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On a Cauchy-Jensen Functional Inequality

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Abstract. In this paper, we investigate the following functional inequality

$$\left\| f(x) + f(y) + 2f\left(\frac{x+y}{2} + z\right) \right\| \leq 2\|f(x+y+z)\|$$

in Banach modules over a C^*-algebra, and prove the generalized Hyers-Ulam stability of additive mappings in Banach modules over a C^*-algebra to approximate homomorphisms.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms: *Let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta(\epsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with*

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism

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near it. Hyers [17] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$. Aoki [2] and Rassias [38] provided a generalization of Hyers' theorem for additive mappings and linear mappings, respectively, which allows the *Cauchy difference to be unbounded* (see also [5] and [12]).

Theorem 1.1. [38] *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.*

For the case $p = 1$, a counter example has been given by Gajda [11] (see also [40]). The generalized Hyers-Ulam stability mentioned in Theorem 1.1 is known as Hyers-Ulam-Rassias stability (cf. the books of Czerwik [10], Hyers, Isac and Rassias [18]).

Theorem 1.2. [35, 36, 37] *Let X be a real normed linear space and Y a real Banach space. Assume that $f : X \to Y$ is a mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the functional inequality* (*Cauchy-Găvruta-Rassias inequality*)

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\|x\|^r$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is linear.

Găvruta [14] showed that Theorem 1.2 is not true when $r = 1$. The stability in Theorem 1.2 involving a product of different powers of norms is called Ulam-Găvruta-Rassias stability (see [4, 29, 30]). During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of Hyers-Ulam-Rassias stability and Ulam-Găvruta-Rassias stability to a number of functional equations and mappings (see [3, 6–9, 13–16, 20–22, 24–34]. We also refer the readers to the books [1, 10, 18, 41, 39].

Park, Cho and Han [33] investigated the functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\| \tag{1.3}$$

in Banach spaces, and proved the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with unit e, unitary group $U(A)$ and norm $|\cdot|$. Assume that X is a normed A-module with norm $\|\cdot\|_X$ and that Y is a Banach A-module with norm $\|\cdot\|_Y$. For $a \in A$, let $a^\dagger = a, a^*$ or $(a+a^*)/2$. An additive mapping $T : X \to Y$ is called A-additive if $T(ax) = a^\dagger T(x)$ for all $a \in A$ and all $x \in X$.

In this paper, we investigate an A-additive mapping associated with the functional inequality

$$\left\|f(x)+f(y)+2f\left(\frac{x+y}{2}+z\right)\right\| \le 2\|f(x+y+z)\| \tag{1.4}$$

and prove the generalized Hyers-Ulam stability of A-additive mappings in Banach A-modules associated with the functional inequality (1.4).

For convenience, we use the following abbreviation for a given $a \in A$ and a mapping $f : X \to Y$

$$D_a f(x,y,z) := f(ax)+f(ay)+2a^\dagger f\left(\frac{x+y}{2}+z\right)$$

for all $x, y, z \in X$.

2. Functional inequalities in Banach modules over a C^*-algebra

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\|D_a f(x,y,z)\|_Y \le 2\|f(ax+ay+az)\|_Y \tag{2.1}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then $f : X \to Y$ is A-additive.

Proof. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (2.1), we get that $f(0) = 0$. Letting $z = 0, y = -x$ and $a = e \in U(A)$ in (2.1), we get

$$\|f(x)+f(-x)\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x-y$ and $a = e \in U(A)$ in (2.1) and using the oddness of f, we get

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x, y \in X$. So

$$2f\left(\frac{x+y}{2}\right) = f(x)+f(y) \tag{2.2}$$

for all $x, y \in X$. Letting $y = 0$ in (2.2), we get $2f(x/2) = f(x)$ for all $x \in X$. Thus (2.2) implies that

$$f(x+y) = f(x)+f(y)$$

for all $x, y \in X$. Hence $f(rx) = rf(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$.

Letting $z = -x$ and $y = 0$ in (2.1) and using the oddness of f, we get

$$\|f(ax) - a^\dagger f(x)\|_Y \leq 2\|f(0)\|_Y = 0$$

for all $x \in X$ and all $a \in U(A)$. Thus

$$f(ax) = a^\dagger f(x) \tag{2.3}$$

for all $a \in U(A)$ and all $x \in X$. It is clear that (2.3) holds for $a = 0$.

Now let $a \in A$ ($a \neq 0$) and m be an integer greater than $4|a|$. Then $|a/m| < 1/4 < 1 - 2/3 = \frac{1}{3}$. By Theorem 1 of [23], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $(3/m)a = u_1 + u_2 + u_3$. So

$$\frac{3}{m}a^\dagger = (\frac{3}{m}a)^\dagger = u_1^\dagger + u_2^\dagger + u_3^\dagger.$$

Hence by (2.3) we have

$$\begin{aligned} f(ax) &= \frac{m}{3} f\left(\frac{3}{m}ax\right) = \frac{m}{3} f\left(u_1x + u_2x + u_3x\right) = \frac{m}{3}\left[f(u_1x) + f(u_2x) + f(u_3x)\right] \\ &= \frac{m}{3}(u_1^\dagger + u_2^\dagger + u_3^\dagger) f(x) = \frac{m}{3} \cdot \frac{3}{m} a^\dagger f(x) = a^\dagger f(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-additive, as desired. ∎

Now we prove the generalized Hyers-Ulam stability of A-additive mappings in Banach A-modules.

Theorem 2.1. *Let $r_i > 1$ and θ_i be non-negative real numbers for all $1 \leq i \leq 3$, and let $f : X \to Y$ be a mapping such that*

$$\|D_a f(x, y, z)\|_Y \leq 2\|f(ax + ay + az)\|_Y + \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3} \tag{2.4}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then there exists a unique A-additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{2^{r_1} + 2}{2^{r_1} - 2}\theta_1\|x\|_X^{r_1} + \frac{2\theta_2}{2^{r_2} - 2}\|x\|_X^{r_2} + \frac{2^{r_3}\theta_3}{2^{r_3} - 2}\|x\|_X^{r_3} \tag{2.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (2.4), we get that $f(0) = 0$. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (2.4), we get

$$\|f(x) + f(-x)\|_Y \leq \theta_1\|x\|_X^{r_1} + \theta_2\|x\|_X^{r_2} \tag{2.6}$$

for all $x \in X$. Letting $a = e \in U(A)$, $y = 0$ and $z = -x$ in (2.4), we get

$$\left\|f(x) + 2f\left(\frac{-x}{2}\right)\right\|_Y \leq \theta_1\|x\|_X^{r_1} + \theta_3\|x\|_X^{r_3} \tag{2.7}$$

for all $x \in X$. It follows from (2.6) and (2.7) that

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \frac{2 + 2^{r_1}}{2^{r_1}}\theta_1\|x\|_X^{r_1} + \frac{2\theta_2}{2^{r_2}}\|x\|_X^{r_2} + \theta_3\|x\|_X^{r_3}$$

for all $x \in X$. Hence

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y \leq \sum_{j=m}^{n-1}\left\|2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right)\right\|_Y$$

$$\leq \frac{2+2^{r_1}}{2^{r_1}}\theta_1\|x\|_X^{r_1}\sum_{j=m}^{n-1}\left(\frac{2}{2^{r_1}}\right)^j + \theta_2\|x\|_X^{r_2}\sum_{j=m}^{n-1}\left(\frac{2}{2^{r_2}}\right)^{j+1}$$

$$+\,\theta_3\|x\|_X^{r_3}\sum_{j=m}^{n-1}\left(\frac{2}{2^{r_3}}\right)^j \tag{2.8}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get (2.5). It follows from (2.4) that

$$\begin{aligned}\|D_a L(x,y,z)\|_Y &= \lim_{n\to\infty} 2^n \left\|D_a f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} 2^{n+1}\left\|f\left(\frac{ax}{2^n}+\frac{ay}{2^n}+\frac{az}{2^n}\right)\right\|_Y \\ &\quad + \lim_{n\to\infty} 2^n\left[\frac{\theta_1}{2^{nr_1}}\|x\|_X^{r_1} + \frac{\theta_2}{2^{nr_2}}\|y\|_X^{r_2} + \frac{\theta_3}{2^{nr_3}}\|z\|_X^{r_3}\right] \\ &= 2\|L(ax+ay+az)\|_Y\end{aligned}$$

for all $x, y, z \in X$ and all $a \in U(A)$. So by Lemma 2.1, the mapping $L : X \to Y$ is A-additive.

Now, let $T : X \to Y$ be another A-additive mapping satisfying (2.5). Then we have

$$\begin{aligned}\|L(x) - T(x)\|_Y &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_Y \\ &\leq \lim_{n\to\infty} 2^n\left[\frac{(2^{r_1}+2)\theta_1}{2^{nr_1}(2^{r_1}-2)}\|x\|_X^{r_1} + \frac{2\theta_2}{2^{nr_2}(2^{r_2}-2)}\|x\|_X^{r_2}\right. \\ &\quad \left. + \frac{2^{r_3}\theta_3}{2^{nr_3}(2^{r_3}-2)}\|x\|_X^{r_3}\right] = 0\end{aligned}$$

for all $x \in X$. So we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L. Thus the mapping $L : X \to Y$ is a unique A-additive mapping satisfying (2.5). ∎

Theorem 2.2. *Let $0 < r_i < 1$ and θ_i, δ be non-negative real numbers for all $1 \leq i \leq 3$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y,z)\|_Y \leq 2\|f(ax+ay+az)\|_Y + \delta + \theta_1\|x\|_X^{r_1} + \theta_2\|y\|_X^{r_2} + \theta_3\|z\|_X^{r_3} \tag{2.9}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then there exists a unique A-additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq 3\delta + \frac{2+2^{r_1}}{2-2^{r_1}}\theta_1\|x\|_X^{r_1} + \frac{2\theta_2}{2-2^{r_2}}\|x\|_X^{r_2} + \frac{2^{r_3}\theta_3}{2-2^{r_3}}\|x\|_X^{r_3}$$

for all $x \in X$.

Proof. Similarly to the proof of Theorem 2.1, it follows from (2.9) that

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_Y \le 3\delta+\frac{2+2^{r_1}}{2^{r_1}}\theta_1\|x\|_X^{r_1}+\frac{2\theta_2}{2^{r_2}}\|x\|_X^{r_2}+\theta_3\|x\|_X^{r_3}$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^n}f(2^nx)-\frac{1}{2^m}f(2^mx)\right\|_Y &\le \sum_{j=m}^{n-1}\left\|\frac{1}{2^{j+1}}f(2^{j+1}x)-\frac{1}{2^j}f(2^jx)\right\|_Y\\ &\le 3\delta\sum_{j=m+1}^{n}\left(\frac{1}{2}\right)^j+\frac{2+2^{r_1}}{2^{r_1}}\theta_1\|x\|_X^{r_1}\sum_{j=m+1}^{n}\left(\frac{2^{r_1}}{2}\right)^j\\ &\quad+\frac{2\theta_2}{2^{r_2}}\|x\|_X^{r_2}\sum_{j=m+1}^{n}\left(\frac{2^{r_2}}{2}\right)^j+\theta_3\|x\|_X^{r_3}\sum_{j=m+1}^{n}\left(\frac{2^{r_3}}{2}\right)^j\end{aligned}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1 and we omit the details. ∎

Theorem 2.3. *Let* $\{r_i\}_{i=1}^3$ *and* θ *be non-negative real numbers such that* $\lambda := r_1+r_2+r_3 \in (0,1)\cup(1,+\infty)$, $r_1+r_2>0$, $r_3>0$, *and let* $f : X \to Y$ *be a mapping such that*

$$\|D_af(x,y,z)\|_Y \le 2\|f(ax+ay+az)\|_Y+\theta\cdot\|x\|_X^{r_1}\cdot\|y\|_X^{r_2}\cdot\|z\|_X^{r_3} \tag{2.10}$$

for all $x, y, z \in X$ *and all* $a \in U(A)$ *(by letting* $\|\cdot\|_X^0 = 1$*). Then* $f : X \to Y$ *is* A*-additive.*

Proof. Since $r_1 + r_2 > 0$, $r_j > 0$ for some $1 \le j \le 2$. Without loss of generality, we may assume that $r_2 > 0$. Letting $x = y = z = 0$ and $a = e \in U(A)$ in (2.10), we get that $f(0) = 0$. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (2.10), we get

$$\|f(x)+f(-x)\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x \in X$. So the mapping f is odd. Letting $a = e \in U(A)$, $y = 0$ and $z = -x$ in (2.10) and using the oddness of f, we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\|_Y \le 2\|f(0)\|_Y = 0$$

for all $x \in X$. Hence $2f(x/2) = f(x)$ and so

$$2^nf\left(\frac{x}{2^n}\right)=f(x) \tag{2.11}$$

for all $n \in \mathbb{Z}$ and all $x \in X$. Let $\lambda > 1$ (we have a similar proof when $0 < \lambda < 1$). It follows from (2.10) and (2.11) that

$$\begin{aligned}\|D_af(x,y,z)\|_Y &= \lim_{n\to\infty}2^n\left\|D_af\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\right\|_Y\\ &\le \lim_{n\to\infty}2^{n+1}\left\|f\left(\frac{ax}{2^n}+\frac{ay}{2^n}+\frac{az}{2^n}\right)\right\|_Y+\lim_{n\to\infty}2^n\frac{\theta}{2^{n\lambda}}\|x\|_X^{r_1}\|y\|_X^{r_2}\|z\|_X^{r_3}\\ &= 2\|f(ax+ay+az)\|_Y\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $f : X \to Y$ is A-additive. ∎

Theorem 2.4. *Let $\{r_i\}_{i=1}^3$ and θ, δ be non-negative real numbers such that $\lambda := r_1 + r_2 + r_3 \in (0,1), r_1 + r_2 > 0, r_3 > 0$, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y,z)\|_Y \le 2\|f(ax+ay+az)\|_Y + \delta + \theta \cdot \|x\|_X^{r_1} \cdot \|y\|_X^{r_2} \cdot \|z\|_X^{r_3} \tag{2.12}$$

for all $x, y, z \in X$ and all $a \in U(A)$ (by letting $\|\cdot\|_X^0 = 1$). Then there exists a unique A-additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le 2\delta$$

for all $x \in X$.

Proof. Without loss of generality, we may assume that $r_2 > 0$. Similarly to the proof of Theorem 2.1, letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (2.12), we get

$$\|f(x) + f(-x)\|_Y \le \delta \tag{2.13}$$

for all $x \in X$. Letting $a = e \in U(A)$, $y = 0$ and $z = -x$ in (2.12), we get

$$\left\| f(x) + 2f\left(\frac{-x}{2}\right) \right\|_Y \le \delta \tag{2.14}$$

for all $x \in X$. It follows from (2.13) and (2.14) that

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_Y \le 2\delta$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\|_Y &\le \sum_{j=m}^{n-1} \left\| \frac{1}{2^{j+1}} f(2^{j+1}x) - \frac{1}{2^j} f(2^j x) \right\|_Y \\ &\le 2\delta \sum_{j=m+1}^{n} \left(\frac{1}{2}\right)^j \end{aligned}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1 and we omit the details. ∎

Theorem 2.5. *Let $\{r_i\}_{i=1}^2$ and θ, δ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y,z)\|_Y \le \begin{cases} 2\|f(ax+ay+az)\|_Y + \delta \\ \quad + \theta \cdot \|x\|_X^{r_1} \cdot \|y\|_X^{r_2} & \text{if } 0 < \lambda < 1, \\ 2\|f(ax+ay+az)\|_Y \\ \quad + \theta \cdot \|x\|_X^{r_1} \cdot \|y\|_X^{r_2} & \text{if } \lambda > 1 \end{cases} \tag{2.15}$$

for all $x, y, z \in X$ and all $a \in U(A)$ (by letting $\|\cdot\|_X^0 = 1$), where $\lambda := r_1 + r_2$. Then there exists a unique A-additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \begin{cases} 3\delta + \frac{2\theta}{2-2^\lambda}\|x\|_X^\lambda & \text{if } 0 < \lambda < 1, \\ \frac{2\theta}{2^\lambda - 2}\|x\|_X^\lambda & \text{if } \lambda > 1 \end{cases} \tag{2.16}$$

for all $x \in X$.

Proof. Since $r_1 + r_2 > 0$, without loss of generality, we may assume that $r_2 > 0$. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (2.15), we get

$$\|f(x) + f(-x)\|_Y \leq \begin{cases} \delta + \theta\|x\|_X^\lambda & \text{if } 0 < \lambda < 1, \\ \theta\|x\|_X^\lambda & \text{if } \lambda > 1 \end{cases} \tag{2.17}$$

for all $x \in X$. Letting $a = e \in U(A)$, $y = 0$ and $z = -x$ in (2.15), we get

$$\left\|f(x) + 2f\left(\frac{-x}{2}\right)\right\|_Y \leq \begin{cases} \delta & \text{if } 0 < \lambda < 1, \\ 0 & \text{if } \lambda > 1 \end{cases} \tag{2.18}$$

for all $x \in X$. It follows from (2.17) and (2.18) that

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \begin{cases} 3\delta + \frac{2\theta}{2^\lambda}\|x\|_X^\lambda & \text{if } 0 < \lambda < 1, \\ \\ \frac{2\theta}{2^\lambda}\|x\|_X^\lambda & \text{if } \lambda > 1 \end{cases}$$

for all $x \in X$. Hence we have the following cases:

Case I. Let $0 < \lambda < 1$. In this case, we get

$$\begin{aligned} \left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\right\|_Y &\leq \sum_{j=m}^{n-1} \left\|\frac{1}{2^{j+1}}f(2^{j+1}x) - \frac{1}{2^j}f(2^j x)\right\|_Y \\ &\leq 3\delta \sum_{j=m+1}^{n} \left(\frac{1}{2}\right)^j + \theta\|x\|_X^\lambda \sum_{j=m}^{n-1} \left(\frac{2^\lambda}{2}\right)^j \end{aligned} \tag{2.19}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$.

Case II. Let $\lambda > 1$. In this case, we get

$$\begin{aligned} \left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_Y &\leq \sum_{j=m}^{n-1} \left\|2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right)\right\|_Y \\ &\leq \theta\|x\|_X^\lambda \sum_{j=m}^{n-1} \left(\frac{2}{2^\lambda}\right)^{j+1} \end{aligned} \tag{2.20}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. It follows from (2.19) (respectively, (2.20)) that the sequence $\{1/2^n f(2^n x)\}$ (respectively, $\{2^n f(x/2^n)\}$) is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{1/2^n f(2^n x)\}$ (respectively, $\{2^n f(x/2^n)\}$) converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \begin{cases} \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) & \text{if } 0 < \lambda < 1, \\ \\ \lim_{n\to\infty} 2^n f(\frac{x}{2^n}) & \text{if } \lambda > 1 \end{cases}$$

for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.19) and (2.20), we get (2.16). The rest of the proof is similar to the proof of Theorem 2.1. ∎

Theorem 2.6. *Let r, δ and θ be non-negative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\|D_a f(x,y,z)\|_Y \le \begin{cases} 2\|f(ax+ay+az)\|_Y + \delta + \theta \cdot \|z\|_X^r & \text{if } 0 < r < 1, \\ 2\|f(ax+ay+az)\|_Y + \theta \cdot \|z\|_X^r & \text{if } r > 1 \end{cases} \tag{2.21}$$

for all $x, y, z \in X$ and all $a \in U(A)$. Then there exists a unique A-additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \begin{cases} 2\delta + \frac{2^r\theta}{2-2^r}\|x\|_X^r & \text{if } 0 < r < 1, \\ \frac{2^r\theta}{2^r-2}\|x\|_X^r & \text{if } r > 1 \end{cases}$$

for all $x \in X$.

Proof. Letting $a = e \in U(A)$, $y = -x$ and $z = 0$ in (2.21), we get

$$\|f(x) + f(-x)\|_Y \le \begin{cases} \delta & \text{if } 0 < r < 1, \\ 0 & \text{if } r > 1 \end{cases} \tag{2.22}$$

for all $x \in X$. Letting $a = e \in U(A)$, $y = 0$ and $z = -x$ in (2.21), we get

$$\left\| f(x) + 2f\left(\frac{-x}{2}\right) \right\|_Y \le \begin{cases} \delta + \theta\|x\|_X^r & \text{if } 0 < r < 1, \\ \theta\|x\|_X^r & \text{if } r > 1 \end{cases} \tag{2.23}$$

for all $x \in X$. It follows from (2.22) and (2.23) that

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\|_Y \le \begin{cases} 2\delta + \theta\|x\|_X^r & \text{if } 0 < r < 1, \\ \theta\|x\|_X^r & \text{if } r > 1 \end{cases}$$

for all $x \in X$. Hence we have the following cases:

Case I. Let $0 < r < 1$. In this case, we get

$$\begin{aligned} \left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\|_Y &\le \sum_{j=m}^{n-1} \left\| \frac{1}{2^{j+1}} f(2^{j+1}x) - \frac{1}{2^j} f(2^j x) \right\|_Y \\ &\le 2\delta \sum_{j=m+1}^{n} \left(\frac{1}{2}\right)^j + \theta\|x\|_X^r \sum_{j=m+1}^{n} \left(\frac{2^r}{2}\right)^j \end{aligned}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$.

Case II. Let $r > 1$. In this case, we get

$$\begin{aligned} \left\| 2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_Y &\le \sum_{j=m}^{n-1} \left\| 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right) \right\|_Y \\ &\le \theta\|x\|_X^r \sum_{j=m}^{n-1} \left(\frac{2}{2^r}\right)^j \end{aligned}$$

for all non-negative integers m and n with $n > m$ and all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1 and Theorem 2.5 and we omit the details. ∎

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