+1$.

Note that $d_F(w_i) \geq 3$ for $w_i \in A$ by the definition of A and assumption that $d_X(w_i) \leq 2m+1 < 2m+4 \leq |V(G_1)|$. Moreover, $|T| \leq 2(m+1)(n+1)$ and the number of leaves in F is at most $|T|-2$ by the construction of F. So, F has at most $2(m+1)(n+1) - \sum_{w_i \in A} d_X(w_i) - 2$ leaves.

On the other hand, we know that any nonempty forest $F \subseteq V(G')$ has leaves no less than

$$\begin{aligned}\textstyle\sum_{w_i \in V(F)}(d_F(w_i)-2) &\geq \textstyle\sum_{w_i \in A}(d_F(w_i)-2)\\ &= \textstyle a(|V(G_1)|-2) - \sum_{w_i \in A} d_X(w_i)\\ &\geq \textstyle (n+1)(2m+2) - \sum_{w_i \in A} d_X(w_i)\\ &> \textstyle 2(m+1)(n+1) - 2 - \sum_{w_i \in A} d_X(w_i),\end{aligned}$$

a contradiction. This completes the proof of Claim 2. ∎

Now we return to the proof of Case 2. Our aim is to construct a set M of $|E(F)|$ independent edges in $(G_2 \circ G_1) - X$ step by step. For any edge $w_iw_j \in E(F)$, we take one and only one edge e between $V(G_1^{v_{2i-1},v_{2i}})$ and $V(G_1^{v_{2j-1},v_{2j}})$ such that e is not covered by X and M constructed already, and put e into M. Suppose $w_iw_j \in E(F) \subseteq E(G_2')$ is the next edge to consider. The vertex set $X \cap V(G_1^{v_{2i-1},v_{2i}})$ (resp. $X \cap V(G_1^{v_{2j-1},v_{2j}})$) together with the already chosen edges of M cover a set of no more than $|V(G_1)|-1$ (resp. $|V(G_1)|-1$) vertices by (3.1). Since the edges between $G_1^{v_{2i-1},v_{2i}}$ and $G_1^{v_{2j-1},v_{2j}}$ together with the vertices constitute a complete bipartite graph, there always exists an edge e with one end vertex in $G_1^{v_{2i-1},v_{2i}} - (X \cup V(M))$ and the other in $G_1^{v_{2j-1},v_{2j}} - (X \cup V(M))$. Then add the edge e to M.

Since F is a T-join of G_2', then $|X \cap V(G_1^{v_{2i-1},v_{2i}})| + |V(M) \cap V(G_1^{v_{2i-1},v_{2i}})|$ is even. By (3.1) and the construction of G_2', $|X \cap V(G_1^{v_{2i-1},v_{2i}})| + |V(M) \cap V(G_1^{v_{2i-1},v_{2i}})| \leq |V(G_1)|$. Then, $G_1^{v_{2i-1},v_{2i}} - X - V(M)$ has a perfect matching M_i for each i. Hence, $M \cup \bigcup_{i=1}^{t} M_i$ is the desired perfect matching of $(G_2 \circ G_1) - X$.

Finally, we deal with the case of $|V(G_1)| = 2m+2$. We prove the assertion by induction on m. When $m=0$, it holds by Theorem 2.2. Suppose it holds for smaller m. If there is an edge $u_1u_2 \in E(G_1)$ for which $|X \cap V(G_2^{u_1,u_2})| \geq 2(n+1)$, then it is similar to the discussion as in Case 1, we can obtain a perfect matching of $(G_2 \circ G_1) - X$. Assume for every edge $u_iu_j \in E(G_1)$, we have $|X \cap V(G_2^{u_i,u_j})| \leq 2n+1$.

Since G_1 is m-extendable, by Theorem 2.4, $\delta(G_1) \geq m+1$. Then G_1 is Hamiltonian by Theorem 2.6. Hence, by Lemma 2.1, $(G_2 \circ G_1) - X$ has a perfect matching. This completes the proof. ∎

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References

[1] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc. (3)* **2** (1952), 69–81.

[2] O. Favaron, On k-factor-critical graphs, *Discuss. Math. Graph Theory* **16** (1996), no. 1, 41–51.

[3] E. Győri and M. D. Plummer, The Cartesian product of a k-extendable and an l-extendable graph is $(k+l+1)$-extendable, *Discrete Math.* **101** (1992), no. 1–3, 87–96.

[4] E. Győri and W. Imrich, On the strong product of a k-extendable and an l-extendable graph, *Graphs Combin.* **17** (2001), no. 2, 245–253.

[5] W. Imrich and S. Klavžar, *Product Graphs*, Wiley-Interscience, New York, 2000.

[6] J. Liu and Q. L. Yu, Matching extensions and products of graphs, in *Quo Vadis, Graph Theory?*, (1993), 191–200, *Ann. Discrete Math.*, 55, North-Holland, Amsterdam.
[7] L. Lovász and M. D. Plummer, *Matching Theory*, 29, North-Holland, Amsterdam, 1986.
[8] M. D. Plummer, On n-extendable graphs, *Discrete Math.* **31** (1980), no. 2, 201–210.
[9] Z. F. Wu, X. Yang and Q. L. Yu, On product of factor-critical graphs, (*submitted*).
[10] Q. L. Yu, Characterizations of various matching extensions in graphs, *Australas. J. Combin.* **7** (1993), 55–64.