BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33***(2) (2010), 211–220*

A Direct Approach to Co-Universal Algebras Associated to Directed Graphs

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Abstract. We prove directly that if E is a directed graph in which every cycle has an entrance, then there exists a C^*-algebra which is co-universal for Toeplitz-Cuntz-Krieger E-families. In particular, our proof does not invoke ideal-structure theory for graph algebras, nor does it involve use of the gauge action or its fixed point algebra.

2000 Mathematics Subject Classification: Primary: 46L05

Key words and phrases: Graph algebra, co-universal property.

1. Introduction

In recent years there has been a great deal of interest in graph C^*-algebras and their generalisations (see [3] for a survey). To associate C^*-algebras to a given generalisation of directed graphs, one assigns partial isometries to the edges of the graph in a way which encodes connectivity in the graph. One then aims to identify relations amongst the partial isometries so that the C^*-algebra universal for these relations satisfies a version of the Cuntz-Krieger uniqueness theorem. For directed graphs, this theorem states that if every cycle has an entrance, then any representation of its Cuntz-Krieger algebra which is nonzero on generators is faithful. In trying to identify appropriate relations, there is typically some analogue of a left-regular representation which points to a natural notion of an abstract representation; in the case of directed graphs, each graph can be represented on the Hilbert space with orthonormal basis indexed by finite paths in the graph, and this gives rise to the notion of a Toeplitz-Cuntz-Krieger family. However, the universal C^*-algebra for such representations is typically too big to satisfy a version of the Cuntz-Krieger uniqueness theorem, and one has to identify an additional relation to correct this.

In [2], in the much more general context of Cuntz-Pimsner algebras associated to Hilbert bimodules, Katsura developed a very elegant solution to this problem. For directed graphs, his results say that given any Toeplitz-Cuntz-Krieger E-family

Communicated by Sriwulan Adji.
Received: July 25, 2009; *Revised:* January 21, 2010.

consisting of nonzero partial isometries, if the C^*-algebra B it generates carries a circle action compatible with the canonical gauge action on the Toeplitz algebra, then there is a canonical homomorphism from B onto the Cuntz-Krieger algebra. We call this a co-universal property: the Cuntz-Krieger algebra is co-universal for Toeplitz-Cuntz-Krieger families consisting of nonzero partial isometries and compatible with the gauge action. Katsura proved this theorem *a posteriori*: The Cuntz-Krieger algebra had already been defined in terms of a universal property. However, he pointed out that this theorem implies that the Cuntz-Krieger algebra could be *defined* to be the algebra co-universal for nonzero Cuntz-Krieger families; one would then have to work to prove that such a co-universal algebra exists.

When every cycle in the graph E has an entrance, it is a consequence of the Toeplitz-Cuntz-Krieger uniqueness theorem that every Cuntz-Krieger E-family consisting of nonzero partial isometries is automatically compatible with the gauge action. In particular, in this case the use of the gauge action in Katsura's analysis should not be necessary. In this article we show that this is indeed the case: We present a direct argument that for an arbitrary directed graph in which every cycle has an entrance, there exists a C^*-algebra which is co-universal for Toeplitz-Cuntz-Krieger families consisting of nonzero partial isometries. In particular, we do not first identify the Cuntz-Krieger relation or the corresponding universal C^*-algebra. We also do not proceed via the machinery of ideal structure of Toeplitz algebras of directed graphs, and we do not deal with the gauge-action of the circle or with an analysis of its fixed-point algebra. Instead we work directly with the abelian subalgebra generated by range projections associated to paths in the graph.

2. Preliminaries

A directed graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \to E^0$ indicating the direction of the edges. We will follow the conventions of [3] so that a path is a sequence $\alpha = \alpha_1\alpha_2\dots\alpha_n$ of edges such that $s(\alpha_i) = r(\alpha_{i+1})$ for all i. We write $|\alpha|$ for n, and if we want to indicate a segment of a path, we shall denote it $\alpha_{[p,q]} = \alpha_{p+1}\alpha_{p+2}\dots\alpha_q$. If $n = \infty$ (so that α is actually a right-infinite string), then we call α an *infinite path.* The range of an infinite path $\alpha = \alpha_1\alpha_2\dots$ is $r(\alpha) := r(\alpha_1)$.

For $n \in \mathbb{N}$, we write E^n for the collection of paths of length n. We write E^* for the category of finite paths in E (we regard vertices as paths of length zero), and E^∞ for the collection of all infinite paths. Given $\alpha \in E^*$ and $X \subset E^* \cup E^\infty$, we denote $\{\alpha\mu : \mu \in X, r(\mu) = s(\alpha)\}$ by αX. In particular, if $v \in E^0$, then

$$\begin{aligned} vE^* &= \{\lambda \in E^* : r(\lambda) = v\}, \text{ and} \\ vE^1 &= \{e \in E^1 : r(e) = v\}. \end{aligned}$$

Definition 2.1. *Given $v \in E^0$, a set $X \subset vE^*$ is said to be* exhaustive *if, for every $\lambda \in vE^*$ there exists $\alpha \in X$ such that either $\lambda = \alpha\lambda'$ or $\alpha = \lambda\alpha'$.*

Given a directed graph E, a *Toeplitz-Cuntz-Krieger E-family* in a C^*-algebra B is a pair (t, q) where $t : e \mapsto t_e$ assigns to each edge a partial isometry in B, and $q : v \mapsto q_v$ assigns to each vertex a projection in B such that

(TCK1) the q_v are mutually orthogonal

(TCK2) each $t_e^* t_e = q_{s(e)}$, and
(TCK3) for each $v \in E^0$ and each finite subset $F \subset vE^1$, we have $q_v \geq \sum_{e \in F} t_e t_e^*$.

There is a C^*-algebra $\mathcal{T}C^*(E)$ generated by a Toeplitz-Cuntz-Krieger family (s, p) which is universal in the sense that given any Toeplitz-Cuntz-Krieger family (t, q) in a C^*-algebra B there is a homomorphism $\pi_{t,q} : \mathcal{T}C^*(E) \to B$ such that $\pi_{t,q}(s_e) = t_e$ and $\pi_{t,q}(p_v) = q_v$ for all $e \in E^1$ and $v \in E^0$.

Given a path $\alpha \in E^*$ and a Toeplitz-Cuntz-Krieger E-family (t, q), we write t_α for $t_{\alpha_1} t_{\alpha_2} \dots t_{\alpha_{|\alpha|}}$.

3. The co-universal algebra

Theorem 3.1. *Let E be a directed graph, and suppose that every cycle in E has an entrance. There exists a Toeplitz-Cuntz-Krieger E-family $(S^{\mathrm{ap}}, P^{\mathrm{ap}})$ consisting of nonzero partial isometries such that $C^*_{min}(E) := C^*(S^{\mathrm{ap}}, P^{\mathrm{ap}})$ is co-universal in the sense that given any other Toeplitz-Cuntz-Krieger E family (t, q) in which each q_v is nonzero, there is a homomorphism $\psi_{t,q} : C^*(t, q) \to C^*_{min}(E)$ such that $\psi_{t,q}(t_e) = S_e^{\mathrm{ap}}$ and $\psi_{t,q}(q_v) = P_v^{\mathrm{ap}}$ for all $e \in E^1$ and $v \in E^0$.*

Moreover, the pair $(C^*_{min}(E), (S^{\mathrm{ap}}, P^{\mathrm{ap}}))$ is unique up to canonical isomorphism in the sense that if A is another C^*-algebra generated by another Toeplitz-Cuntz-Krieger E-family (t, q) with the same co-universal property, then there is an isomorphism $C^*_{min}(E) \cong A$ which carries each S_e^{ap} to t_e and each P_v^{ap} to q_v.

Our proof relies on understanding the structure the C^*-algebra generated by the projections $\{t_\lambda t_\lambda^* : \lambda \in E^*\}$ for a Toeplitz-Cuntz-Krieger E-family (t, q). We begin with the following definition.

Definition 3.1. *Let E be a directed graph. A* Boolean representation *of E in a C^*-algebra B is a map $p : \lambda \mapsto p_\lambda$ from E^* to B such that each p_λ is a projection, and*

$$p_\mu p_\nu = \begin{cases} p_\nu & \text{if } \nu = \mu\nu' \\ p_\mu & \text{if } \mu = \nu\mu' \\ 0 & \text{otherwise.} \end{cases}$$

If p is a Boolean representation of E, then the p_λ commute, so $\overline{\operatorname{span}}\{p_\lambda : \lambda \in E^*\}$ is a commutative C^*-algebra.

Lemma 3.1. *Let E be a directed graph, let p be a Boolean representation of E, and fix a finite subset $F \subset E^*$. For $\mu \in F$, define*

$$q_\mu^F := p_\mu \prod_{\mu\mu' \in F \setminus \{\mu\}} (p_\mu - p_{\mu\mu'}).$$

Then the q_μ^F are mutually orthogonal projections and for each $\mu \in E^*$,

$$p_\mu = \sum_{\mu\mu' \in F} q_{\mu\mu'}^F \tag{3.1}$$

Proof. We proceed by induction on $|F|$. If $|F| = 1$ then (3.1) is trivial. Suppose (3.1) holds whenever $|F| < n$, and fix F with $|F| = n$. Let $\lambda \in F$ be of maximal length, and let $G = F \setminus \{\lambda\}$. Then $q^F_\lambda = p_\lambda$, and for $\mu \in G$,

$$q^F_\mu = \begin{cases} q^G_\mu & \text{if } \lambda \neq \mu\mu' \\ q^G_\mu - q^G_\mu p_\lambda & \text{if } \lambda = \mu\mu'. \end{cases}$$

Fix $\mu \in F$. If $\lambda \neq \mu\mu'$, then the inductive hypothesis implies that $\sum_{\mu\mu' \in F} q^F_{\mu\mu'} = \sum_{\mu\mu' \in G} q^G_{\mu\mu'} = p_\mu$. If $\lambda = \mu\mu'$, then

$$\begin{aligned} \sum_{\mu\mu' \in F} q^F_{\mu\mu'} &= \sum_{\mu\mu' \in G} (q^G_\mu - q^G_\mu p_\lambda) + q^F_\lambda \\ &= \sum_{\mu\mu' \in G} q^G_\mu - \sum_{\mu\mu' \in G} q^G_\mu p_\lambda + p_\lambda \\ &= p_\mu - p_\mu p_\lambda + p_\lambda \qquad \text{(by the inductive hypothesis)} \\ &= p_\mu - p_\lambda + p_\lambda \qquad \text{(since } \lambda = \mu\mu'\text{)} \\ &= p_\mu. \end{aligned}$$

∎

Recall that for $\lambda \in E^*$, we write $s(\lambda)E^1$ for $\{e \in E^1 : r(e) = s(\lambda)\}$. Given a directed graph E, we define the set of *aperiodic boundary paths* in E by

$$\partial E^{\mathrm{ap}} := \big\{\lambda \in E^* : |s(\lambda)E^1| \in \{0, \infty\}\big\} \sqcup E^\infty \setminus \{\lambda\mu^\infty : s(\lambda) = r(\mu) = s(\mu)\}.$$

So an aperiodic boundary path is either a finite path whose source receives either infinitely many edges or no edges, or else is an infinite path which is aperiodic in the sense that it does not eventually repeatedly traverse some fixed cycle.

If $x \in \partial E^{\mathrm{ap}}$ and $\lambda \in E^*$ with $s(\lambda) = r(x)$, then $\lambda x := \lambda_1 \dots \lambda_{|\lambda|} x_1 x_2 \dots$ also belongs to ∂E^{ap}.

Let $\mathcal{H} := \ell^2(\partial E^{\mathrm{ap}})$, with orthonormal basis $\{\xi_x : x \in \partial E^{\mathrm{ap}}\}$, and define $\{P^{\mathrm{ap}}_\lambda : \lambda \in E^*\} \subset \mathcal{B}(\mathcal{H})$ by

$$P^{\mathrm{ap}}_\lambda \xi_x = \begin{cases} \xi_x & \text{if } x \in \lambda\partial E^{\mathrm{ap}} \\ 0 & \text{otherwise.} \end{cases}$$

Since each $P^{\mathrm{ap}}_\lambda = \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_x : x \in \lambda\partial E^{\mathrm{ap}}\}}$, it is straightforward to check that P^{ap} is a Boolean representation of E.

Suppose that every cycle in E has an entrance. We claim that each P^{ap}_λ is nonzero. Indeed, fix $\lambda \in E^*$. Since every cycle in E has an entrance, there exists an $x \in \partial E^{\mathrm{ap}}$ with $r(x) = s(\lambda)$. Then $\lambda x \in \partial E^{\mathrm{ap}}$ so $P^{\mathrm{ap}}_\lambda \xi_{\lambda x} = \xi_{\lambda x} \neq 0$.

Lemma 3.2. *Let E be a directed graph. Let $\lambda \in E^*$, and suppose that $F \subset s(\lambda)E^*$ is finite and exhaustive. Then*

$$\prod_{\mu \in F} (P^{\mathrm{ap}}_\lambda - P^{\mathrm{ap}}_{\lambda\mu}) = 0.$$

Proof. Let $x \in \partial E^{\mathrm{ap}}$. We seek $\mu \in F$ such that $(P^{\mathrm{ap}}_\lambda - P^{\mathrm{ap}}_{\lambda\mu})\xi_x = 0$.

If $P^{\mathrm{ap}}_\lambda \xi_x = 0$, then any $\mu \in F$ will suffice, so suppose that $P^{\mathrm{ap}}_\lambda \xi_x \neq 0$. Then $x_{[0,|\lambda|]} = \lambda$. If there exists $\mu \in F$ such that $x = \lambda\mu x'$, then $(P^{\mathrm{ap}}_\lambda - P^{\mathrm{ap}}_{\lambda\mu})\xi_x = \xi_x - \xi_x = 0$, so we suppose that $x \neq \lambda\mu x'$ for all $\mu \in F$ and seek a contradiction. Since F is exhaustive, there exists $\mu \in F$ such that $\lambda\mu = x\mu'$ for some μ' of nonzero length. In particular, x is a finite path $\alpha \in E^*$ and since μ' has nonzero length,

$s(\alpha)E^1 \neq \emptyset$. Hence $\alpha \in \partial E^{\mathrm{ap}}$ forces $|s(\alpha)E^1| = \infty$. Since no initial segment of α belongs to λF, that F is exhaustive implies that for each $e \in s(\alpha)E^1$, there exists $\mu_e \in F$ such that $\lambda\mu_e = \alpha e \alpha'_e$ for some $\alpha_e \in E^*$, and this contradicts that F is a finite set. ∎

Lemma 3.3. *Let E be a directed graph and p be a Boolean representation of E such that $p_\lambda \neq 0$ for each $\lambda \in E^*$. If $\{\alpha' : \alpha\alpha' \in F\}$ is not exhaustive, then $q_\alpha^F \neq 0$.*

Proof. Since $\{\alpha' : \alpha\alpha' \in F\}$ is not exhaustive, there exists $\tau \in E^*$ such that $\tau \neq \alpha'\alpha''$ and $\alpha' \neq \tau\tau'$ for each α' such that $\alpha\alpha' \in F$. In particular, each $p_{\alpha\alpha'}p_{\alpha\tau} = 0$, and hence $q_\alpha^F p_{\alpha\tau} = p_{\alpha\tau} \neq 0$, whence $q_\alpha^F \neq 0$. ∎

Proposition 3.1. *Let E be a directed graph and let p be a Boolean representation of E such that $p_\lambda \neq 0$ for each $\lambda \in E^*$. Then there is a homomorphism $\psi_p : \overline{\operatorname{span}}\{p_\lambda : \lambda \in E^*\} \to \overline{\operatorname{span}}\{P_\lambda^{\mathrm{ap}} : \lambda \in E^*\}$ satisfying $\psi_p(p_\lambda) = P_\lambda^{\mathrm{ap}}$ for all $\lambda \in E^*$. Moreover, ψ_p is injective if and only if $\prod_{\mu\in F}(p_\lambda - p_{\lambda\mu}) = 0$ for all $\lambda \in E^*$ and finite exhaustive $F \subset s(\lambda)E^*$.*

Proof. For the first assertion it suffices to show that for every finite subset F of E^* and every collection of scalars $\{a_\lambda : \lambda \in F\}$,

$$\Big\| \sum_{\lambda\in F} a_\lambda P_\lambda^{\mathrm{ap}} \Big\| \leq \Big\| \sum_{\lambda\in F} a_\lambda p_\lambda \Big\| \tag{3.2}$$

Fix a finite subset $F \subset E^*$, and for $\alpha \in F$, define

$$Q_\alpha^F := P_\alpha^{\mathrm{ap}} \prod_{\alpha\alpha' \in F\setminus\{\alpha\}} (P_\alpha^{\mathrm{ap}} - P_{\alpha\alpha'}^{\mathrm{ap}}).$$

Lemma 3.1 gives

$$\sum_{\lambda\in F} a_\lambda P_\lambda^{\mathrm{ap}} = \sum_{\alpha\in F} \Big(\sum_{\substack{\mu\in F\\ \alpha=\mu\mu'}} a_\mu \Big) Q_\alpha^F, \qquad \text{and} \qquad \sum_{\lambda\in F} a_\lambda p_\lambda = \sum_{\alpha\in F} \Big(\sum_{\substack{\mu\in F\\ \alpha=\mu\mu'}} a_\mu \Big) q_\alpha^F.$$

By Lemmas 3.2 and 3.3, we have $\{\alpha \in F : Q_\alpha^F \neq 0\} \subset \{\alpha \in F : q_\alpha^F \neq 0\}$. Hence

$$\Big\| \sum_{\lambda\in F} a_\lambda P_\lambda^{\mathrm{ap}} \Big\| = \max_{Q_\alpha^F \neq 0} \Big| \sum_{\substack{\mu\in F\\ \alpha=\mu\mu'}} a_\mu \Big| \leq \max_{q_\alpha^F \neq 0} \Big| \sum_{\substack{\mu\in F\\ \alpha=\mu\mu'}} a_\mu \Big| = \Big\| \sum_{\lambda\in F} a_\lambda p_\lambda \Big\|,$$

and the first assertion follows.

For the second assertion, note that if $\prod_{\mu\in F}(p_\lambda - p_{\lambda\mu}) = 0$ for all $\lambda \in E^*$ and finite exhaustive $F \subset s(\lambda)E^*$, then for each finite exhaustive F, we have $\{\alpha \in F : Q_\alpha^F \neq 0\} = \{\alpha \in F : q_\alpha^F \neq 0\}$ so the calculation above shows that ψ_p is isometric. ∎

We now show that the C^*-algebra generated by any Toeplitz-Cuntz-Krieger E-family in which all the partial isometries are nonzero admits a conditional expectation onto the subalgebra spanned by the range projections $s_\lambda s_\lambda^*$. Recall that given a Toeplitz-Cuntz-Krieger E-family (t, q), we write $\pi_{t,q}$ for the canonical homomorphism from $\mathcal{T}C^*(E)$ to $C^*(t, q)$ induced by the universal property of the former. We first need a technical lemma.

Lemma 3.4. *Suppose $\lambda, \mu, \nu \in E$ satisfy $|\lambda| \geq |\nu| > |\mu|$ and*

$$t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* \neq 0. \tag{3.3}$$

Then $\lambda = \nu\nu' = \mu\mu'\nu'$ for some μ', ν', and

$$t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = t_\lambda t_{\lambda\rho}^*$$

for some cycle $\rho \in E$.

Proof. Equation (3.3) forces $t_\lambda t_\lambda^* t_\mu t_\mu^* \neq 0$ and $t_\nu t_\nu^* t_\lambda t_\lambda^* \neq 0$. Hence $\lambda = \nu\nu'$ and $\lambda = \mu\alpha$ for some ν' and α, and since $|\mu| < |\nu|$, this forces $\nu = \mu\mu'$ and hence $\lambda = \mu\mu'\nu'$ for some μ'.

We have

$$0 \neq t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = t_\lambda t_\lambda^* t_\mu (t_{\mu'}^* t_\mu^*)(t_\mu t_{\mu'} t_{\nu'}) t_\lambda^* = t_\lambda t_\lambda^* t_\mu t_{\nu'} t_\lambda^*, \tag{3.4}$$

forcing $s(\mu) = r(\nu')$. Since $r(\mu') = s(\mu)$ and $s(\mu') = r(\nu')$, μ' is a cycle, and has nonzero length since $|\nu| > |\mu|$. Furthermore, continuing from (3.4)

$$0 \neq t_\lambda t_\lambda^* t_\mu t_{\nu'} t_\lambda^* = (t_\mu t_{\mu'\nu'})(t_{\mu'\nu'}^* t_\mu^*) t_\mu t_{\nu'} (t_{\nu'}^* t_{\mu\mu'}^*) = t_\mu t_{\mu'\nu'} t_{\mu'\nu'}^* t_{\nu'} t_{\nu'}^* t_{\mu\mu'}^*,$$

so $t_{\mu'\nu'}^* t_{\nu'}$ is nonzero, forcing

$$\mu'\nu' = \nu'\rho \qquad \text{for some } \rho \in E. \tag{3.5}$$

We claim ρ is a cycle. We proceed by induction on $|\nu'|$. As a base case, suppose that $|\nu'| \leq |\mu'|$. Then $\nu' = \mu'_{[0,|\nu'|]}$, so

$$\mu'\nu' = \nu'\rho \implies \rho = \mu'_{[|\nu'|,|\mu'|]}\mu'_{[0,|\nu'|]},$$

whence $r(\rho) = s(\rho) = s(\nu')$.

Now suppose as an inductive hypothesis that ρ is a cycle whenever (3.5) holds with $|\nu'| < n$ for some $n > |\mu'|$, and fix ν' with $|\nu'| = n$ satisfying (3.5). In particular, $|\nu'| > |\mu'|$, so $\mu' = \nu'_{[0,|\mu'|]}$ and $\mu'\nu'_{[|\mu'|,|\nu'|]} = \nu' = \nu'_{[|\mu'|,|\nu'|]}\rho$. Since $|\nu'_{[|\mu'|,|\nu'|]}| = |\nu'| - |\mu'| < n$, the inductive hypothesis now implies that ρ is a cycle.

Finally, from (3.4),

$$\begin{aligned} 0 \neq t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* &= t_\lambda t_\lambda^* t_\mu t_{\nu'} t_\lambda^* \\ &= t_\lambda (t_{\mu'\nu'}^* t_\mu^*) t_\mu t_{\nu'} t_\lambda^* = t_\lambda t_{\nu'\rho}^* t_{\nu'} t_\lambda^* = t_\lambda t_\rho^* t_{\nu'}^* t_{\nu'} t_\lambda^* = t_\lambda t_\rho^* t_\lambda^* = t_\lambda t_{\lambda\rho}^* \end{aligned}$$

as required. ∎

Given a directed graph E and $e \in E^1$, define $S_e^{\mathrm{ap}} \in \mathcal{B}(\ell^2(\partial E^{\mathrm{ap}}))$ by

$$S_e^{\mathrm{ap}}\xi_x = \begin{cases} \xi_{ex} & \text{if } s(e) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

Then with $\{P_v^{\mathrm{ap}} : v \in E^0\}$ as on page 214, $(S^{\mathrm{ap}}, P^{\mathrm{ap}})$ is a Toeplitz-Cuntz-Krieger E-family, and $S_\lambda^{\mathrm{ap}}(S_\lambda^{\mathrm{ap}})^* = P_\lambda^{\mathrm{ap}}$ for all $\lambda \in E^*$. We denote $C^*(S^{\mathrm{ap}}, P^{\mathrm{ap}})$ by $C^*_{\min}(E)$.

Proposition 3.2. *Let E be a directed graph, and suppose that every cycle in E has an entrance. Let (t, q) be a Toeplitz-Cuntz-Krieger E-family. Then $q : \lambda \mapsto t_\lambda t_\lambda^*$ is a Boolean representation of E, and there exists a conditional expectation $\Phi_{t,q} : C^*(t, q) \to \overline{\operatorname{span}}\{q_\lambda : \lambda \in E^*\}$ satisfying $\Phi_{t,q}(t_\mu t_\nu^*) = \delta_{\mu,\nu} q_\mu$. In particular we have*

$$\psi_q \circ \Phi_{t,q} \circ \pi_{t,q} = \Phi_{S^{\mathrm{ap}},P^{\mathrm{ap}}} \circ \pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}. \tag{3.6}$$

Proof. A standard argument shows that the q_λ form a Boolean representation of E.

Claim 1. Given a finite subset F of E^* and a collection $\{a_{\mu,\nu} : \mu,\nu \in F\}$ of scalars,

$$\Big\| \sum_{\mu\in F} a_{\mu,\mu} q_\mu \Big\| \leq \Big\| \sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\|.$$

By enlarging F (and setting the extra scalars equal to zero), we may assume that F is closed under initial segments in the sense that if $\mu\nu \in F$ then $\mu \in F$.

For each $\lambda \in F$, let $T_\lambda^F := \{\lambda' \in s(\lambda)E^* : \lambda\lambda' \in F, |\lambda'| > 0\}$. For each $\lambda \in F$ such that T_λ^F is not exhaustive, fix a path α^λ such that $\alpha^\lambda \neq \mu\mu'$ and $\mu \neq \alpha^\lambda\alpha'$ for all $\mu \in T_\lambda^F$. Since every cycle in E has an entrance, [3, Lemma 3.7] implies that for each v such that $v = s(\alpha^\lambda)$ for some λ, there exists $\tau^v \in vE^*$ such that either:

(1) $s(\tau^v)E^1 = \emptyset$; or
(2) $|\tau^v| > \max\{|\lambda| : \lambda \in F, T_\lambda^F \text{ is not exhaustive}\}$, and $\tau_k^v \neq \tau_{|\tau^v|}^v$ for all $k < |\tau^v|$.

We write τ^λ for $\tau^{s(\alpha^\lambda)}$.

For each $\lambda \in F$ we define

$$\phi_\lambda^F := \begin{cases} q_{\lambda\alpha^\lambda\tau^\lambda} & \text{if } T_\lambda^F \text{ is not exhaustive} \\ q_\lambda^F & \text{otherwise.} \end{cases}$$

By definition we have each $\phi_\lambda^F \leq q_\lambda^F$, and since the q_λ^F are mutually orthogonal, it follows that the ϕ_λ^F are also. Hence

$$\Big\| \sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\| \geq \Big\| \sum_{\lambda\in F} \phi_\lambda^F \Big(\sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^* \Big) \phi_\lambda^F \Big\|.$$

Fix $\lambda,\mu,\nu \in F$. We claim that

$$\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = \begin{cases} \phi_\lambda^F & \text{if } \mu = \nu \text{ and } \lambda = \mu\lambda' \text{ for some } \lambda' \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

To see this, suppose first that $\mu = \nu$. If $\lambda = \mu\lambda'$ then $\phi_\lambda^F \leq t_\lambda t_\lambda^* \leq t_\mu t_\mu^*$ by definition of ϕ_λ^F. If $\lambda \neq \mu\lambda'$, then either $\mu = \lambda\mu'$ in which case $\phi_\lambda^F \leq (t_\lambda t_\lambda^* - t_{\lambda\mu'} t_{\lambda\mu'}^*) \perp t_\mu t_\mu^*$, or else $\mu \neq \lambda\mu'$ in which case $\phi_\lambda^F \leq t_\lambda t_\lambda^* \perp t_\mu t_\mu^*$.

Now suppose that $\mu \neq \nu$; by symmetry under adjoints, we may assume that $|\mu| < |\nu|$. We must show that $\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = 0$. Since $\phi_\lambda^F \leq t_\lambda t_\lambda^*$, if $t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = 0$ then we are done, so we may assume that $t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* \neq 0$. Then Lemma 3.4 implies that $\lambda = \nu\nu' = \mu\mu'\nu'$ and that $t_\lambda t_\lambda^* t_\mu t_\nu^* t_\lambda t_\lambda^* = t_\lambda t_{\lambda\rho}^*$ for some cycle $\rho \in E$. Hence $\phi_\lambda^F \leq t_\lambda t_\lambda^*$ forces

$$\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = \phi_\lambda^F t_\lambda t_{\lambda\rho}^* \phi_\lambda^F.$$

We consider two cases: T_λ^F is exhaustive, or it is not.

First suppose that T_λ^F is exhaustive. Fix n such that $n|\rho| > \max\{|\lambda'| : \lambda' \in T_\lambda^F\}$. Then $\rho^n = \lambda'\beta$ for some $\lambda' \in T_\lambda^F$ and $\beta \in E^*$. In particular, $\lambda' = \rho^n_{[0,|\lambda'|]}$, forcing $\rho_1 = \lambda'_1$. Since $\lambda\rho_1\lambda'_{[1,|\lambda'|]} = \lambda\lambda' \in F$ and F is closed under initial segments, $\lambda\rho_1 \in F$ and then

$$t_{\lambda\rho} t_{\lambda\rho}^* \phi_\lambda^F \leq t_{\lambda\rho}^* t_{\lambda\rho}^* (t_\lambda t_\lambda^* - t_{\lambda\rho_1} t_{\lambda\rho_1}^*) = 0,$$

giving $\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = 0$.

Now suppose that T_λ^F is not exhaustive. Then $\phi_\lambda^F = q_{\lambda\alpha^\lambda\tau^\lambda}$. We then have

$$\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = t_{\lambda\alpha^\lambda\tau^\lambda} t^*_{\mu'\nu'\alpha^\lambda\tau^\lambda} t_{\nu'\alpha^\lambda\tau^\lambda} t^*_{\lambda\alpha^\lambda\tau^\lambda}.$$

This is nonzero only if $\mu'\nu'\alpha^\lambda\tau^\lambda = \nu'\alpha^\lambda\tau^\lambda\zeta$ for some ζ. This is impossible if $s(\tau^\lambda)E^1 = \emptyset$, so suppose that $s(\tau^\lambda)E^1 \neq \emptyset$. By choice of τ^λ, we have $|\tau^\lambda| > |\mu'|$. Let $m = |\nu'\alpha^\lambda\tau^\lambda|$. Then

$$(\mu'\nu'\alpha^\lambda\tau^\lambda)_m = \tau^\lambda_{|\tau^\lambda|-|\mu'|} \neq \tau^\lambda_{|\tau^\lambda|} = (\nu'\alpha^\lambda\tau^\lambda)_m,$$

so $\mu'\nu'\alpha^\lambda\tau^\lambda \neq \nu'\alpha^\lambda\tau^\lambda\zeta$ for all ζ, and hence $\phi_\lambda^F t_\mu t_\nu^* \phi_\lambda^F = 0$, establishing (3.7).

We now have

$$\begin{aligned}\Big\|\sum_{\mu\in F} a_{\mu,\mu} q_\mu\Big\| &= \Big\|\sum_{\mu\in F}\Big(\sum_{n\le|\mu|} a_{\mu_{[0,n]},\mu_{[0,n]}}\Big) q_\mu^F\Big\| \\ &= \max_{\mu\in F}\Big|\sum_{n\le|\mu|} a_{\mu_{[0,n]},\mu_{[0,n]}}\Big| \\ &= \Big\|\sum_{\mu\in F}\Big(\sum_{n\le|\mu|} a_{\mu_{[0,n]},\mu_{[0,n]}}\Big) \phi_\mu^F\Big\| \\ &= \Big\|\sum_{\lambda\in F}\phi_\lambda^F\Big(\sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^*\Big)\phi_\lambda^F\Big\| \qquad \text{by (3.7)} \\ &\le \Big\|\sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^*\Big\|\end{aligned}$$

completing the proof of Claim 1.

Claim 1 implies that the formula $t_\mu t_\nu^* \mapsto \delta_{\mu,\nu} t_\mu t_\mu^*$ extends to a well-defined linear map $\Phi_{t,q}$ from $C^*(t,q)$ to $\overline{\operatorname{span}}\{q_\lambda : \lambda \in E^*\}$. This $\Phi_{t,q}$ is a linear idempotent of norm 1, and hence a conditional expectation (see for example [1, Definition II.6.10.2 and Theorem II.6.10.2]).

The final statement is straightforward to check since the two maps in question agree on spanning elements. ∎

Lemma 3.5. *Let E be a directed graph in which every cycle has an entrance. Then the expectation $\Phi_{S^{\mathrm{ap}},P^{\mathrm{ap}}} : C^*_{\min}(E) \to \overline{\operatorname{span}}\{P^{\mathrm{ap}}_\lambda : \lambda \in E^*\}$ obtained from Proposition 3.2 is faithful on positive elements.*

Proof. It suffices to show that for $a \in C^*_{\min}(E)$, $\Phi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}(a)$ is equal to the strong-operator limit $\sum_{x\in\partial E^{\mathrm{ap}}}(a\xi_x|\xi_x)\theta_{\xi_x,\xi_x}$ for all $a \in C^*_{\min}(E)$, where the ξ_x are the canonical orthonormal basis for $\ell^2(\partial E^{\mathrm{ap}})$ and θ_{ξ_x,ξ_x} is the rank-one projection onto $\mathbb{C}\xi_x$. Fix $\mu,\nu \in E^*$ and $x \in \partial E^{\mathrm{ap}}$. We have

$$\big(S^{\mathrm{ap}}_\mu (S^{\mathrm{ap}}_\nu)^*\xi_x\big|\xi_x\big) = \big((S^{\mathrm{ap}}_\nu)^*\xi_x\big|(S^{\mathrm{ap}}_\mu)^*\xi_x\big) = \begin{cases}1 & \text{if } x=\nu y=\mu y \text{ for some } y\\ 0 & \text{otherwise.}\end{cases}$$

Since $x \in \partial E^{\mathrm{ap}}$, we have $y \neq \rho^\infty$ for any cycle ρ, so $\mu y = \nu y$ forces $|\mu| = |\nu|$, and hence $\mu = \nu$.

Hence

$$\sum_{x\in\partial E^{\mathrm{ap}}} \big(S^{\mathrm{ap}}_\mu (S^{\mathrm{ap}}_\nu)^*\xi_x|\xi_x\big)\theta_{\xi_x,\xi_x} = \delta_{\mu,\nu}\,\mathrm{proj}_{\overline{\mathrm{span}}\{\xi_x : x\in\mu\partial E^{\mathrm{ap}}\}} = \delta_{\mu,\nu}P^{\mathrm{ap}}_\mu$$

as required. ∎

We now have the tools we need to prove the main theorem.

Proof of Theorem 3.1. We showed that the P^{ap}_λ, and in particular the P^{ap}_v are nonzero immediately subsequent to their definition.

Fix a Toeplitz-Cuntz-Krieger E-family (t,q) with each q_v nonzero. We will show that $\ker(\pi_{t,q}) \subset \ker(\pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}})$ in $\mathcal{T}C^*(E)$.

Since each $t^*_\lambda t_\lambda = q_{s(\lambda)}$, each q_λ is nonzero, so Proposition 3.1 implies that there is a homomorphism $\psi_q : \overline{\mathrm{span}}\{q_\lambda : \lambda\in E^*\} \to \overline{\mathrm{span}}\{P^{\mathrm{ap}}_\lambda : \lambda \in E^*\}$ taking each q_λ to P^{ap}_λ.

We calculate

$$\begin{aligned}
\pi_{t,q}(a) = 0 &\iff \pi_{t,q}(a^*a) = 0\\
&\implies \psi_q\circ\Phi_{t,q}\circ\pi_{t,q}(a^*a) = 0 \qquad (3.8)\\
&\iff \Phi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}(\pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}(a^*a)) = 0 \quad \text{by (3.6)}\\
&\iff \pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}(a^*a) = 0 \quad \text{by Lemma 3.5}\\
&\iff \pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}(a) = 0.
\end{aligned}$$

Thus $\ker(\pi_{t,q}) \subset \ker(\pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}})$ as claimed, and it follows that $\pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}}$ descends to the desired homomorphism $\psi_{t,q} : C^*(t,q) \to C^*_{\min}(E)$.

For the final statement of the theorem, observe that if A is a C^*-algebra and (t,q) a generating Toeplitz-Cuntz-Krieger E-family in A with the same co-universal property, then the co-universal property of A yields an inverse for the homomorphism $\psi_{t,q} : A \to C^*_{\min}(E)$ obtained from the co-universal property of E. ∎

It is, of course, interesting to know when the homomorphism $\psi_{t,q}$ of Theorem 3.1 is injective.

Theorem 3.2. *Let E be a directed graph in which every cycle has an entrance.*

(1) *If (t,q) is a Toeplitz-Cuntz-Krieger family with each q_v nonzero, then the homomorphism $\psi_{t,q}$ of Theorem 3.1 is injective if and only if*
   (a) *$\prod_{\lambda\in F}(q_v - q_\lambda) = 0$ whenever $v\in E^0$ and $F\subset vE^*$ is finite exhaustive; and*
   (b) *the expectation $\Phi_{t,q}$ is faithful.*

(2) *If π is homomorphism from $C^*_{\min}(E)$ to a C^*-algebra C such that each $\pi_{P^{\mathrm{ap}}_v}$ is nonzero, then π is injective.*

Proof. (1) Since conditions (a) and (b) hold in $C^*_{\min}(E)$, the “only if” implication is trivial. For the “if” implication, note that given a Toeplitz-Cuntz-Krieger E-family (t,q), we have $\ker(\pi_{t,q}) = \ker(\pi_{S^{\mathrm{ap}},P^{\mathrm{ap}}})$ whenever the implication (3.8) is equivalence. Condition (a) implies that ψ_q is faithful by the final statement of Proposition 3.1, and this combined with (b) implies that $\psi_q\circ\Phi_{t,q}$ is faithful on positive elements, giving

$$\pi_{t,q}(a^*a) = 0 \iff \psi_q\circ\Phi_{t,q}\circ\pi_{t,q}(a^*a) = 0$$

as required.

(2) Define a Toeplitz-Cuntz-Krieger E-family by $t_e := \pi(S_e^{\mathrm{ap}})$ and $q_v := \pi(P_v^{\mathrm{ap}})$. Theorem 3.1 supplies a homomorphism $\psi_{t,q} : C^*(t,q) \to C^*_{\min}(E)$ which is an inverse for π. ∎

Acknowledgement. This research was supported by the Australian Research Council.

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