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On Graph C^*-Algebras with a Linear Ideal Lattice

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Abstract. At the cost of restricting the nature of the involved K-groups, we prove a classification result for a hitherto unexplored class of graph C^*-algebras, allowing us to classify all graph C^*-algebras on finitely many vertices with a finite linear ideal lattice if all pair of vertices are connected by infinitely many edges when they are connected at all.

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1. Introduction

In a series of papers [8, 9, 5, 7] reported on at the ACM, much insight has been obtained about the classification theory of graph C^*-algebras with finitely many ideals, leading to classification results for a lot of special cases by a computable invariant. Outside of the simple case and the case with one non-trivial ideal (resolved in [10, 13, 9]) several open questions remain, but we are beginning to believe in the following:

Working conjecture 1.1. Graph C^*-algebras $C^*(E)$ with finitely many ideals are classified up to stable isomorphism by their filtrated, ordered K-theory $FK^+_{\mathsf{X}}(C^*(E))$.

Here, the filtrated, ordered K-theory over $\mathsf{X} = \operatorname{Prim}(C^*(E))$ is simply the collection of all K_0- and K_1-groups of subquotients of the C^*-algebra in question, taking into account all the natural transformations among them, and the order on each K_0-group. Details will be given below.

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It is the purpose of this note to give further evidence for the validity of this conjecture in a class complementing our earlier work under some, unfortunately, rather draconian assumptions about the nature of the involved K-groups.

As in [7] and [16] we focus on the case of a linear ideal lattice of arbitrary length:

Definition 1.1. *For each n the class $\mathcal{LG}[n]$ is the family of graph C^*-algebras with a linear ideal lattice of length n;*

$$0 = \mathfrak{I}_0 \lhd \mathfrak{I}_1 \lhd \cdots \lhd \mathfrak{I}_n = C^*(E)$$

with each $\mathfrak{I}_{k-1}$ being the largest proper ideal of $\mathfrak{I}_k$.

The key technical concepts of importance when studying classification in $\mathcal{LG}[n]$ are

- Kirchberg's isomorphism result [13]
- The corona factorization property [14]
- The notion of full extensions [11]
- The Universal Coefficient Theorem of Meyer and Nest [16]
- Rørdam's calculus of extensions [18] (refined in [8])

In [7] we consider the following subclass of graph C^*-algebras with "separated" finite and infinite parts:

Definition 1.2. *We define for each n the class $\mathcal{LSG}[n] \subseteq \mathcal{LG}[n]$ consisting of graph C^*-algebras such that there exists a $k \in \{0, \ldots, n\}$ with the property that $\mathfrak{I}_k$ is an AF algebra or $\mathcal{O}_\infty$-absorbing and $C^*(E)/\mathfrak{I}_k$ is an AF algebra or $\mathcal{O}_\infty$-absorbing.*

The main result of [7] is:

Theorem 1.1. *Let E_1 and E_2 be graphs such that $C^*(E_1)$ and $C^*(E_2)$ are in $\mathcal{LSG}[n]$. Then the following are equivalent:*

(1) $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$,
(2) $\mathrm{FK}^+_{\mathsf{X}_n}(C^*(E_1)) \cong \mathrm{FK}^+_{\mathsf{X}_n}(C^*(E_2))$.

Here and in the following X_n denotes the primitive ideal space of any element of $\mathcal{LG}[n]$, i.e. the set $\{1, 2, \ldots, n\}$ equipped with the Alexandrov topology, where the open subsets are

$$[a, n] = \{x \in X \,:\, a \leq x \leq n\}\,.$$

We describe the invariant more carefully below.

The separation which we are assuming in [7] is automatic when we work with $\mathcal{L}G[1]$ and $\mathcal{L}G[2]$, but becomes an unwanted assumption already in $\mathcal{L}G[3]$. We believe it may be removed, but this will require the development of entirely new methods and is not within immediate reach. In the present paper, at the cost of restricting the nature of the involved K-groups, we shall demonstrate that classification is still possible in certain situations where there is no such separation. This will allow us to classify all graph C^*-algebras on finitely many vertices with a finite linear ideal lattice if all pairs of vertices are connected by infinitely many edges if they are connected at all. We believe that this lends credibility to our working conjecture 1.1.

2. Notation and conventions

To keep this note short we refer to [8] and [7] for details of notation and only present the key concepts here.

The work of the second named author [17] on the classification of non-simple Cuntz-Krieger C^*-algebras showed the importance of the so-called K-web considered in symbolic dynamics [2] for the problem at hand. More recently, Meyer and Nest [16] have provided an extremely useful algebraic context for studying invariants of this type, and pointed out several complications with regards to obtaining universal coefficient theorems and determining all natural transformations of relevance to the invariant. We believe – but are far from able to prove – that the particular algebraic nature of the K-groups associated to graph C^*-algebras implies that these complications are not relevant in our setting. Hence we conjecture that the following invariant suffices:

Definition 2.1. *Let $\mathfrak{A}$ be a C^*-algebra with finitely many ideals and set $\mathsf{X} = \mathrm{Prim}(\mathfrak{A})$. Note that for any three ideals $\mathfrak{I} \trianglelefteq \mathfrak{J} \trianglelefteq \mathfrak{K} \trianglelefteq C^*(E)$, we have a six term exact sequence*

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \xrightarrow{\iota_*} & K_0(\mathfrak{K}/\mathfrak{I}) & \xrightarrow{\pi_*} & K_0(\mathfrak{K}/\mathfrak{J}) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(\mathfrak{K}/\mathfrak{J}) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{K}/\mathfrak{I}) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

The filtrated, ordered K-theory $\mathrm{FK}^+_{\mathsf{X}}(\mathfrak{A})$ of $\mathfrak{A}$ is the collection of all K-groups thus occurring, equipped with order on K_0 and the natural transformations $\{\iota_*, \pi_*, \partial\}$.

Consequently, if also $\mathrm{Prim}(\mathfrak{B}) = \mathsf{X}$, leading to a pairing $\mathfrak{I} \mapsto \mathfrak{I}'$ of the ideals of $\mathfrak{A}$ and $\mathfrak{B}$, we will say that $\mathrm{FK}^+_{\mathsf{X}}(\mathfrak{A}) \cong \mathrm{FK}^+_{\mathsf{X}}(\mathfrak{B})$ if there exist group isomorphisms

$$\alpha_*^{\mathfrak{K},\mathfrak{J}} : K_*(\mathfrak{K}/\mathfrak{J}) \to K_*(\mathfrak{K}'/\mathfrak{J}')$$

preserving all natural transformations in such a way that all $\alpha_0^{\mathfrak{K},\mathfrak{J}}$ are also order isomorphisms.

All components of this invariant are readily computable [5], and often, much of it is redundant, as is certainly the case in the setting we shall focus on below. Indeed, in the case we shall study here (because of the draconian assumption mentioned above) none of the connecting maps carry information, and only the K_0-groups of the simple subquotients are necessary. In other cases we have studied, various other components of this invariant are necessary, and we do not think that a large reduction of the invariant is possible in full generality.

We define concepts of fullness for essential extensions of C^*-algebras

$$\mathfrak{e}: \ 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$$

as follows:

Definition 2.2. *We say that x is* norm-full *in a C^*-algebra $\mathfrak{X}$ if x is not contained in any norm-closed proper ideal of $\mathfrak{X}$.*

An extension $\mathfrak{e}$ is said to be full *if the associated Busby invariant $\tau_{\mathfrak{e}}$ has the property that $\tau_{\mathfrak{e}}(a)$ is a norm-full element of $M(\mathfrak{B})/\mathfrak{B}$ for every $a \in \mathfrak{A}\backslash\{0\}$.*

Let $\mathfrak{X}$ be a separable stable C^*-algebra. Then $\mathfrak{X}$ is said to have the *corona factorization property* [14] if every norm-full projection in $M(\mathfrak{X})$ is Murray-von Neumann equivalent to $1_{M(\mathfrak{X})}$. Kucerovsky and Ng proved that for C^*-algebras satisfying the corona factorization property, any full and essential extension is absorbing in the sense of [4], cf. [1, §15.12]

We observe the following in [7]:

Proposition 2.1. *Assume that every stabilized simple subquotient of a C^*-algebra $\mathfrak{A}$ with finitely many ideals has the corona factorization property. Then also $\mathfrak{A} \otimes \mathbb{K}$ has the corona factorization property.*

With this in hand, the following is easy to see:

Observation 2.1. Let E be a graph such that $C^*(E)$ has finitely many ideals and assume that $\mathfrak{I} \lhd \mathfrak{J} \unlhd C^*(E)$ are ideals. Then

(i) $(\mathfrak{J}/\mathfrak{I}) \otimes \mathbb{K}$ is of the form $C^*(F) \otimes \mathbb{K}$ for some graph F;
(ii) $(\mathfrak{J}/\mathfrak{I}) \otimes \mathbb{K}$ has the corona factorization property when $\mathfrak{J}/\mathfrak{I}$ is simple;
(iii) $C^*(E) \otimes \mathbb{K}$ has the corona factorization property; and
(iv) $(\mathfrak{J}/\mathfrak{I}) \otimes \mathbb{K}$ always has the corona factorization property.

F can be chosen as a subgraph of the Drinen-Tomforde desingularization of E [6].

3. Proving fullness of extensions

We quote the following key result from [7]:

Proposition 3.1. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{I}$ and $\mathfrak{D}$ be ideals of $\mathfrak{A}$. Suppose $\mathfrak{D}/\mathfrak{I}$ is an essential ideal of $\mathfrak{A}/\mathfrak{I}$. Then*

$$\mathfrak{e}_1 : \ 0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$$

is a full extension if and only if

$$\mathfrak{e}_2 : \ 0 \to \mathfrak{I} \to \mathfrak{D} \to \mathfrak{D}/\mathfrak{I} \to 0$$

is a full extension.

With this in hand we prove our key technical result, allowing us to conclude fullness of certain extensions.

Proposition 3.2. *Let $\mathfrak{A}$ be a C^*-algebra with a norm-full projection p in $\mathfrak{A}$ and let $\mathfrak{I} \lhd \mathfrak{D} \lhd \mathfrak{A}$ be ideals such that $\mathfrak{D}/\mathfrak{I}$ is an essential ideal of $\mathfrak{A}/\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{D}$ is a simple C^*-algebra.*

Suppose $\mathfrak{I} \otimes \mathbb{K}$ and $\mathfrak{D} \otimes \mathbb{K}$ satisfy the corona factorization property and suppose

$$\mathfrak{e}_1 : 0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{D} \otimes \mathbb{K} \to \mathfrak{D}/\mathfrak{I} \otimes \mathbb{K} \to 0$$
$$\mathfrak{e}_2 : 0 \to \mathfrak{D}/\mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{D} \otimes \mathbb{K} \to 0$$

are both full extensions. Then

$$\mathfrak{e} : 0 \to \mathfrak{D} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{D} \otimes \mathbb{K} \to 0$$

is a full extension.

Proof. By Proposition 3.1, we have that

$$\mathfrak{e}_3 \ : \ 0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I} \otimes \mathbb{K} \to 0$$

is a full extension.

Note that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathfrak{I} \otimes \mathbb{K} & = & \mathfrak{I} \otimes \mathbb{K} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathfrak{D} \otimes \mathbb{K} & \to & \mathfrak{A} \otimes \mathbb{K} & \to & \mathfrak{A}/\mathfrak{D} \otimes \mathbb{K} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathfrak{D}/\mathfrak{I} \otimes \mathbb{K} & \to & \mathfrak{A}/\mathfrak{I} \otimes \mathbb{K} & \to & \mathfrak{A}/\mathfrak{D} \otimes \mathbb{K} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

is commutative. Let $\{e_{ij}\}$ be a system of matrix units of $\mathbb{K}$. Since $q = p \otimes e_{11}$ is norm-full in $\mathfrak{A} \otimes \mathbb{K}$, the image of q in $\mathfrak{A}/\mathfrak{D} \otimes \mathbb{K}$ is norm-full in $\mathfrak{A}/\mathfrak{D} \otimes \mathbb{K}$. To show that $\mathfrak{e}$ is a full extension it is enough to show that q is norm-full in $M(\mathfrak{D} \otimes \mathbb{K})$.

Let e be the image of q in $\mathfrak{A}/\mathfrak{I} \otimes \mathbb{K}$. Since $\mathfrak{e}_2$ is a full extension, we have that e is norm-full in $M(\mathfrak{D}/\mathfrak{I} \otimes \mathbb{K})$. Hence, e is Murray-von Neumann equivalent to $1_{M(\mathfrak{D}/\mathfrak{I}\otimes\mathbb{K})}$, since by Theorem 3.1 (2) of [15], $\mathfrak{D}/\mathfrak{I} \otimes \mathbb{K}$ has the corona factorization property. So, $e(\mathfrak{D}/\mathfrak{I} \otimes \mathbb{K})e$ is a stable C^*-algebra.

Since $\mathfrak{e}_3$ is a full extension, q is a norm-full projection in $M(\mathfrak{I}\otimes\mathbb{K})$. Since $\mathfrak{I}\otimes\mathbb{K}$ has the corona factorization property, q is Murray-von Neumann equivalent to $1_{M(\mathfrak{I}\otimes\mathbb{K})}$. Hence, $q(\mathfrak{I} \otimes \mathbb{K})q$ is a stable C^*-algebra which is isomorphic to $\mathfrak{I} \otimes \mathbb{K}$. By Theorem 2.4 of [15], $q(\mathfrak{D}\otimes\mathbb{K})q$ is a stable C^*-algebra. Since $q(\mathfrak{D}\otimes\mathbb{K})q$ is norm-full in $\mathfrak{D}\otimes\mathbb{K}$, by Theorem 4.23 of [3], q is Murray-von Neumann equivalent to $1_{M(\mathfrak{D}\otimes\mathbb{K})}$. In particular, q is norm-full in $M(\mathfrak{D} \otimes \mathbb{K})$. ∎

The following lemma was essentially proven in [9].

Lemma 3.1. [9, 4.5] *Let E be a graph such that $C^*(E)$ has exactly one non-trivial ideal $\mathfrak{I}$. Suppose $\mathfrak{I}$ and $C^*(E)/\mathfrak{I}$ are not both AF algebras. Then the extension*

$$\mathfrak{e} \ : \ 0 \to \mathfrak{I} \otimes \mathbb{K} \to C^*(E) \otimes \mathbb{K} \to C^*(E)/\mathfrak{I} \otimes \mathbb{K} \to 0$$

is a full extension.

Theorem 3.1. *Let E be a graph such that $\mathfrak{A} = C^*(E) \in \mathcal{LG}[n]$ for $n \geq 2$. Suppose that for any $k \in \{1, \ldots, n-1\}$ if $\mathfrak{I}_k/\mathfrak{I}_{k-1}$ and $\mathfrak{I}_{k+1}/\mathfrak{I}_k$ are both AF algebras, then $\mathfrak{I}_k/\mathfrak{I}_{k-1} = \mathbb{K}$. Then for any $\ell \in \{1, \ldots, n-1\}$*

$$0 \to \mathfrak{I}_\ell \otimes \mathbb{K} \to \mathfrak{I}_{\ell+1} \otimes \mathbb{K} \to (\mathfrak{I}_{\ell+1}/\mathfrak{I}_\ell) \otimes \mathbb{K} \to 0$$

is a full extension.

Proof. We prove the claim by induction on n. For $n = 2$ the conclusion follows from Lemma 3.1 unless we are in the case when both ideal and quotient are AF. In that case, we use that since the ideal is $\mathbb{K}$, the corona algebra is simple and hence any essential extension is full.

Let $n > 2$. By Observation 2.1(i), $\mathfrak{I}_{\ell+1}$ is a graph C^*-algebra and as above,

$$0 \to \mathfrak{I}_\ell \otimes \mathbb{K} \to \mathfrak{I}_{\ell+1} \otimes \mathbb{K} \to \mathfrak{I}_{\ell+1}/\mathfrak{I}_\ell \otimes \mathbb{K} \to 0$$

the induction assumption and Lemma 3.1 combine to establish that is full for any $\ell \in \{1, \dots, n-2\}$. We need to prove fullness at $\ell = n - 1$ as well.

We have seen that

$$0 \to \mathfrak{I}_{n-2} \otimes \mathbb{K} \to \mathfrak{I}_{n-1} \otimes \mathbb{K} \to \mathfrak{I}_{n-1}/\mathfrak{I}_{n-2} \otimes \mathbb{K} \to 0 \tag{3.1}$$

is a full extension. Since also $\mathfrak{A}/\mathfrak{I}_{n-2} \otimes \mathbb{K}$ is a graph C^*-algebra according to Observation 2.1(i), as in the case $n = 2$,

$$0 \to \mathfrak{I}_{n-1}/\mathfrak{I}_{n-2} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I}_{n-2} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I}_{n-1} \otimes \mathbb{K} \to 0 \tag{3.2}$$

is a full extension. Since the C^*-algebras considered here have real rank zero (cf. [12]), the fact that there is a largest proper ideal easily gives the existence of a full projection, so we may apply Proposition 3.2 to the sequences (3.1) and (3.2) above to deduce that

$$0 \to \mathfrak{I}_{n-1} \otimes \mathbb{K} \to \mathfrak{I}_n \otimes \mathbb{K} \to \mathfrak{I}_n/\mathfrak{I}_{n-1} \otimes \mathbb{K} \to 0$$

is a full extension. ∎

4. Classification

We cite from [7]:

Lemma 4.1. *Let*

$$\mathfrak{e}_1 : \ 0 \to \mathfrak{I}_1 \to \mathfrak{E}_1 \to \mathfrak{A}_1 \to 0$$
$$\mathfrak{e}_2 : \ 0 \to \mathfrak{I}_2 \to \mathfrak{E}_2 \to \mathfrak{A}_2 \to 0$$

be non-unital full essential extensions. Suppose $\mathfrak{I}_i$ is a stable C^*-algebra satisfying the corona factorization property. If there exists an isomorphism $\phi_0 : \mathfrak{I}_1 \to \mathfrak{I}_2$ and an isomorphism $\phi_2 : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $KK(\phi_2) \times [\tau_{\mathfrak{e}_2}] = [\tau_{\mathfrak{e}_1}] \times KK(\phi_0)$, then $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$.

Theorem 4.1. *Let $\mathfrak{A}^1$ and $\mathfrak{A}^2$ be separable nuclear C^*-algebras with linear ideal structures*

$$0 = \mathfrak{I}_0^i \lhd \mathfrak{I}_1^i \lhd \cdots \lhd \mathfrak{I}_n^i = \mathfrak{A}^i$$

for $i \in \{1, 2\}$, respectively. Suppose that for all $i \in \{1, 2\}$, $k \in \{1, \dots, n\}$ and all $\ell \in \{1, \dots, n-1\}$, we have

1. $K_0(\mathfrak{I}_k^i/\mathfrak{I}_{k-1}^i)$ *is a free group*
2. $K_1(\mathfrak{I}_k^i/\mathfrak{I}_{k-1}^i) = 0$
3. $\mathfrak{I}_k^i/\mathfrak{I}_{k-1}^i$ *is a purely infinite simple C^*-algebra satisfying the UCT, or a simple AF algebra*

(4)
$$0 \to \mathfrak{I}^i_\ell \otimes \mathbb{K} \to \mathfrak{I}^i_{\ell+1} \otimes \mathbb{K} \to (\mathfrak{I}^i_{\ell+1}/\mathfrak{I}^i_\ell) \otimes \mathbb{K} \to 0$$
is a full extension.

Then $\mathfrak{A}^1 \otimes \mathbb{K} \cong \mathfrak{A}^2 \otimes \mathbb{K}$ *if and only if there exist order isomorphisms*
$$\alpha_k : K_*(\mathfrak{I}^1_k/\mathfrak{I}^1_{k-1}) \to K_*(\mathfrak{I}^2_k/\mathfrak{I}^2_{k-1})$$
for each $k \in \{1, \ldots, n\}$.

Proof. As noted in Proposition 2.1 and by Theorem 2.4 of [15], $\mathfrak{I}^i_k \otimes \mathbb{K}$ satisfies the corona factorization property because all stable purely infinite algebras and simple AF algebras do. It is clear that if $\mathfrak{A}^1 \otimes \mathbb{K} \cong \mathfrak{A}^2 \otimes \mathbb{K}$, then there exist order isomorphisms
$$\alpha_k : K_*(\mathfrak{I}^1_k/\mathfrak{I}^1_{k-1}) \to K_*(\mathfrak{I}^2_k/\mathfrak{I}^2_{k-1})$$
for each $k \in \{1, \ldots, n\}$.

We now proceed by induction on n. If $n = 1$, then $\mathfrak{A}^1$ and $\mathfrak{A}^2$ are simple C^*-algebras. The conclusion of the theorem now follows from the classification of AF algebras or the Kirchberg-Phillips classification result. For other n, suppose $\mathfrak{A}^1$ and $\mathfrak{A}^2$ satisfy the assumptions (1)–(4) and that there exist order isomorphisms
$$\alpha_k : K_*(\mathfrak{I}^1_k/\mathfrak{I}^1_{k-1}) \to K_*(\mathfrak{I}^2_k/\mathfrak{I}^2_{k-1})$$
for each $k \in \{1, \ldots, n\}$.

Consider the extensions
$$\mathfrak{e}_i \;:\; 0 \to \mathfrak{I}^i_{n-1} \otimes \mathbb{K} \to \mathfrak{A}^i \otimes \mathbb{K} \to (\mathfrak{A}^i/\mathfrak{I}^i_{n-1}) \otimes \mathbb{K} \to 0$$
Since $\mathfrak{I}^i_{n-1}$ satisfies the assumptions of the theorem at $n-1$, from the induction hypothesis, there exists an isomorphism $\psi : \mathfrak{I}^1_{n-1} \otimes \mathbb{K} \to \mathfrak{I}^2_{n-1} \otimes \mathbb{K}$. Since there exists an order isomorphism $K_*(\mathfrak{A}^1/\mathfrak{I}^1_{n-1})$ and $K_*(\mathfrak{A}^2/\mathfrak{I}^2_{n-1})$, there exists an isomorphism $\phi : (\mathfrak{A}^1/\mathfrak{I}^1_{n-1}) \otimes \mathbb{K} \to (\mathfrak{A}^2/\mathfrak{I}^2_{n-1}) \otimes \mathbb{K}$. Since $K_0(\mathfrak{A}^1/\mathfrak{I}^1_{n-1})$ is free, $K_1(\mathfrak{A}^1/\mathfrak{I}^1_{n-1}) = 0$, and $K_1(\mathfrak{I}^2_{n-1}) = 0$, we have that $KK^1(\mathfrak{A}^1/\mathfrak{I}^1_{n-1}, \mathfrak{I}^2_{n-1}) = 0$ by the universal coefficient theorem, since
$$\mathrm{Ext}(K_*(\mathfrak{A}^1/\mathfrak{I}^1_{n-1}), K_*(\mathfrak{I}^2_{n-1})) = 0, \quad \mathrm{Hom}(K_*(\mathfrak{A}^1/\mathfrak{I}^1_{n-1}), K_{*+1}(\mathfrak{I}^2_{n-1})) = 0.$$
Thus, $KK(\psi) \times [\tau_{\mathfrak{e}_1}] = [\tau_{\mathfrak{e}_2}] \times KK(\phi)$ in $KK^1(\mathfrak{A}^1/\mathfrak{I}^1_{n-1}, \mathfrak{I}^2_{n-1})$. Thus, by Lemma 4.1, $\mathfrak{A}^1 \otimes \mathbb{K} \cong \mathfrak{A}^2 \otimes \mathbb{K}$. ∎

Corollary 4.1. *Let* $\mathfrak{A}^1$ *and* $\mathfrak{A}^2$ *be graph* C^**-algebras in* $\mathcal{LG}[n]$. *Suppose, for* $i \in \{1, 2\}$, *that* $\mathfrak{I}^i_k/\mathfrak{I}^i_{k-1}$ *and* $\mathfrak{I}^i_{k+1}/\mathfrak{I}^i_k$ *are only both* AF *algebras in the case when*
$$\mathfrak{I}^i_k/\mathfrak{I}^i_{k-1} = \mathbb{K}$$
and suppose $\mathfrak{I}^i_k/\mathfrak{I}^i_{k-1}$ *have free* K_0*-group and trivial* K_1*-group for any* k. *Then* $\mathfrak{A}^1 \otimes \mathbb{K} \cong \mathfrak{A}^2 \otimes \mathbb{K}$ *if and only if there exist order isomorphisms*
$$\alpha_k : K_*(\mathfrak{I}^1_k/\mathfrak{I}^1_{k-1}) \to K_*(\mathfrak{I}^2_k/\mathfrak{I}^2_{k-1})$$
for each k.

Proof. The corollary follows from Theorem 3.1 combined with Theorem 4.1. ∎

Example 4.1. Any graph C^*-algebra in $\mathcal{LG}[n]$ given by a finite adjacency matrix in which all entries are either 0 or ∞ falls in the class covered by the corollary above. Indeed, each stable, simple subquotient will be isomorphic to one of the C^*-algebras

$$\mathbb{K}, \mathcal{O}_\infty \otimes \mathbb{K}, \mathcal{O}_{\infty^2} \otimes \mathbb{K}, \mathcal{O}_{\infty^3} \otimes \mathbb{K}, \ldots$$

where $\mathcal{O}_{\infty^k} \otimes \mathbb{K}$ is the stable Kirchberg algebra with

$$K_0(\mathcal{O}_{\infty^k} \otimes \mathbb{K}) = \mathbb{Z}^k, \quad K_1(\mathcal{O}_{\infty^k} \otimes \mathbb{K}) = 0.$$

For instance, the matrices

$$\begin{bmatrix} 0 & \infty & 0 & 0 & 0 \\ \infty & 0 & \infty & 0 & 0 \\ 0 & 0 & 0 & \infty & 0 \\ 0 & 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & \infty \end{bmatrix}$$

and

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & 0 & \infty & \infty \\ 0 & 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & \infty \end{bmatrix}$$

give rise to stably isomorphic C^*-algebras in $\mathcal{L}G[4]\backslash\mathcal{L}SG[4]$.

References

[1] B. Blackadar, *K-Theory for Operator Algebras*, Second edition, Cambridge Univ. Press, Cambridge, 1998.

[2] M. Boyle and D. Huang, Poset block equivalence of integral matrices, *Trans. Amer. Math. Soc.* **355** (2003), no. 10, 3861–3886 (electronic).

[3] L. G. Brown, Semicontinuity and multipliers of C^*-algebras, *Canad. J. Math.* **40** (1988), no. 4, 865–988.

[4] L. G. Brown, R. G. Douglas and P. A. Fillmore, Extensions of C^*-algebras and K-homology, *Ann. of Math.* (2) **105** (1977), no. 2, 265–324.

[5] T. Carlsen, S. Eilers and M. Tomforde, *Index maps in the K-theory of graph algebras*, Preprint, 2010.

[6] D. Drinen and M. Tomforde, The C^*-algebras of arbitrary graphs, *Rocky Mountain J. Math.* **35** (2005), no. 1, 105–135.

[7] S. Eilers, G. Restorff and E. Ruiz, *Classifying C^*-algebras with both finite and infinite subquotients*, In preparation.

[8] S. Eilers, G. Restorff and E. Ruiz, Classification of extensions of classifiable C^*-algebras, *Adv. Math.* **222** (2009), no. 6, 2153–2172.

[9] S. Eilers and M. Tomforde, On the classification of nonsimple graph algebras, *Math. Ann.*, **346** (2010), pp. 393–418.

[10] G. A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra* **38** (1976), no. 1, 29–44.

[11] G. A. Elliott and D. Kucerovsky, An abstract Voiculescu-Brown-Douglas-Fillmore absorption theorem, *Pacific J. Math.* **198** (2001), no. 2, 385–409.

[12] J. H. Hong and W. Szymański, Purely infinite Cuntz-Krieger algebras of directed graphs, *Bull. London Math. Soc.* **35** (2003), no. 5, 689–696.

[13] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, Preprint, third draft, 1994.

[14] D. Kucerovsky and P. W. Ng, The corona factorization property and approximate unitary equivalence, *Houston J. Math.* **32** (2006), no. 2, 531–550 (electronic).

[15] D. Kucerovsky and P. W. Ng, S-regularity and the corona factorization property, *Math. Scand.* **99** (2006), no. 2, 204–216.

[16] R. Meyer and R. Nest, *C^*-algebras over topological spaces: Filtrated K-theory*, Preprint, arXiv:0810.0096v2.

[17] G. Restorff, Classification of Cuntz-Krieger algebras up to stable isomorphism, *J. Reine Angew. Math.* **598** (2006), 185–210.

[18] M. Rørdam, Classification of extensions of certain C^*-algebras by their six term exact sequences in K-theory, *Math. Ann.* **308** (1997), no. 1, 93–117.