BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **33**(3) (2010), 355–360

Degree Conditions of Fractional ID-k-Factor-Critical Graphs

[1]Renying Chang, [2]Guizhen Liu and [3]Yan Zhu

[1]Department of Mathematics, Linyi Normal University, Linyi, Shandong, 276005 P. R. China
[2]School of Mathematics and System Sciences, Shandong University, Jinan, Shandong, 250100 P. R. China
[3]Department of Mathematics, East China University of Science and Technology, Shanghai, 200237 P. R. China
[1]changrysd@163.com, [2]gzliu@sdu.edu.cn, [3]operationzy@163.com

Abstract. We say that a simple graph G is fractional independent-set-deletable k-factor-critical, shortly, fractional ID-k-factor-critical, if $G - I$ has a fractional k-factor for every independent set I of G. Some sufficient conditions for a graph to be fractional ID-k-factor-critical are studied in this paper. Furthermore, we show that the result is best possible in some sense.

2010 Mathematics Subject Classification: 05C70

Key words and phrases: fractional k-factor, independent set, fractional ID-k-factor-critical.

1. Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree of G is denoted by $\delta(G)$. For any vertex x of G, the neighborhood of x is denoted by $N_G(x)$, the degree of x is denoted by $d_G(x)$, and we write $N_G[x]$ for $N_G(x) \bigcup \{x\}$. We use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$, respectively, for $S \subseteq V(G)$. The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. Notations and definitions not given in this paper can be found in [1].

A subset I of $V(G)$ is said to be *independent* if no two distinct vertices in I are adjacent. A *matching* in a graph is a set of edges, no two of which meet a common vertex. A matching is *perfect* if it covers all vertices of the graph. A graph G is *factor-critical* [5] if $G - v$ has a perfect matching for every vertex $v \in V(G)$. In [7], the concept of factor-critical graph was generalized to the ID-factor-critical graph. We say that G is *independent-set-deletable factor-critical* (shortly, ID-factor-critical)

Communicated by Xueliang Li.
Received: June 1, 2009; *Revised:* July 21, 2009.

if for every independent set I of G which has the same parity with $|V(G)|$, $G - I$ has a perfect matching.

Let $h : E(G) \longrightarrow [0, 1]$ be a function, and let $k \geq 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for each vertex $x \in V(G)$, we call $G[F_h]$ a *factional k-factor* of G with indicator function h where $F_h = \{e \in E(G) |\ h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [6]. We say that G is *fractional ID-k-factor-critical* if for every independent set I of G, $G - I$ has a fractional k-factor. When $k = 1$, we say that G is *fractional ID-factor-critical* if for every independent set I of G, $G - I$ has a fractional perfect matching.

Liu and Zhang gave a necessary and sufficient condition for a graph to have fractional (g, f)-factor and a k-factor in [4] and [8], respectively.

Lemma 1.1. *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$, $\Phi_G(S; k) = k|S| - k|T| + d_{G-S}(T) \geq 0$, where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq k - 1\}$.*

Lemma 1.2. *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$, $k|S| - \sum_{i=0}^{k-1}(k - i)p_i(G - S) \geq 0$, where $p_i(G - S) = |\{x : x \in V(G) - S, d_{G-S}(x) = i\}|$.*

The degree condition of ID-factor-critical graphs was studied in [3].

Lemma 1.3. *Let G be a graph with n vertices. Then G is ID-factor-critical if $\delta(G) \geq (2n - 1)/3$.*

In this paper, we discuss the degree conditions of fractional ID-k-factor-critical graphs. The main results will be given in the next section.

2. Main results

We begin our discussion with a well-known theorem of Dirac [2].

Lemma 2.1. *Let G be a graph on $n \geq 3$ vertices with $\delta(G) \geq n/2$. Then G is hamiltonian.*

The next result follows easily from Lemma 2.1.

Lemma 2.2. *If G is a graph of order n and $\delta(G) \geq 2n/3$, then G is fractional ID-k-factor-critical when $k = 1,\ 2$.*

Proof. Let I be an independent set of G. It is easy to see that $n - |I| \geq \delta(G)$. Hence

$$\begin{aligned} 2\delta(G) - |I| - n &= 2\delta(G) + n - |I| - 2n \\ &\geq 3\delta(G) - 2n \ \geq \ 0. \end{aligned}$$

It follows that $\delta(G) - |I| \geq (n - |I|)/2$.

Let $H = G - I$. Then $|V(H)| = n - |I|$, and $\delta(H) \geq \delta(G) - |I| \geq |V(H)|/2$. By Lemma 2.1, H has a hamiltonian cycle C. C is also a fractional 2-factor and C also contains a fractional perfect matching. Thus Lemma 2.2 holds. ∎

Theorem 2.1. *Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq 2n/3$, then G is fractional ID-k-factor-critical.*

Proof. Let X be an independent set of G and $H = G - X$. We have that $|V(H)| = n - |X|$ and $\delta(H) \geq |V(H)|/2$ by the same argument of Lemma 2.2. Clearly, Theorem 2.1 holds when $k = 1$ or $k = 2$. Therefore, we may assume $k \geq 3$.

We prove the theorem by contradiction. Suppose H has no fractional k-factor. Then by Lemma 1.1, there exists some subset $S \subseteq V(H)$ such that $\Phi_H(S;k) = k|S| - k|T| + d_{H-S}(T) \leq -1$, where $T = \{x|\ x \in V(H) - S, d_{H-S}(x) \leq k-1\}$. Set $\Psi_H(S;k) = \Phi_H(S;k) + 1$. It follows that $\Psi_H(S;k) \leq 0$.

Let $h_1 = \min\{d_{H-S}(x)|\ x \in T\}$. Choose $x_1 \in T$ such that $d_{H-S}(x_1) = h_1$. If $T - N_T[x_1] \neq \emptyset$, let $h_2 = \min\{d_{H-S}(x)|\ x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{H-S}(x_2) = h_2$.

Set $|S| = s$, $|T| = t$, and $|\ N_T[x_1]\ | = p$. We have $p \leq h_1 + 1$, $d_{H-S}(T) \geq h_1 p + h_2(t-p)$, and

$$\begin{aligned} 0 \geq \Psi_H(S;k) &= ks - kt + d_{H-S}(T) + 1 \\ &\geq ks - kt + h_1 p + h_2(t-p) + 1. \end{aligned}$$

Set $|V(H)| = m$. Then $m = n - |X| \geq \delta(G) \geq 2n/3 \geq (12k-16)/3 = 4k - 16/3$. Since m is an integer, we have that $m \geq 4k - 5$.

We consider the following cases.

Case 1. $T = N_T[x_1]$.

In this case, we have $t = p \leq h_1 + 1$, $0 \leq h_1 \leq k-1$, $h_2 = 0$. By $\delta(H) \geq (n - |X|)/2 \geq n/3 \geq (6k-8)/3 \geq k$ $(k \geq 3)$ and $d_H(x_1) \leq s + h_1$, we have $s \geq k - h_1$ and

$$\begin{aligned} \Psi_H(S;k) &\geq ks - kt + h_1 p + h_2(t-p) + 1 \\ &= ks + (h_1 - k)t + 1 \\ &\geq k(k - h_1) + (h_1 - k)t + 1 \\ &= (k - h_1)(k - t) + 1 \ \geq\ 1. \end{aligned}$$

Then we get a contradiction.

Case 2. $T - N_T[x_1] \neq \emptyset$.

Subcase 2.1. $0 \leq h_1 \leq 2$.

In this case, we have $t > p$, $0 \leq h_1 \leq h_2$, $m/2 \leq d_H(x_1) \leq s + h_1$. Then $s \geq m/2 - h_1 \geq (4k-5)/2 - h_1 = 2k - 5/2 - h_1$. Since s is an integer and $m - s - t \geq 0$, we have $s \geq 2k - 2 - h_1$, $t \leq m - s \leq s + 2h_1$. Then we obtain that

$$\begin{aligned} 0 \ \geq\ \Psi_H(S;k) &\geq ks - kt + h_1 p + h_2(t-p) + 1 \\ &\geq ks - kt + h_1 t + 1 \\ &= ks + (h_1 - k)t + 1 \\ &\geq ks + (h_1 - k)(s + 2h_1) + 1 \\ &= h_1 s + 2h_1^2 - 2h_1 k + 1 \\ &\geq h_1(2k - 2 - h_1) + 2h_1^2 - 2h_1 k + 1 \\ &= h_1^2 - 2h_1 + 1 = (h_1 - 1)^2. \end{aligned}$$

When $h_1 = 0$ or $h_1 = 2$ (since $0 \leq h_1 \leq 2$ and h_1 is an integer), we have

$$0 \geq \Psi_H(S;k) \geq 1,$$

a contradiction.

When $h_1 = 1$, we have $\Psi_H(S;k) \geq 0$ and we notice that $\Psi_H(S;k) = 0$ holds if and only if $s = 2k-2-h_1 = 2k-3$ and $t = s+2h_1 = 2k-1$. Then $m \leq 2s+2h_1 = 4k-4$ and $m \geq s+t = 4k-4$, so $m = 4k-4 = s+t$. Therefore $H = G[S \cup T]$ and $\mid N_T[x_1] \mid = p = h_1 + 1 = 2$, $\mid N_T(x_1) \mid = 1$.

So for every vertex $v \in T$, $\mid N_T(v) \mid \geq \mid N_T(x_1) \mid \geq 1$, and $t = 2k-1$ is odd, it follows that there exists a vertex $u \in T$ such that $\mid N_T(u) \mid \geq 2$.

$$\begin{aligned} 0 \ \geq \ \Psi_H(S;k) &= ks - kt + d_{H-S}(T) + 1 \\ &\geq ks - kt + (t-1) + 2 + 1 \\ &= k(2k-3) - k(2k-1) + (2k-1-1) + 3 \\ &= 1, \end{aligned}$$

a contradiction, too.

Subcase 2.2. $h_1 \geq 3$.

In this case, $3 \leq h_1 \leq h_2 \leq k-1$. Then $k - h_2 \geq 1$ and $m - s - t \geq 0$. Thus $(k-h_2)(m-s-t) \geq 0$. So

$$\begin{aligned} (k-h_2)(m-s-t) &\geq \Psi_H(S;k) \\ &\geq ks - kt + h_1 p + h_2(t-p) + 1 \\ &= ks + (h_1 - k)p + (h_2 - k)(t-p) + 1. \end{aligned}$$

It follows that

$$(k-h_2)(m-s) - ks \geq (h_1 - h_2)(h_1 + 1) + 1. \tag{2.1}$$

Since $m \geq 4k-5$, we have

$$h_2 m \geq h_2(4k-5). \tag{2.2}$$

Furthermore, since $m/2 \leq d_H(x_1) \leq s + h_1$ and $m/2 \leq d_H(x_2) \leq s + h_2$, we have $2s - m \geq -(h_1 + h_2)$. Then we can obtain that

$$(2s-m)(2k-h_2) \geq -(h_1+h_2)(2k-h_2). \tag{2.3}$$

By $(2.2) + (2.3) + 2 \times (2.1)$, we get

$$\begin{aligned} 0 &\geq h_2(4k-5) - (h_1+h_2)(2k-h_2) + 2(h_1-h_2)(h_1+1) + 2 \\ &= 2(h_2-h_1)k + h_2(h_2+h_1-5) + 2(h_1-h_2)(h_1+1) + 2. \end{aligned}$$

Set $\Omega(k) = 2(h_2-h_1)k + h_2(h_2+h_1-5) + 2(h_1-h_2)(h_1+1) + 2$. Then we obtain

$$0 \geq \Omega(k). \tag{2.4}$$

Since $k \geq h_2 + 1$ and $\Omega(k)$ is a nondecreasing function for k ($h_2 \geq h_1$), then we obtain that $\Omega(k) \geq \Omega(h_2+1) = 3h_2^2 - (3h_1+5)h_2 + 2h_1^2 + 2$. Set $\Delta = (3h_1 + 5)^2 - 12(2h_1^2+2) = -15(h_1-1)^2 + 16$. And $\Delta < 0$ when $h_1 \geq 3$. It follows that $\Omega(h_2+1) > 0$ and $\Omega(k) \ \geq \ \Omega(h_2+1) \ > \ 0$, which contradicts (2.4).

The above arguments yield that H has a fractional k-factor and G is fractional ID-k-factor-critical. The proof is completed. ∎

In [8] we have the following result about fractional k-factors.

Theorem 2.2. *If G has fractional k-factors, then G has fractional m-factor for $1 \le m \le k$.*

Theorem 2.2 implies immediately the following result.

Theorem 2.3. *If a graph G is fractional ID-k-factor-critical, then G is fractional ID-m-factor-critical for $1 \le m \le k$.*

3. The sharpness of the bounds in Theorem 2.1

In this section we show that the conditions in Theorem 2.1 are best possible.

Let $G = (2k-4)K_1 \vee (2k-3)K_1 \vee (k-1)K_2$. Then we have $n = |V(G)| = 6k-9$ and $\delta(G) = 4k-6 \ge 2n/3$. Clearly, $A = (2k-3)K_1$ is an independent set of G. Let $H = G - A = (2k-4)K_1 \ \vee \ (k-1)K_2$. Choose $S = (2k-4)K_1$. Then $\sum_{i=0}^{k-1}(k-i)p_i(H-S) = (k-1)(2k-2) = k(2k-4)+2 > k(2k-4) = k|S|$. Therefore, by Lemma 1.2, H has no fractional k-factor. Hence G is not fractional ID-k-factor-critical. In this sense, the bound of n is best possible.

This bound of $\delta(G)$ is sharp indeed. To see this, we construct a graph G with $\delta(G) = \lceil 2n/3 \rceil - 1$ which is not fractional ID-k-factor-critical as follows.

Case 1. $n = 3m$.

In this case, let $G = (m-1)K_1 \ \vee \ mK_1 \ \vee \ (m+1)K_1$, $n = |V(G)| = 3m$ and $\delta(G) = 2m-1 = \lceil 2n/3 \rceil - 1$.

Clearly, $A = (m-1)K_1$ is an independent set of G. Let $H = G - A = mK_1 \vee (m+1)K_1$. Choose $S = mK_1$. Then $\sum_{i=0}^{k-1}(k-i)p_i(H-S) = k(m+1) > km = k|S|$. By Lemma 1.2, H has no fractional k-factor. So G is not fractional ID-k-factor-critical.

Case 2. $n = 3m+1$.

In this case, let $G = mK_1 \ \vee \ mK_1 \ \vee \ (m+1)K_1$, $n = |V(G)| = 3m+1$ and $\delta(G) = 2m = \lceil 2n/3 \rceil - 1$. Clearly, $A = mK_1$ is an independent set of G. Let $H = G - A = mk_1 \ \vee \ (m+1)K_1$. By the same argument as above, H has no fractional k-factor. Thus G is not fractional ID-k-factor-critical.

Case 3. $n = 3m+2$.

In this case, let $G = (m+1)K_1 \ \vee \ mK_1 \ \vee \ (m+1)K_1$, $n = |V(G)| = 3m+2$ and $\delta(G) = 2m+1 = \lceil 2n/3 \rceil - 1$. Clearly, $A = (m+1)K_1$ is an independent set of G. We obtain that G is not fractional ID-k-factor-critical by the same argument as above.

When $k = 1$, let G be a graph and let I be an arbitrary independent set of G. If I has the same parity with $|V(G)|$, we have known that if $\delta(G) \ge (2n-1)/3$, then G is ID-factor-critical, that is, $G - I$ has a perfect matching [3]. Obviously, $G - I$ has a fractional perfect matching. If I does not have the same parity with $|V(G)|$, we have known that if $\delta(G) \ge 2n/3$, G is fractional ID-factor-critical, that is, $G - I$

has a fractional perfect matching. Hence the bound of $\delta(G)$ is sharp by the above argument.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.
[2] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* (*3*) **2** (1952), 69–81.
[3] Y. Liu and Y. Ma, Degree conditions of ID-factor-critical graphs, *Southeast Asian Bull. Math.* **27** (2003), no. 4, 641–648.
[4] G. Liu and L. Zhang, Fractional (g, f)-factors of graphs, *Acta Math. Sci. Ser. B Engl. Ed.* **21** (2001), no. 4, 541–545.
[5] L. Lovász and M. D. Plummer, Matching Theory, *Ann. Discrete Math.*, 29, North-Holland, Amsterdam, 1986.
[6] E. R. Scheinerman and D. H. Ullman, *Fractional Graph Theory*, Wiley, New York, 1997.
[7] J. Yuan, Independent-set-deletable factor-critical power graphs, *Acta Math. Sci. Ser. B Engl. Ed.* **26** (2006), no. 4, 577–584.
[8] L. J. Zhang and G. Z. Liu, Fractional k-factors of graphs, *J. Systems Sci. Math. Sci.* **21** (2001), no. 1, 88–92.