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Weak Potency of Fundamental Groups of Graphs of Groups

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Abstract. In this note we shall prove a characterization for the fundamental group of a graph of polycyclic-by-finite groups and free-by-finite groups with infinite cyclic edge subgroups to be weakly potent. We shall show also that all one-relator groups with non-trivial center are weakly potent.

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1. Introduction

In this note, we shall prove a characterization for the fundamental groups of graphs of polycyclic-by-finite groups and free-by-finite groups with infinite cyclic edge subgroups to be again weakly potent. Let F be a graph and T be a maximal subtree of F. The fundamental group G of the graph F of groups with infinite cyclic edge subgroups can be considered as an HNN extension of the form $G = \langle A, t_1, \ldots, t_n; t_i^{-1} h_i t_i = k_i, i = 1, \ldots, n\rangle$ where A is a tree product of the vertex groups according to the maximal subtree T and the h_i, k_i are in the vertex groups. Since one of the simplest type of HNN extensions, the Baumslag-Solitar group, $\langle h, t; t^{-1}h^2t = h^3\rangle$ is not even residually finite (see Baumslag and Solitar [3]), the residual properties of fundamental groups of graphs are difficult to determine. Shirvani in [13] first gave conditions for the residual finiteness of fundamental groups of graphs. Recently, Raptis and Varsos [11], Varsos [16] proved the residual nilpotence and subgroup separability of fundamental groups of graphs where the edge group have finite index in the containing vertex groups. More recently, Kim [8] obtained characterizations for

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the fundamental groups of graphs of certain cyclic subgroup separable groups with infinite cyclic edge subgroups to be again cyclic subgroup separable.

Weak potency is a strong form of residual finiteness and was first introduced by Evans [4] with the name regular quotients and he showed that free groups and finitely generated torsion-free nilpotent groups are weakly potent. Later, Tang [15] independently defined weak potency and he proved that finite extensions of free groups and finitely generated torsion-free nilpotent groups are weakly potent. Evans [4] used weak potency to show the cyclic subgroup separability of certain generalised free products while more recently Kim and Tang [9] and Tang [15] used it to determine the conjugacy separability of certain generalised free products of conjugacy separable groups. It is known that polycyclic-by-finite groups and free-by-finite groups are weakly potent [4, 15].

A fundamental group of a graph F of groups can be described as follows: (See Kim [8]) Let $F = (V, E)$ be a graph where V is a set of vertices and E is a set of edges. To each vertex v in V, we assign a group G_v. To each edge e in E, we assign a group G_e together with monomorphisms α_e and β_e embedding G_e into the two vertex groups at the end of e. Then for a maximal subtree T of F, the fundamental group of the graph F of groups G_v amalgamating the edge subgroups G_e is defined to be the group generated by all the generators of the vertex groups and additional generators t_e for each edge $e \in E$. The defining relations are given by the defining relations of all the vertex groups together with the relations ${t_e}^{-1}(g_e\alpha_e)t_e = g_e\beta_e$ for each g_e of G_e where $t_e = 1$ if e is an edge of T. Each of the subgroups $G_e\alpha_e$ and $G_e\beta_e$ is called an edge subgroup in its containing vertex group. It is well known that the fundamental group of a graph of groups is independent from the choice of the maximal subtree (Serre [12]). In particular if the graph F is a tree, then the fundamental group of graph of groups is called a tree product.

Our main result is a characterization for the fundamental group of a graph of weakly potent groups with infinite cyclic edge subgroups to be weakly potent (Theorem 5.1). We then apply this result to graphs of polycyclic-by-finite groups and free-by-finite groups (Theorem 5.2). We prove Theorem 5.1 in two parts. First we prove a characterization for certain HNN extensions of weakly potent groups with infinite cyclic associated subgroups to be weakly potent (Theorem 3.3). Then we show that the tree products of weakly potent groups with infinite cyclic edge subgroups are again weakly potent (Theorem 4.2). From these results we can also show that all one-relator groups with non-trivial center are weakly potent (Theorem 6.1).

The notations used here are standard. In addition, the following will be used. Let G be a group.

- $N \triangleleft_f G$ means N is a normal subgroup of finite index in G.
- $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ denotes the HNN extension where K is the base group, A, B are the associated subgroups and φ is the associated isomorphism $\varphi : A \longrightarrow B$.
- If $x \in G = \langle t, K; t^{-1}At = B, \varphi\rangle$ is reduced, we shall express x in the form

$$x = x_0 t^{e_1} x_1 \cdots x_{n-1} t^{e_n} x_n$$

where $x_0, x_i \in K$ and $e_i = \pm 1, 1 \leq i \leq n$.

- $G = A \underset{H}{*} B$ denotes the generalised product of the groups A and B amalgamating the subgroup H.
- If $x \in A \underset{H}{*} B$ is reduced, we shall express x in the form
$$x = a_1 b_1 \dots a_n b_n$$
where $a_i \in A \backslash H$ and $b_i \in B \backslash H$ for $1 \le i \le n$.
- $\|x\|$ will denote the usual reduced length of x in the HNN extension or generalised free product G.

2. Preliminaries

We begin with two definitions.

Definition 2.1. [15] *A group G is called weakly potent if for any element x of infinite order in G, we can find a positive integer r with the property that for every positive integer n, there exists a normal subgroup M_n of finite index in G such that xM_n has order exactly rn.*

Similarly a group G is called potent if for any element x of infinite order in G and every positive integer n, there exists a normal subgroup M_n of finite index in G such that xM_n has order exactly n.

From the definition above, potency is a much stronger property than weak potency. Stebe [14] first proved that free groups are potent (before Evan introduced weak potency) but he did not give it a name. Indeed Stebe proved that if n is a power of a prime q then the factor group G/N_n can be chosen to be a q-group. (Stebe acknowledged that the proof of this result was suggested by D. S. Passman.) Later Allenby and Tang independently introduced potency in [2] and they used it to prove proved finite extensions of certain generalised free products. In [1] Allenby showed that cyclically pinched one-relator groups are potent.

Definition 2.2. *A group G is called H-separable for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.*

- *G is termed π_c if G is $\langle h \rangle$-separable for every cyclic subgroup $\langle h \rangle$.*

Free-by-finite groups and polycyclic-by-finite groups are weakly potent and π_c. (See [4, 15]). Indeed certain HNN extensions of polycyclic-by-finite groups with central associated subgroups are π_c and even subgroup separable. (See [17, 18]).

3. HNN extensions of weakly potent groups

In this section we give conditions for an HNN extension of weakly potent group with cyclic associated subgroups to be weakly potent.

Lemma 3.1. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ where K is finite. Then G is free-by-finite* (*see* [5]) *and hence weakly potent and π_c* (*see* [4, 15]).

Theorem 3.1. *Let $G = \langle t, A; t^{-1}ht = k \rangle$ where A is a weakly potent group and $\langle h \rangle, \langle k \rangle$ are infinite cyclic subgroups of A such that $\langle h \rangle \cap \langle k \rangle \neq 1$. Suppose that A is $\langle h \rangle$-separable and $\langle k \rangle$-separable. Then G is weakly potent if and only if $h^m = k^{\pm m}$ for some $m > 0$.*

Proof. First we note that since A is weakly potent, we can find positive integers r_1, r_2 such that for each positive integer n, there exist $P \lhd_f A, Q \lhd_f A$ such that $P \cap \langle h \rangle = \langle h^{r_1 n} \rangle, Q \cap \langle k \rangle = \langle h^{r_2 n} \rangle$.

Suppose that $h^m = k^{\pm m}$ for some $m > 0$. Let x be an element of infinite order in G.

Case 1. $\|x\| = 0$, that is, $x \in A$. From above, let $N_1 \lhd_f A, N_2 \lhd_f A$ be such that $N_1 \cap \langle h \rangle = \langle h^{r_1 r_2 m} \rangle, N_2 \cap \langle k \rangle = \langle k^{r_1 r_2 m} \rangle$. Let $Q = N_1 \cap N_2$. Then $Q \lhd_f A$ and $Q \cap \langle h \rangle = \langle h^{r_1 r_2 m} \rangle = \langle k^{r_1 r_2 m} \rangle = Q \cap \langle k \rangle$.

Suppose $Q \cap \langle x \rangle = \langle x^\alpha \rangle$ for some integer α. Since A is weakly potent, we can find a positive integer r such that for each positive integer n, there exists $P \lhd_f A$ such that $P \cap \langle x \rangle = \langle x^{r\alpha n} \rangle$. Let $N = P \cap Q$. Then $N \lhd_f A$ is such that $N \cap \langle x \rangle = \langle x^{r\alpha n} \rangle$ and $N \cap \langle h \rangle = N \cap \langle k \rangle$.

Now we form $\bar{G} = \langle t, \bar{A}; t^{-1}\bar{h}t = \bar{k} \rangle$ where $\bar{A} = A/N, \bar{h} = hN, \bar{k} = kN$. Clearly $\bar{G}$ is a homomorphic image of G. Since $\bar{A}$ is finite, $\bar{G}$ is residually finite by Lemma 3.1 and hence there exists $\bar{M} \lhd_f \bar{G}$ such that $\bar{x}, \bar{x}^2, \ldots, \bar{x}^{r\alpha n-1} \notin \bar{M}$. Thus $\bar{x}\bar{M}$ has order exactly $r\alpha n$ in the finite group $\bar{G}/\bar{M}$. Let M be the preimage of $\bar{M}$ in G. Then xM has order exactly $r\alpha n$ in G/M and we are done.

Case 2. $\|x\| \geq 1$. Let $x = x_0 t^{e_1} x_1 t^{e_2} \ldots t^{e_n} x_n$ where $x_i \in A$ and $n \geq 1$. Since A is $\langle h \rangle$-separable and $\langle k \rangle$-separable, there exists $M \lhd_f A$ such that $x_i \notin \langle h \rangle M$ if $x_i \notin \langle h \rangle$ and $x_i \notin \langle k \rangle M$ if $x_i \notin \langle k \rangle$. Suppose $M \cap \langle h \rangle = \langle h^{\alpha_1} \rangle$ and $M \cap \langle k \rangle = \langle k^{\alpha_2} \rangle$ for some integers α_1, α_2. Let $\alpha = \alpha_1 \alpha_2$. From above let $N_1 \lhd_f A, N_2 \lhd_f A$ be such that $N_1 \cap \langle h \rangle = \langle h^{\alpha r_1 r_2 m} \rangle$ and $N_2 \cap \langle k \rangle = \langle k^{\alpha r_1 r_2 m} \rangle$. Let $N = M \cap N_1 \cap N_2$. Then $N \lhd_f A$ and $N \cap \langle h \rangle = \langle h^{\alpha r_1 r_2 m} \rangle = \langle k^{\alpha r_1 r_2 m} \rangle = N \cap \langle k \rangle$. As in Case 1, we form $\bar{G}$. Then $\bar{x}$ is reduced in $\bar{G}$ and $\|\bar{x}\| = \|x\|$. It follows that $\bar{x}$ has infinite order in $\bar{G}$. Since $\bar{A}$ is finite, $\bar{G}$ is weakly potent by Lemma 3.1 and our result follows.

Conversely, suppose that G is weakly potent. Since $\langle h \rangle \cap \langle k \rangle \neq 1$, it follows that $h^m = k^p$ where $p > 0$. Since G is weakly potent, the subgroup $G_1 = \langle t, h; t^{-1}h^p t = h^m \rangle$ is weakly potent. Then there exists a positive integer r with the property that for every positive integer n, we can find $N \lhd_f G_1$ such that hN has order exactly rn. We choose $n = \mid p \mid\mid m \mid$. Then $\mid h^m \mid = r \mid p \mid$ and $\mid h^p \mid = r \mid m \mid$. Since $\bar{h}^p$ is conjugate to $\bar{h}^m$ in $\bar{G}_1 = G_1/N$, $r \mid m \mid = r \mid p \mid$ which implies that $\mid m \mid = \mid p \mid$. ∎

By using Theorem 3.1, we can easily obtain a characterization for the Baumslag-Solitar groups to be weakly potent.

Theorem 3.2. *Let $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l \rangle$. Then $G_{k,l}$ is weakly potent if and only if $\mid k \mid = \mid l \mid$.*

Next we extend Theorem 3.1 to HNN extensions of the form $G = \langle A, t_1, \ldots, t_n; t_i^{-1} h_i t_i = k_i, i = 1, \ldots, n \rangle$.

Lemma 3.2. *Let $G = \langle t, A; t^{-1}ht = k \rangle$ where A is a weakly potent group and $\langle h \rangle, \langle k \rangle$ are infinite cyclic subgroups of A such that $h^m = k^{\pm m}$ for some $m > 0$. Suppose that A is $\langle h \rangle$-separable and $\langle k \rangle$-separable. Let $a \in A$ such that A is $\langle a \rangle$-separable. Then $G = \langle t, A; t^{-1}ht = k \rangle$ is $\langle a \rangle$-separable.*

Proof. Let $x \in G \backslash \langle a \rangle$ be a reduced element in G.

Case 1. $\mid x \mid= 0$, that is, $x \in A$. Since A is $\langle a\rangle$-separable, there exists $M \lhd_f A$ such that $x \notin \langle a\rangle M$. As in the Case 2 of Theorem 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and $N \cap \langle h\rangle = N \cap \langle k\rangle$. We form $\bar{G} = \langle t, \bar{A}; t^{-1}\bar{h}t = \bar{k}\rangle$ where $\bar{A} = A/N, \bar{h} = hN, \bar{k} = kN$. Clearly $\bar{G}$ is a homomorphic image of G. Since $\bar{x} \notin \langle\bar{a}\rangle$ and $\bar{G}$ is π_c by Lemma 3.1, the result now follows.

Case 2. $\mid x \mid\geq 1$. Let $x = x_0t_1^{e_1}x_1t_1^{e_2}\dots t_1^{e_n}x_n$ where $x_i \in A$ and $n \geq 1$. Since A is $\langle h\rangle$-separable and $\langle k\rangle$-separable, there exists $M \lhd_f A$ such that $x_i \notin \langle h\rangle M$ if $x_i \notin \langle h\rangle$ and $x_i \notin \langle k\rangle M$ if $x_i \notin \langle k\rangle$. Again as in Case 2 of Theorem 3.1, there exists $N \lhd_f A$ such that $N \subseteq M$ and $N \cap \langle h\rangle = N \cap \langle k\rangle$. Again we form $\bar{G}$. Then $\bar{x}$ is reduced in $\bar{G}$ and $\|\bar{x}\| = \|x\|$. Since $\bar{x} \notin \langle\bar{a}\rangle$ and $\bar{G}$ is π_c by Lemma 3.1, again we are done. ∎

Theorem 3.3. *Let A be weakly potent and $h_i, k_i \in A$ be elements of infinite order such that $\langle h_i\rangle \cap \langle k_i\rangle \neq 1$ for each $i = 1, \dots, n$. Suppose A is $\langle h_i\rangle$-separable and $\langle k_i\rangle$-separable. Then $G = \langle A, t_1, \dots, t_n; t_i^{-1}h_it_i = k_i, i = 1, \dots, n\rangle$ is weakly potent if and only if for each $i = 1, \dots, n$, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.*

Proof. Suppose that G is weakly potent. Since $\langle A, t_i; t_i^{-1}h_it_i = k_i\rangle$ is a subgroup of G, then it must be weakly potent. Hence by Theorem 3.1, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.

Conversely, suppose that for each $i = 1, \dots, n$, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$. Let $G_1 = \langle A, t_1; t_1^{-1}h_1t_1 = k_1\rangle$. Then G_1 is weakly potent by Theorem 3.1 and G_1 is $\langle h_i\rangle$-separable and $\langle k_i\rangle$-separable for each $i = 2, \dots, n$ by Lemma 3.2. Let $G_j = \langle A, t_1, \dots, t_j; t_i^{-1}h_it_i = k_i, i = 1, \dots, j\rangle$. Then $G_j = \langle G_{j-1}, t_j; t_j^{-1}h_jt_j = k_j\rangle$. Inductively, we assume that $G_{n-1} = \langle A, t_1, \dots, t_{n-1}; t_i^{-1}h_it_i = k_i, i = 1, \dots, n-1\rangle$ is weakly potent and $\langle h_n\rangle$-separable and $\langle k_n\rangle$-separable. Then $G = G_n = \langle G_{n-1}, t_n; t_n^{-1}h_nt_n = k_n\rangle$ is weakly potent by Theorem 3.1. ∎

Since polycyclic-by-finite groups and free-by-finite groups are weakly potent and π_c (Evans [4], Tang [15]), we have immediately the following:

Theorem 3.4. *Let A be a free-by-finite or polycyclic-by-finite group and $h_i, k_i \in A$ be elements of infinite order such that $\langle h_i\rangle \cap \langle k_i\rangle \neq 1$ for each $i = 1, \dots, n$. Then $G = \langle A, t_1, \dots, t_n; t_i^{-1}h_it_i = k_i, i = 1, \dots, n\rangle$ is weakly potent if and only if for each $i = 1, \dots, n$, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.*

Note that in Theorem 3.3, if A is a tree product where h_i and k_i may not be in the same vertex group, then G can be considered as a fundamental group of a graph of groups amalgamating cyclic edge subgroups. In order to do this, we next prove the weak potency of tree products of weakly potent groups with infinite cyclic edge subgroups in the next section.

4. Tree products of weakly potent groups

In this section we will show that tree products of finitely many weakly potent groups amalgamating infinite cyclic subgroups are weakly potent.

Lemma 4.1. *Let $G = G_1 \underset{H}{*} G_2$ where G_1, G_2 are finite. Then G is free-by-finite* (*see* [6]) *and hence weakly potent and π_c* (*see* [4, 15]).

Theorem 4.1. *Let $G = G_1 \underset{H}{*} G_2$ where G_1, G_2 are weakly potent and $H = \langle h \rangle$ is infinite cyclic. Suppose G_1, G_2 are $\langle h \rangle$-separable. Then G is weakly potent.*

Proof. First we note that since G_1, G_2 are weakly potent, we can find positive integers r_1, r_2 such that for each positive integer n, there exist $P \lhd_f G_1, Q \lhd_f G_2$ such that $P \cap \langle h \rangle = \langle h^{r_1 n} \rangle, Q \cap \langle h \rangle = \langle h^{r_2 n} \rangle$. Let g be an element of infinite order in G.

Case 1. $\|g\| \leq 1$, that is, $g \in G_1 \cup G_2$. Without loss of generality (WLOG), assume $g \in G_1$. From above let $N_1 \lhd_f G_1$ be such that $N_1 \cap \langle h \rangle = \langle h^{r_1 r_2} \rangle$. Suppose $N_1 \cap \langle g \rangle = \langle g^s \rangle$ for some positive integer s. By the weak potency of G_1, we can find a positive integer r such that for each positive integer n, there exists $N_2 \lhd_f G_1$ such that $N_2 \cap \langle g \rangle = \langle g^{rsn} \rangle$. Let $N = N_1 \cap N_2$. Then $N \lhd_f G_1$, $N \cap \langle g \rangle = \langle g^{rsn} \rangle$ and $N \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle$ for some positive integer t. Let $M \lhd_f G_2$ be such that $M \cap \langle h \rangle = \langle h^{r_1 r_2 t} \rangle$. Now we form $\bar{G} = \bar{G}_1 \underset{\bar{H}}{*} \bar{G}_2$ where $\bar{G}_1 = G_1/N$, $\bar{G}_2 = G_2/M$ and $\bar{H} = \langle h \rangle N/N = \langle h \rangle M/M$. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of g in $\bar{G}$. Then $\bar{g}$ has order exactly rsn in $\bar{G}$. Since $\bar{G}$ is residually finite by Lemma 4.1, there exists $\bar{P} \lhd_f \bar{G}$ such that $\bar{g}, \ldots, \bar{g}^{rsn-1} \notin \bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ and gP has order exactly rsn in G/P and we are done.

Case 2. $\|g\| > 1$, that is, $g \notin G_1 \cup G_2$. WLOG, assume $g = a_1 b_1 \ldots a_n b_n$ where $a_i \in G_1 \backslash \langle h \rangle$ and $b_i \in G_2 \backslash \langle h \rangle$ for all i. Since G_1, G_2 are $\langle h \rangle$-separable, there exist $N_1 \lhd_f G_1$, $M_1 \lhd_f G_2$ such that $a_i \notin \langle h \rangle N_1$ and $b_i \notin \langle h \rangle M_1$ for all i. Suppose $N_1 \cap \langle h \rangle = \langle h^{s_1} \rangle$ and $M_1 \cap \langle h \rangle = \langle h^{s_2} \rangle$ for some positive integers s_1 and s_2. Let $N_2 \lhd_f G_1$, $M_2 \lhd_f G_2$ be such that $N_2 \cap \langle h \rangle = \langle h^{r_1 r_2 s_1 s_2} \rangle = M_2 \cap \langle h \rangle$. Let $N = N_1 \cap N_2$ and $M = M_1 \cap M_2$. Then $N \lhd_f G_1, M \lhd_f G_2$ and $N \cap \langle h \rangle = M \cap \langle h \rangle$. As in Case 1, we form $\bar{G}$. Then $\|\bar{g}\| = \|g\|$ and hence $\bar{g}$ has infinite order in $\bar{G}$. By Lemma 4.1, $\bar{G}$ is weakly potent and the result follows. ∎

To extend Theorem 4.1 to a tree product, we need the next few lemmas.

Lemma 4.2. [7] *Let $G = G_1 \underset{H}{*} G_2$. Suppose that*

(a) *G_1, G_2 are H-separable;*
(b) *for each $R \lhd_f H$, there exist $N \lhd_f G_1, M \lhd_f G_2$ such that $N \cap H = M \cap H \subseteq R$.*

- *Let K be a subgroup of G_1 and G_1 is K-separable. Then G is K-separable.*

The next lemma can easily be derived from Lemma 4.2.

Lemma 4.3. *Let $G = G_1 \underset{H}{*} G_2$ where $H = \langle h \rangle$ is infinite cyclic. Suppose that G_1, G_2 are weakly potent and $\langle h \rangle$-separable. Let K be a subgroup of G_1 and G_1 is K-separable. Then G is K-separable.*

Lemma 4.3 can be extended to a tree product with the additional condition that the group G is weakly potent.

Lemma 4.4. *Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij} \rangle$ of G_i and $\langle a_{ji} \rangle$ of G_j. Suppose G is weakly potent and each G_i is $\langle a_{ij} \rangle$-separable. Let K be a subgroup of G_r and G_r is K-separable. Then G is K-separable.*

Proof. We use induction on n. The case $n = 2$ follows from Lemma 4.3. Now, let $n > 2$. The tree product G has an extremal vertex, say G_n, which is joined to a unique vertex, say G_{n-1}. The subgroup of G generated by $G_1, G_2, \ldots, G_{n-1}$ is just their tree product. Let G' denote this subgroup. Then $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)} \rangle$. By the inductive hypothesis, G' is $\langle a_{(n-1)n} \rangle$-separable and by assumption, G_n is $\langle a_{n(n-1)} \rangle$-separable. Furthermore, by assumption G is weakly potent. Hence G' is weakly potent and G_n is weakly potent.

Case 1. $K \subseteq G'$. By inductive hypothesis, G' is K-separable and we are done by Lemma 4.3.

Case 2. $K \subseteq G_n$. By assumption, G_n is K-separable and we are done by Lemma 4.3. ∎

Now, Theorem 4.1 can be extended to a tree product as follows:

Theorem 4.2. *Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij} \rangle$ of G_i and $\langle a_{ji} \rangle$ of G_j. Suppose each G_i is weakly potent and $\langle a_{ij} \rangle$-separable. Then G is weakly potent.*

Proof. We use induction on n. The case $n = 2$ follows from Theorem 4.1. Now, let $n > 2$. As in Lemma 4.4, we write $G = \langle G', G_n; a_{(n-1)n} = a_{n(n-1)} \rangle$ where G' is the tree product generated by $G_1, G_2, \ldots, G_{n-1}$. By inductive hypothesis, G' is weakly potent. Hence G' is $\langle a_{(n-1)n} \rangle$-separable by Lemma 4.4. Furthermore by assumption, G_n is weakly potent and G_n is $\langle a_{n(n-1)} \rangle$-separable. Therefore G is weakly potent by Theorem 4.1. ∎

Corollary 4.1. *Let $G_1, G_2, \ldots, G_n$ be free-by-finite groups or polycyclic-by-finite groups. Let $G = \langle G_1, G_2, \ldots, G_n; a_{ij} = a_{ji} \rangle$ be a tree product of $G_1, G_2, \ldots, G_n$, amalgamating the infinite cyclic subgroups $\langle a_{ij} \rangle$ of G_i and $\langle a_{ji} \rangle$ G_j. Then G is weakly potent.*

5. Fundamental groups of graphs of weakly potent groups

From Theorem 3.3 and Theorem 4.2, we obtain our main results.

Theorem 5.1. *Let A_v be weakly potent groups. Let G be a fundamental group of a graph of the groups A_v amalgamating cyclic edge subgroups, presented by $G = \langle A, t_1, \ldots, t_n; t_i^{-1} h_i t_i = k_i, i = 1, \ldots, n \rangle$ where A is a tree product of the groups A_v according to a maximal subtree of the graph and where $\langle h_i \rangle \cap \langle k_i \rangle \neq 1$ for each $i = 1, \ldots, n$. Then G is weakly potent if and only if for each $i = 1, \ldots, n$, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.*

Theorem 5.2. *Let A_v be free-by-finite or polycyclic-by-finite groups. Let G be a fundamental group of a graph of the groups A_v amalgamating cyclic edge subgroups, presented by $G = \langle A, t_1, \ldots, t_n; t_i^{-1} h_i t_i = k_i, i = 1, \ldots, n \rangle$ where A is a tree product of the groups A_v according to a maximal subtree of the graph and where $\langle h_i \rangle \cap \langle k_i \rangle \neq 1$ for each $i = 1, \ldots, n$. Then G is weakly potent if and only if for each $i = 1, \ldots, n$, $h_i^{m_i} = k_i^{\pm m_i}$ for some $m_i > 0$.*

6. One-relator groups with non-trivial center

Next, we apply Theorem 3.1 and Theorem 4.2 to show that all one-relator groups with non-trivial center are weakly potent.

Theorem 6.1. *Let G be a one-relator group with non-trivial centre. Then G is weakly potent.*

Proof. First suppose that the abelianisation of G is not free abelian of rank two. Then by Pietrowski [10, Theorem 1], G has a presentation of the form

$$\langle a_1, a_2, \ldots, a_m; a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$$

where $m, p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$. Clearly G is a tree product of infinite cyclic groups and hence G is weakly potent by Theorem 4.2.

Now suppose that the abelianisation of G is free abelian of rank two. Again by Pietrowski [10, Theorem 3], G has a presentation of the form

$$\langle t, a_1, a_2, \ldots, a_m; t^{-1}a_1t = a_m, a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \ldots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$$

where $m, p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$ such that $p_1p_2 \ldots p_{m-1} = q_1q_2 \ldots q_{m-1}$. Then $G = \langle t, B; t^{-1}a_1t = a_m \rangle$ is an HNN extension where $B = \langle a_1, a_2, \cdots, a_m; a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \cdots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$ and $a_1^{\delta} = a_m^{\delta}$ where $\delta = p_1p_2 \ldots p_{m-1} = q_1q_2 \ldots q_{m-1}$. Now B is a tree product of infinite cyclic groups and hence is weakly potent by Theorem 4.2. By Theorem 2.1 of Kim [6], B is π_c and hence B is $\langle a_1 \rangle$-separable and $\langle a_m \rangle$-separable. Therefore G is weakly potent by Theorem 3.1. ∎

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