BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(2) (2010), 205–210

The Connections Between Continued Fraction Representations of Units and Certain Hecke Groups

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Abstract. Let $\lambda = \sqrt{D}$ where D is a square free integer such that $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$, or $D = n^2 - 1$ for $n = 2, 3, 4, 5, \cdots$. Also, let $H(\lambda)$ be the Hecke group associated to λ. In this paper, we show that the units in $H(\lambda)$ are infinite pure periodic λ-continued fraction for a certain set of integer D, and hence can not be cusp points.

2000 Mathematics Subject Classification: 20H10, 11K55

Key words and phrases: Hecke group, Fuchsian group, continued fraction, cusp point.

1. Introduction

Hecke groups $H(\lambda)$ are, in some sense, generalizations of the well-known modular group

$$\Gamma = \left\{ \frac{az+b}{cz+d} \middle| \, a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

They are the discrete subgroups of $PSL(2, \mathbb{R})$ (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \quad \text{and} \quad T(z) = z + \lambda,$$

Communicated by Lee See Keong.
Received: September 24, 2008; *Revised:* June 15, 2009.

where λ is a fixed positive real number. Hecke groups $H(\lambda)$ were introduced by Erich Hecke in [2]. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos(\pi/q)$, where $q \geq 3$, $q \in \mathbb{N}$, or $\lambda \geq 2$. These groups are called *Hecke groups.*

Actually, the modular group $H(\lambda_3)$ has been worked intensively as an Hecke group. In this $q = 3$ case, $\lambda_3 = 2\cos(\pi/3) = 1$, that is, all coefficients of the elements of $H(\lambda_3)$ are rational integers. Therefore $H(\lambda_3) = PSL(2,\mathbb{Z})$. The next two most important Hecke groups are those for $q = 4$ and 6. As $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$, $H(\sqrt{2})$ and $H(\sqrt{3})$, denote the Hecke groups corresponding to λ_4 and λ_6, respectively. For these groups, the underlying fields are the quadratic extensions of $\mathbb{Q}$ by the algebraic numbers $\sqrt{2}$ and $\sqrt{3}$, that is, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$. It is known (see [1]) that $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, respectively, i.e.

$$H(\lambda_q) \cong C_2 * C_q.$$

Also in case $\lambda \geq 2$, the element $S = RT$ is parabolic when $\lambda = 2$, or hyperbolic (boundary) when $\lambda > 2$. It is known that when $\lambda \geq 2$, $H(\lambda)$ is a free product of two cyclic groups of orders 2 and infinity, see [10], so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) \cong C_2 * \mathbb{Z}.$$

Here we deal with the cases $\lambda = \sqrt{D}$ where D is a squarefree integer such that $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$, or $D = n^2 - 1$ for $n = 2, 3, 4, 5, \cdots$ and denote the group by $H(\sqrt{D})$ (see, [8, 9, 11].

The entries of elements of $H(\sqrt{D})$ are in $\mathbb{Z}\left[\sqrt{D}\right]$, which for each D is the ring of algebraic integers for $\mathbb{Q}(\sqrt{D})$. For $m \geq 1$, $\mathbb{Q}(\sqrt{D})$ has nontrivial units and may have a nontrivial class group.

It is clear from the above that $H(\sqrt{D}) \subset PSL(2, \mathbb{Z}[\sqrt{D}])$, (the inclusion is strict when $\lambda > 1$) and hence the cusps are contained in $\mathbb{Q}(\sqrt{D}) \cup \{\infty\}$ [3].

Hecke groups and continued fractions play a crucial role in many different aspects of number theory and have been extensively studied in [5], [6] and [7]. Rosen introduced a class of continued fractions closely associated with the Hecke groups in [5]. He expanded any λ_q, $q = 4$, 5, 6 and 7 into a continued fraction using a nearest integral multiple of λ algorithm (in fact that he expanded any real number into a continued fraction using a nearest integral multiple of λ algorithm).

It is well-known that a λ-continued fraction (λcf) has a form

$$[r_0\lambda,\ \epsilon_1/r_1\lambda,\ \epsilon_2/r_2\lambda, \cdots] = r_0\lambda + \cfrac{\epsilon_1}{r_1\lambda + \frac{\epsilon_2}{r_2\lambda} + \cdots}$$

where $\epsilon_i = \pm 1$, r_i are positive rational integer for $i \geq 1$ and r_0 is a rational integer. We write a λcf as

$$[r_0\lambda,\ \epsilon_1/r_1\lambda,\ \epsilon_2/r_2\lambda, \cdots].$$

Also every real number has a unique reduced λcf obtained by the nearest integer algorithm. Here we are interested in finding λcf representations of the element of $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{D})$.

Apart from $H(1)$, $H(\sqrt{2})$ and $H(\sqrt{3})$, the elements of $H(\lambda)$ are not completely known. It is important to know the general structure of the elements of $H(\lambda)$ to find the cusp set, the set of real numbers which are images of ∞ under the group elements. The next best result is given by Rosen.

Rosen [6], gave the necessary and sufficient condition for an element $V(z) = (Az+B)/(Cz+D)$ of $H(\lambda)$ to be A/C is a finite λcf. He showed that $\alpha \in \mathbb{Q}(\lambda)$ if α has a finite λcf. Also in [7], Rosen and Towse proved that when $D = 2$ or 3, the units in $\mathbb{Z}[\sqrt{D}]$ are infinite pure periodic λcf's, and hence cannot be cusp points.

Now we briefly mention the concept of the fundamental unit. The fundamental units for real quadratic fields $\mathbb{Q}(\sqrt{D})$ may be computed from the fundamental solution of the Pell equation

$$T^2 - DU^2 = \pm 4$$

where the sign is taken such that the solution (T, U) has smallest possible positive integer T. If the positive sign is taken, then one solution is simply given by $(T, U) = (2x, 2y)$, where (x, y) is the solution to the Pell equation

$$x^2 - Dy^2 = 1.$$

Given a minimal (T, U), the fundamental unit is given by

$$\eta = \frac{1}{2}(T + U\sqrt{D}) \qquad [4].$$

Let $\eta = \frac{1}{2}(T + U\sqrt{D})$ be the fundamental unit where T is chosen to be the smallest positive integer and D is a square free integer such that $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$, or $D = n^2 - 1$ for $n = 2, 3, 4, 5, \cdots$.

In this paper, we are interested especially in the cases $U = 2$ and $T = 2m$, $m = 1, 3, 4, 5, \cdots$ or $U = 2$ and $T = 2n$, $n = 2, 3, 4, 5, \cdots$. In these cases, the fundamental unit is $\eta = m + \sqrt{m^2+1}$ or $\eta = n + \sqrt{n^2-1}$. The only problem is when $m = 2$. If $m = 2$, then $T = 1$ and $U = 1$. But in this case the fundamental unit is different from our fundamental unit form. Therefore, for all $m \neq 2$, we have an even value of T, giving fundamental unit $a + b\sqrt{D}$, where $a, b \in \mathbb{Z}^+$.

We generalize some results in the Hecke groups $H(\sqrt{2})$ and $H(\sqrt{3})$ to the Hecke groups $H(\sqrt{D})$ where D is a squarefree integer such that $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$, or $D = n^2 - 1$ for $n = 2, 3, 4, 5, \cdots$. We consider units in the ring of integers of the fields $\mathbb{Q}(\sqrt{D})$. We show that for a certain set of integer D, the Hecke group $H(\lambda)$ of translation length $\sqrt{D}$ is such that no unit of $\mathbb{Z}[\lambda]$ is a cusp value. In order to do these we use one of the Rosen's continued fraction expansions of the units, in which we expand $\lambda = \sqrt{D}$ into continued fractions using a nearest integral multiple of λ algorithm.

2. Main results

Before giving the main theorem, we give the fundamental units of $\mathbb{Q}(\sqrt{D})$ where D is a squarefree integer such that $D = m^2+1$, for $m = 1, 3, 4, 5, \cdots$ or $D = n^2-1$, for $n = 2, 3, 4, 5, \cdots$. We know that a fundamental unit for $\mathbb{Q}(\sqrt{D})$ is $\eta = m + \sqrt{m^2+1}$ or $\eta = n + \sqrt{n^2-1}$ for m and n, respectively.

Theorem 2.1. *Let D be a squarefree integer such that $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$. Then all units in $\mathbb{Z}(\sqrt{D})$ have infinite repeating λcf representations.*

Proof. If $D = m^2 + 1$ for $m = 1, 3, 4, 5, \cdots$, then a fundamental unit in $\mathbb{Q}(\sqrt{D})$ is $\eta = m + \sqrt{D}$. Thus all units in $\mathbb{Z}[\sqrt{D}]$ can be expressed in the form $\pm(m + \sqrt{D})^k$.

We will make use of the nearest integer algorithm to obtain the λcf representation of $(m + \sqrt{D})^k$ for $k \geq 2$. Also, we will denote the nearest integer function with $[|\cdot|]$ and the nearest integer multiple of λ to α with $[|\alpha/\lambda|]$.

Firstly let $\eta^k = (a_k + b_k\sqrt{D})$. To complete the proof, we need to calculate $[|(a_k + b_k\sqrt{D})/\sqrt{D}|]$. We now have

$$\left|a_k^2 - D.b_k^2\right| = 1 \tag{2.1}$$

as $a_k + b_k\sqrt{D}$ is a unit. After simple calculatings, we have $a_{k+1} = m.a_k + D.b_k$ and $\mathrm{b}_{k+1} = a_k + m.b_k$. It is clearly seen that $b_{k+1} > b_k$ and $b_k \geq 2$ for $k \geq 2$. Thus we find $b_k^2 \geq 2$.

If we divide both sides of equality (2.1) by b_k^2, we obtain

$$\left|\frac{a_k^2}{b_k^2} - D\right| = \frac{1}{b_k^2} \leq \frac{1}{4}.$$

Thus a_k may be approximated by $b_k\sqrt{D}$. The nearest (rational) integer multiple of $\sqrt{D}$ to $a_k + b_k\sqrt{D}$ is given by

$$\left[\left|\frac{a_k + b_k\sqrt{D}}{\sqrt{D}}\right|\right] \approx \left[\left|\frac{b_k\sqrt{D} + b_k\sqrt{D}}{\sqrt{D}}\right|\right] = 2b_k.$$

Therefore we get

$$a_k + b_k\sqrt{D} = 2b_k\sqrt{D} - (-a_k + b_k\sqrt{D}) = 2b_k\sqrt{D} + \frac{a_k^2 - D.b_k^2}{a_k + b_k\sqrt{D}}.$$

Then

$$\eta^k = a_k + b_k\sqrt{D} = 2b_k\sqrt{D} + \frac{\epsilon}{a_k + b_k\sqrt{D}},$$

where $\epsilon = a_k^2 - D.\ b_k^2 = \pm 1$. It can be easily seen that $\epsilon = (-1)^k$.

We then get $\eta^k = [2b_k\lambda, \epsilon/2b_k\lambda, \epsilon/2b_k\lambda, \cdots]$ as the λcf expansion, which is infinite pure periodic.

If $k = 0$, we get a result from [5].

Now we find the λcf of $\eta = (m+\sqrt{D})$ for $k = 1$. Firstly we get, $[|(m + \sqrt{D})/\sqrt{D}|] \approx 2$, and later

$$m + \sqrt{D} = 2\sqrt{D} - (-m + \sqrt{D}) = 2\sqrt{D} - \frac{1}{m + \sqrt{D}} = 2\sqrt{D} - 1/\eta.$$

Thus we obtain,

$$m + \sqrt{D} = [2\lambda, -1/2\lambda, -1/2\lambda, \cdots].$$

Finally the algorithm works similarly for $k < 0$. We have

$$\eta^k = 1/\eta^{-k} = [0; +1/2b_k\lambda, \epsilon/2b_k\lambda, \epsilon/2b_k\lambda, \cdots].$$ ∎

Notice that for $m = 1$, this result coincides with the Theorem 1 given in [7] for the Hecke group $H(\sqrt{2})$.

Theorem 2.2. *Let D be a squarefree integer such that $D = n^2 - 1$, for $n = 2, 3, 4, 5, \cdots$. Then all units in $\mathbb{Z}(\sqrt{D})$ have infinite repeating λcf representations.*

Proof. For $D = n^2 - 1$, $n = 2, 4, 5, \cdots$, the fundamental unit is $n + \sqrt{D}$. Then all units in $\mathbb{Z}(\sqrt{D})$ are of the form $\pm(n + \sqrt{D})^k$. Using the method in the Theorem 2.1, it follows that a_k is well-approximated by $b_k\sqrt{D}$, since $b_k \geq 4$ for $k \geq 2$. Thus,

$$\left[\left|\frac{a_k + b_k\sqrt{D}}{\sqrt{D}}\right|\right] \approx \left[\left|\frac{b_k\sqrt{D} + b_k\sqrt{D}}{\sqrt{D}}\right|\right] = 2b_k.$$

Therefore we have

$$(n + \sqrt{D})^k = [2b_k\lambda,\ \epsilon/2b_k\lambda,\ \epsilon/2b_k\lambda, \cdots].$$

Here it is easy to check that ϵ is always $+1$.

The cases $k \leq 0$ are dealt with as in Theorem 2.1. We find the fundamental unit $n + \sqrt{D}$ for $k = 1$. We obtain

$$\begin{aligned} n + \sqrt{n^2 - 1} &= 2\sqrt{n^2 - 1} + (n - \sqrt{n^2 - 1}) \\ &= 2\sqrt{n^2 - 1} + \frac{1}{n + \sqrt{n^2 - 1}\ldots} \\ &= [2\lambda, +1/2\lambda, +1/2\lambda, \cdots]. \end{aligned}$$

∎

Notice that for $n = 2$, this result coincides with the Theorem 2 given in [7] for the Hecke group $H(\sqrt{3})$.

From the Theorems 2.1 and 2.2, we have the following corollary.

Corollary 2.1. *Let $u = a + b\sqrt{D}$ be a unit in $\mathbb{Z}(\sqrt{D})$, with $u > 1$. Then u has a purely periodic λcf expansion ($\lambda = \sqrt{D}$) :*

$$u = [2b\lambda; \epsilon/2b\lambda, \epsilon/2b\lambda, \cdots].$$

Proof. Since the norm of u has absolute value 1, $(a + b\sqrt{D})/\sqrt{D}$ has absolute value less than $1/\sqrt{D}$. From this, as in the proof of the Theorem 2.1, it follows that the nearest (rational) integer multiple of $\sqrt{D}$ to u is $2b$. But $u = 2b\sqrt{D} + \epsilon/u$, with ϵ the sign of the norm of u. The result follows. ∎

In fact that the proof of the Corollary 2.1 is a variation of the method we used, but allowing to avoid our various cases and it covers setting that we exclude.

Therefore in the it is easily seen that the units in $\mathbb{Q}(\sqrt{D})$ are not cusp points, since they are not finite λcf's.

The continued fractions can easily be checked to show that they represent the unit. For example, the number represented by $[24\lambda,\ +1/24\lambda,\ +1/24\lambda,\ \cdots]$ is a solution to $x = 24\sqrt{35} + 1/x$. This leads to $x^2 - 24\sqrt{35}x - 1 = 0$, which we can solve to get $x = 12\sqrt{35} \pm 71$. We choose the $+$ sign, since $x > 1$.

We note that the results of [5] concern all elements of the field $\mathbb{Q}(\lambda_5)$, not just the units in the ring of integers. However, even in $\mathbb{Q}(\sqrt{D})$ there exist non-units with infinite λcf's. For example, if $m = 1, 3, 4, 5, \cdots$ then

$$2m^2 + 1 = [2m\lambda, +1/4m\lambda, +1/4m\lambda, \cdots]$$

or if $n = 2, 4, 5, 6, \cdots$ then

$$2n^2 - 1 = [2n\lambda, +1/4n\lambda, +1/4n\lambda, \cdots].$$

In the following, we list the fundamental units of $\mathbb{Z}(\sqrt{D})$ where m and n are given above.

m	$D = m^2 + 1$	Fundamental units	n	$D = n^2 - 1$	Fundamental units
1	2	$1 + \sqrt{2}$	2	3	$2 + \sqrt{3}$
3	10	$3 + \sqrt{10}$	3	8	–
4	17	$4 + \sqrt{17}$	4	15	$4 + \sqrt{15}$
5	26	$5 + \sqrt{26}$	5	24	–
6	37	$6 + \sqrt{37}$	6	35	$6 + \sqrt{35}$
7	50	–	7	48	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$m^2 + 1$	$m + \sqrt{m^2 + 1}$	n	$n^2 - 1$	$n + \sqrt{n^2 - 1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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